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Quasi-Invariant Cylindrical Measures

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In this paper we introduce quasi-invariant cylindrical measures on Banach spaces. The definition generalizes the notion of quasi-invariant measures. However, contrary to the measure case, there exist non-trivial quasi-invariant cylindrical measures on infinite dimensional Banach spaces. It is shown that a Banach space is isomorphic to a Hilbert space if and only if it admits a quasi-invariant cylindrical measure of type 2. Moreover, we prove that each rotation-invariant cylindrical measure on an infinite dimensional Hilbert space is quasi-invariant, whenever the cylindrical measure satisfies an additional assumption. For instance the canonical Gaussian cylindrical measure on a Hilbert space is quasi-invariant.

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Throughout the paper E is a real Banach space and H is a real Hilbert space. By E' we denote the dual space. Let $a_1, \dots, a_n$ be a finite system of elements of E'. Then by $\mathbf{a}_i$ we denote the operator from E into R_n mapping x onto the vector $(\langle x, a_1 \rangle, \dots, \langle x, a_n \rangle)$. We write $\langle x, \mathbf{a}_i \rangle$ instead of $\mathbf{a}_i(x)$. By $\mathscr{B}(R_n)$ we denote the Borel sets of R_n. A set $Z \subseteq E$ is said to be a *cylindrical set* if there are $a_1, \dots, a_n \in E'$ and $B \in \mathscr{B}(R_n)$ such that $Z = (\mathbf{a}_i)^{-1}(B)$. A map μ from the algebra of all cylindrical sets into $[0, 1]$ is called a *cylindrical measure* if it satisfies the two following conditions:

(1) $\mu(E) = 1$

(2) Restrict μ to the σ-algebra of cylindrical sets which are generated by a fixed system of functionals. Then each such restriction is σ-additiv.

By putting

$$\mu_{a_1, \dots, a_n}(B) = \mu((\mathbf{a}_i)^{-1}(B))$$

each cylindrical measure μ defines a family of normed Borel measures. Then the family $\{\mu_{a_1, \dots, a_n}\}$ satisfies the following consistency condition:

Let $a_1, \dots, a_n$ and $b_1, \dots, b_m$ be two finite systems of functionals on E. Moreover let A and B be two Borel sets of R_n and R_m respectively. Then $(\mathbf{a}_i)^{-1}(A) = (\mathbf{b}_j)^{-1}(B)$ implies that $\mu_{a_1, \dots, a_n}(A) = \mu_{b_1, \dots, b_m}(B)$.

Conversely each family of normed Borel measures satisfying the consistency condition defines a cylindrical measure and the correspondence is one-to-one.

A cylindrical measure μ is of *type* r (for $r > 0$) if there is a constant c such that

$$\left(\int_{-\infty}^{\infty} |t|^r \, d\mu_a(t)\right)^{1/r} \leqq c \, \|a\|$$

for all $a \in E'$. A cylindrical measure μ is of *type* 0 if $\mu_{a_n}[-\rho, \rho]$ tends to 1 for all $\rho > 0$, whenever a_n tends to 0 in E'.

For each cylindrical measure μ the complex valued function φ_μ from E' denotes its characteristic function. It is known that μ is of type 0 if and only if φ_μ is continuous with respect to the norm-topology of E'.

Suppose that for every $\varepsilon > 0$ and every $x_0 \in E$ there is a $\delta > 0$ such that

$$\mu(Z + x_0) < \varepsilon$$

for every cylindrical set Z of E for which

$$\mu(Z) < \delta.$$

Then μ is said to be *quasi-invariant.*

Now let us state two equivalent definitions for later use.

(1) Let $\varepsilon > 0$ and $x_0 \in E$ be given. Then there exists a $\delta > 0$ such that

$$\mu_{a_1, \dots, a_n}(B + \langle x_0, \mathbf{a}_i \rangle) < \varepsilon$$

for every finite system $a_1, \dots, a_n \in E'$ and every $B \in \mathscr{B}(R_n)$ for which

$$\mu_{a_1, \dots, a_n}(B) < \delta.$$

(2) Same as in (1) with "$B \in \mathscr{B}(R_n)$" replaced by "$B \subseteq R_n$ and B compact".

Let μ be a quasi-invariant cylindrical measure on E. Let us further assume that μ is σ-additiv. Then μ has an unique extension to the σ-algebra generated by the cylindrical sets. Of course this extension is a quasi-invariant measure. Hence, by a theorem of Sudakov [9], E is finite dimensional. This shows that quasi-invariant cylindrical measures on infinite dimensional Banach spaces cannot be σ-additiv.

The following theorem generalizes a result of Xia (cf. [11]). We prove it for cylindrical measures instead of measures.

Theorem 1. *Assume that E admits a quasi-invariant cylindrical measure μ. Then the mapping*

$$d(a, b) = \int_E \min\{1, |\langle x, a - b \rangle|\} \, d\mu(x), \qquad a, b \in E',$$

defines a metric on E'. The topology generated by d is stronger than the norm topology. In particularly, if μ is of type r, $0<r<\infty$, then there is a constant $c\geqq 0$ such that

$$\|a\| \leqq c\{\int_E |\langle x, a\rangle|^r \, d\mu(x)\}^{1/r}$$

for all $a \in E'$.

Proof.[1] Let us first show that d satisfies the axioms of a metric. Of course, d is a pseudometric. If

$$\int_E \min\{1, |\langle x, a\rangle|\} \, d\mu(x) = 0$$

then

$$\mu\{x \in E; |\langle x, a\rangle| > \rho\} = 0$$

for all $\rho > 0$.

Hence,

$$\mu\{x \in E; |\langle x + x_0, a\rangle| > \rho\} = 0$$

whenever $x_0 \in E$.

This implies $\langle x_0, a\rangle = 0$ and, since x_0 was arbitrary, it follows $a = 0$.

Now, let $a_n \in E'$ be a sequence tending to zero with respect to d, that is

$$\lim_{n\to\infty} \mu\{x \in E; |\langle x, a_n\rangle| > \rho\} = 0$$

for all $\rho > 0$.

By assumption we get

$$\lim_{n\to\infty} \mu\{x \in E; |\langle x_0 + x, a_n\rangle| > \rho\} = 0$$

for each $x_0 \in E$. Consequently, there are elements $x_n \in E$, $n \geqq N$, such that the inequalities

$$|\langle x_n, a_n\rangle| \leqq \rho \quad \text{and} \quad |\langle x_0 + x_n, a_n\rangle| \leqq \rho$$

hold.

From this it follows $\lim_{n\to\infty} \langle x_0, a_n\rangle = 0$ and by the Banach-Steinhaus-theorem we get $\sup_n \|a_n\| < \infty$.

Now, there exists a sequence of positive real numbers α_n such that $\lim_{n\to\infty} \alpha_n = \infty$ and $\alpha_n a_n$ tends to zero with respect to d (cf. [6], p. 40). This and the result above imply $\lim_{n\to\infty} \|a_n\| = 0$. This proves the first part of the theorem. The second statement of the theorem follows immediately from the first one.

Remark. The following example shows that the converse of Theorem 1 is not true in general.

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[1] This proof is due to C. Borell. It is easier than the original one and it includes the case $r=0$. The main idea of the proof can be found in [2]

Let r_n be the n-th Rademacher function. By setting

$$X(a)(t)=\sum_{n=1}^{\infty}\alpha_n r_n(t)$$

for $a=(\alpha_n)\in l_2$ we define a continuous operator X from l_2 into $L_r[0,1]$ for each $r\geqq 0$.

The well-known Hincin inequality asserts that the cylindrical measure μ on l_2 satisfies the statements of Theorem 1, where μ is defined by

$$\mu(Z)=\lambda\{t\in[0,1];(X(a_1)(t),\ldots,X(a_n)(t))\in B\}$$

for $Z=(\mathbf{a}_i)^{-1}(B)$, the Lebesgue measure λ and $a_1,\ldots,a_n\in l_2$. But μ is of course not quasi-invariant.

As an easy consequence of Theorem 1 we get

Theorem 2. *Suppose that the Banach space E admits a quasi-invariant cylindrical measure of type r for $r\geqq 0$. Then there exists a finite measure space (Ω,ν) such that E' is isomorphic to a subspace of $L_r(\Omega,\nu)$.*

Proof. Let μ be a quasi-invariant cylindrical measure of type r defined on E. Then there exists a continuous operator X from E' into a suitable space $L_r(\Omega,\nu)$ such that $\nu(\Omega)=1$ and that

$$\mu(Z)=\nu\{\omega\in\Omega;(X(a_1)(\omega),\ldots,X(a_n)(\omega))\in B\}$$

for all cylindrical sets Z with $Z=(\mathbf{a}_i)^{-1}(B)$, (cf. [7]). Consequently

$$\int_E|\langle x,a\rangle|^r\,d\mu(x)=\int_\Omega|X(a)(\omega)|^r\,d\nu(\omega)=\|X(a)\|^r,\qquad r>0.$$

By Theorem 1 it follows that X is an injection. This concludes the proof of Theorem 2, since the case $r=0$ follows in the same way.

Taking $r=2$ we get an interesting corollary.

Corollary 1. *Suppose that there exists a quasi-invariant cylindrical measure of type 2 on E. Then E is isomorphic to a Hilbert space.*

We will see later on that the converse of Corollary 1 is true, as well. Before we are able to prove this, we need some lemmas which are interesting in their own right. To start with, we introduce some notations.

Let D be a subset of E'. Then a cylindrical measure μ is said to be *D-quasi-invariant* if for each $\varepsilon>0$ and each $x_0\in E$ there exists a $\delta>0$ such that $\mu(Z+x_0)<\varepsilon$ for all cylindrical sets Z for which $\mu(Z)<\delta$, where $Z=(\mathbf{d}_i)^{-1}(B)$ with $d_1,\ldots,d_n\in D$ and $B\in\mathscr{B}(R_n)$.

If $K\subseteq R_n$ and if $\rho>0$, the symbol $[K]_\rho$ denotes the set

$$[K]_\rho=\{\xi\in R_n;\inf_{\eta\in K}\|\xi-\eta\|\leqq\rho\}.$$

Here and in the following by $\|\cdot\|$ we mean the Euclidean norm. Now we can state a useful lemma.

Lemma 1. *Suppose that D is a dense subset of E'. Then each D-quasi-invariant cylindrical measure of type* 0 *on E is quasi-invariant.*

Proof. Let $a_1, \ldots, a_n$ be a fixed finite system of elements of E'. Choose $\delta > 0$ such that $\mu(Z + x_0) < \varepsilon/2$ for

$$\mu(Z) < \delta \quad \text{and} \quad Z = (\mathbf{d}_i)^{-1}(B),$$

where $x_0 \in E$, $d_1, \ldots, d_n \in D$ and $B \in \mathscr{B}(R_n)$.

Now let $K \subseteq R_n$ be a compact subset with $\mu_{a_1, \ldots, a_n}(K) < \delta/3$. Then there exists a $\sigma > 0$ such that

$$\mu_{a_1, \ldots, a_n}([K]_{2\sigma} \setminus K) < \delta/3.$$

Moreover, there are elements $d_1, \ldots, d_n \in D$ such that

$$\mu(Z_\sigma) \geqq 1 - \min(\delta/3, \varepsilon/2),$$

and

$$\|\langle x_0, \mathbf{a}_i - \mathbf{d}_i \rangle\| < \sigma/2,$$

where

$$Z_\sigma = \{x \in E; \|\langle x, \mathbf{a}_i - \mathbf{d}_i \rangle\| < \sigma/2\}.$$

Now we define the following cylindrical sets:

$$Z_0 = (\mathbf{a}_i)^{-1}(K),$$

$$Z_1 = (\mathbf{d}_i)^{-1}([K]_\sigma),$$

$$Z_2 = (\mathbf{a}_i)^{-1}([K]_{2\sigma}).$$

By $Z_1 \cap Z_\sigma \subseteq Z_2$ it follows that

$$\mu(Z_1) \leqq \mu(Z_1 \cap Z_\sigma) + \mu(E \setminus Z_\sigma) \leqq \mu(Z_2) + \delta/3$$
$$\leqq \mu(Z_0) + (2/3)\,\delta < \delta.$$

Thus

$$\mu(Z_1 + x_0) < \varepsilon/2.$$

Now let us assume for a moment that the inclusion

$$(*) \quad (Z_0 + x_0) \cap Z_\sigma \subseteq Z_1 + x_0$$

is already proven.

Then it follows immediately that

$$\mu(Z_0 + x_0) = \mu_{a_1, \ldots, a_n}(K + \langle x_0, \mathbf{a}_i \rangle) < \varepsilon.$$

Obviously, μ is quasi-invariant.

Hence the proof is finished when we show (*).

Put $x \in (Z_0 + x_0) \cap Z_\sigma$. Then from

$$\|\langle x - x_0, \mathbf{a}_i - \mathbf{d}_i \rangle\| \leqq \|\langle x, \mathbf{a}_i - \mathbf{d}_i \rangle\| + \|\langle x_0, \mathbf{a}_i - \mathbf{d}_i \rangle\| < \sigma$$

and from $\langle x - x_0, \mathbf{a}_i \rangle \in K$ we conclude that

$$\langle x - x_0, \mathbf{d}_i \rangle \in [K]_\sigma.$$

So $x \in (Z_1 + x_0)$.

The next lemma can be easily proven. Thus, we omit the proof.

Lemma 2. *Let A be an arbitrary subset of E'. Then each A-quasi-invariant cylindrical measure is* (span *A*)*-quasi-invariant.*

From Lemma 1 and Lemma 2 we derive:

Corollary 2. *Suppose that the linear hull of a subset $A \subseteq E'$ is dense. Then every A-quasi-invariant cylindrical measure of type* 0 *on E is quasi-invariant.*

In the following by γ_n we denote the Gaussian measure on R_n with density

$$(2\pi)^{-n/2} \exp(-\|\xi\|^2/2) \quad \text{for } \xi \in R_n.$$

Lemma 3. *Let $B \in \mathscr{B}(R_n)$, let $\xi_0 \in R_n$ and let $\sigma > 0$. Then the following inequality holds:*

$$\gamma_n(B + \xi_0) \leqq e^\sigma \gamma_n(B) + \int_{|t| \geqq \sigma/\|\xi_0\|} e^{|t|\,\|\xi_0\|} \, d\gamma_1(t).$$

Proof. Using the well-known fact that

$$\gamma_n\{\xi \in R_n; (\xi, \xi_0) \in B\} = \gamma_1\{t \in R_1; t \cdot \|\xi_0\| \in B\}$$

and some elementary inequalities it follows that

$$\begin{aligned}
\gamma_n(B + \xi_0) &= \int_B \exp[(\xi, \xi_0) - \|\xi_0\|^2/2] \, d\gamma_n(\xi) \\
&\leqq \int_B \exp[|(\xi, \xi_0)|] \, d\gamma_n(\xi) \\
&\leqq \int_{B \cap \{\xi; |(\xi, \xi_0)| \leqq \sigma\}} \exp|(\xi, \xi_0)| \, d\gamma_n(\xi) \\
&\quad + \int_{\{\xi; |(\xi, \xi_0)| \geqq \sigma\}} \exp|(\xi, \xi_0)| \, d\gamma_n(\xi) \\
&\leqq e^\sigma \gamma_n(B) + \int_{|t| \geqq \sigma/\|\xi_0\|} e^{|t|\,\|\xi_0\|} \, d\gamma_1(t).
\end{aligned}$$

This proves Lemma 3.

We denote the cylindrical measure μ, uniquely defined by $\mu_{e_1, \ldots, e_n} = \gamma_n$ for $(e_i, e_j) = \delta_{ij}$ (Kronecker's symbol), by γ, and call it *canonical Gaussian cylindrical measure.*

Remark. γ is quasi-invariant.

This follows immediately from Lemma 1, Lemma 2 and Lemma 3. This remark has an interesting consequence.

Let T be an operator (linear and continuous) from a real Banach space E into a real Banach space F. Moreover, let μ be a quasi-invariant cylindrical measure on E. Hence the cylindrical measure $T(\mu)$ (cf. [7] for the definition) is quasi-invariant under translations by elements of F, which are in the image of E with respect to T.

If we apply this statement to the canonical Gaussian cylindrical measure on $L_2[0,1]$ and to the integral operator T from $L_2[0,1]$ into $C[0,1]$ with

$$T(f)(t)=\int_0^t f(s)\,ds \qquad \text{for } t\in[0,1],$$

then we receive the well-known result, due to Maruyama [5], of quasi-invariance of the Wiener measure under translation by any function $g\in C[0,1]$, $g(0)=0$, which is absolutely continuous and for which $g'\in L_2[0,1]$ (cf. [4]).

Next we want to improve the remark above. More precisely, we show the quasi-invariance of a class of cylindrical measures including the canonical Gaussian cylindrical measure.

A cylindrical measure μ on H is *rotation-invariant* if whenever Z is a cylindrical set and U an isometric operator from H onto H,

$$\mu(Z)=\mu(U(Z)).$$

Note that the canonical Gaussian cylindrical measure is rotation-invariant. Since the (cylindrical) measure δ_0 defined by $\delta_0(Z)=0$ for $0\notin Z$ and $\delta_0(Z)=1$ for $0\in Z$, is rotation-invariant, but of course not quasi-invariant, rotation invariance does not imply quasi-invariance in general. However, we want to show that under some additional assumptions rotation-invariance implies quasi-invariance. Let us start with a lemma (cf. [1], p. 170 and p. 172, resp. [10]).

Lemma 4. *Suppose that H is infinite dimensional. Let μ be a rotation-invariant cylindrical measure on H. Then there exists a finite Borel measure λ_μ on $[0,\infty)$ such that*

$$\mu_{e_1,\ldots,e_n}(B)=\int_{t>0}\gamma_n\left(\frac{B}{t}\right)d\lambda_\mu(t)+\lambda_\mu(\{0\})\,\delta_0(B)$$

for every $B\in\mathscr{B}(R_n)$ and every finite system $e_1,\ldots,e_n\in H$ for which $(e_i,e_j)=\delta_{ij}$.

Moreover, $\lambda_\mu(\{0\})=0$ if and only if $\lim_{t\to\infty}\varphi_\mu(tx)=0$ for all $x\in H$, $x\neq 0$. Here φ_μ denotes the characteristic function of μ.

Now we are in position to prove the above mentioned connection between rotation-invariance and quasi-invariance.

Theorem 3. *Let μ be a rotation-invariant cylindrical measure on the infinite dimensional Hilbert space H. If $\lim_{t\to\infty}\varphi_\mu(tx)=0$ for all $x\in H$, $x\neq 0$, then μ is quasi-invariant.*

Proof.[2] Let μ be a cylindrical measure on H satisfying the assumptions of the theorem and let H_0 be any complete orthonormal system of H. Since μ is of type 0, by Corollary 2 it follows that it is enough to prove the H_0-quasi-invariance of μ. Consider $\varepsilon>0$, $x_0\in H$ and $e_1,\dots,e_n\in H_0$. Then by virtue of Lemma 4 there exists a finite Borel measure λ_μ on $(0,\infty)$ such that

$$\mu_{e_1,\dots,e_n}(B)=\int_{t>0}\gamma_n((1/t)B)\,d\lambda_\mu(t)$$

for each $B\in\mathscr{B}(R_n)$.

Now we choose a number $t_0>0$ such that

$$\lambda_\mu(0,t_0)<\varepsilon/3.$$

Then we get the following inequality:

$$\begin{aligned}\mu_{e_1,\dots,e_n}&(B+(x_0,\mathbf{e}_i))\\&=\int_{t>0}\gamma_n\left(\frac{B+(x_0,\mathbf{e}_i)}{t}\right)d\lambda_\mu(t)\\&\leqq\varepsilon/3+e^\sigma\int_{t>0}\gamma_n((1/t)B)\,d\lambda_\mu(t)\\&\quad+\int_{t_0}^{\infty}\int_{|s|\geqq\sigma t/\|(x_0,\mathbf{e}_i)\|}\exp(|s|\,\|(x_0,\mathbf{e}_i)\|/t)\,d\gamma_1(s)\,d\lambda_\mu(t)\\&\leqq\varepsilon/3+e^\sigma\mu_{e_1,\dots,e_n}(B)+\lambda_\mu\{(0,\infty)\}\int_{|s|\geqq\sigma t_0/\|x_0\|}\exp(|s|\,\|x_0\|/t_0)\,d\gamma_1(s).\end{aligned}$$

This inequality is true for any $\sigma>0$. Now choose σ so large that the last term in the last line becomes less than $\varepsilon/3$. Putting $\delta=e^{-\sigma}\varepsilon/3$ we obtain

$$\mu_{e_1,\dots,e_n}(B+(x_0,\mathbf{e}_i))<\varepsilon$$

for $\mu_{e_1,\dots,e_n}(B)<\delta$.

This proves the statement of the theorem.

Remark. The last theorem yields the quasi-invariance of the cylindrical measures μ_p corresponding to the characteristic functions $\varphi_p(x)=\exp(-\|x\|^p)$ for $0<p\leqq 2$ and $x\in H$. Since μ_p is of type r but not of type p for $0<r<p<2$ we conclude the existence of quasi-invariant cylindrical measures not of type q for arbitrary small $q>0$. Compare this with problem 2.

Now we state the main result of this paper. This is an easy consequence of Corollary 1 and Theorem 3, resp. the remark after Lemma 3.

Theorem 4. *A Banach space E is isomorphic to a Hilbert space if and only if there exists a quasi-invariant cylindrical measure of type 2 on E.*

Finally, we state some open problems.

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[2] The author is grateful to A. Tortrat for pointing out a completely different proof of Theorem 3. Although this version is easier we give ours since it proves that the absolute continuity is uniform whenever $\|x_0\|\leqq\rho$

Since all known examples of quasi-invariant cylindrical measures are defined on Banach spaces isomorphic to a Hilbert space the following problem arises:

Problem 1. Are there quasi-invariant cylindrical measures on Banach spaces which are not isomorphic to a Hilbert space

By a result of Shepp [8] the cylindrical measures on l_p, $2<p<\infty$, or c_0 with characteristic functions $\exp(-\|a\|^{p'})$ for $a\in l_{p'}$ or $\exp(-\|a\|)$ for $a\in l_1$ are not quasi-invariant.

Problem 2. Is every quasi-invariant cylindrical measure of type 0?

If there exists a quasi-invariant cylindrical measure on a Hilbert space which is not of type 0, we would get a solution of the following problem:

Problem 3. Is every quasi-invariant cylindrical measure on a Hilbert space the translation of a rotation-invariant cylindrical measure

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