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Spectral Orders, Conditional Expectations and Martingales

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1. Introduction

In this paper, we characterize conditional expectations and martingales in terms of the strong spectral order $\prec$ and submartingales in terms of the weak spectral order $\preccurlyeq$. Using these characterizations, we prove that the conditional form of Jensen's inequality is, in fact, a particular case of an extended form of a theorem of Hardy-Littlewood-Pólya obtained in [3, Theorem 2.5], and we give conditions for equality which we have not found mentioned in the literature before. Moreover, we also obtain a new proof of Doob's optional sampling theorem.

If X and Y are integrable random variables defined on a probability space $(\Omega, \mathscr{F}, P)$, then we write $X \preccurlyeq Y$ whenever

$$E[(X-t)^+] \leqq E[(Y-t)^+] \tag{1}$$

for all $t \in R$, and we write $X \prec Y$ to mean $X \preccurlyeq Y$ and $E[X]=E[Y]$.

The notation $\prec$ and $\preccurlyeq$ are respectively referred to as the *strong* and *weak spectral order*.

If $X_i \prec Y_i$ (respectively $X_i \preccurlyeq Y_i$) and $P[X_i \neq 0, X_j \neq 0]=0$ and $P[Y_i \neq 0, Y_j \neq 0]=0$, $1 \leqq i < j \leqq n$, then it is easy to see that

$$\sum_{i=1}^{n} X_i \prec \sum_{i=1}^{n} Y_i \tag{2}$$

(respectively $\sum_{i=1}^{n} X_i \preccurlyeq \sum_{i=1}^{n} Y_i$). Moreover, if $X \prec Y$ (respectively $X \preccurlyeq Y$) and if A is an event containing $\{\omega : X(\omega) \neq Y(\omega)\}$, then

$$X 1_A \prec Y 1_A \tag{3}$$

(respectively $X 1_A \preccurlyeq Y 1_A$).

2. Spectral Orders and Conditional Expectations

Theorem 1. *If $X \in L^1(\Omega, \mathscr{F}, P)$ is any random variable, if $\mathscr{G}$ is any sub-σ-algebra of $\mathscr{F}$, and Y a $\mathscr{G}$-measurable random variable, then $Y \leqq E(X|\mathscr{G})$ (respectively $Y = E(X|G)$) if and only if*

$$Y 1_A \preccurlyeq X 1_A \quad (Y 1_A \prec X 1_A) \tag{4}$$

for all $A \in \mathscr{G}$.

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Proof. The condition is clearly sufficient.

To prove its necessity, suppose $Y \leqq E(X|\mathscr{G})$. Then $E[Y 1_A] \leqq E[X 1_A]$ for all $A \in \mathscr{G}$. Since Y is $\mathscr{G}$-measurable, the set $A \cap \{Y 1_A > t\}$ belongs to $\mathscr{G}$ for any $A \in \mathscr{G}$ and any $t \in R$. Thus, $E[Y 1_{A \cap \{Y 1_A > t\}}] \leqq E[X 1_{A \cap \{Y 1_A > t\}}]$ which implies that $E[(Y 1_A - t) 1_{\{Y 1_A > t\}}] \leqq E[(X 1_A - t) 1_{\{Y 1_A > t\}}] \leqq E[(X 1_A - t)^+]$, i.e., $E[(Y 1_A - t)^+] \leqq E[(X 1_A - t)^+]$ for all $A \in \mathscr{G}$ and for all $t \in R$. Hence (4) follows.

The following corollaries are obtained by the usual standard argument.

Corollary 1. *$Y = E(X|\mathscr{G})$ if and only if $YZ \prec XZ$ for all $\mathscr{G}$-measurable random variables Z such that $XZ \in L^1(\Omega, \mathscr{F}, P)$.*

Corollary 2. *$E(XZ|\mathscr{G}) = ZE(X|\mathscr{G})$ for all $\mathscr{G}$-measurable random variables Z such that $XZ \in L^1(\Omega, \mathscr{F}, P)$.*

Theorem 2. (Jensen's Inequality). *Suppose $X \in L^1(\Omega, \mathscr{F}, P)$. Then*

$$\Phi \circ E(X|\mathscr{G}) \leqq E(\Phi \circ X|\mathscr{G}) \quad a.s. \tag{5}$$

for all convex functions $\Phi: R \to R$ such that $\Phi \circ X \in L^1(\Omega, \mathscr{F}, P)$.

If, in addition, Φ is strictly convex, then equality holds in (5) if and only if $E(X|\mathscr{G}) 1_A$ and $X 1_A$ are identically distributed for each $A \in \mathscr{G}$ or, equivalently, $E(X|\mathscr{G})$ and X are identically distributed.

Proof. Let $Y = E(X|\mathscr{G})$. Since $Y 1_A \prec X 1_A$ for all $A \in \mathscr{G}$, we have, by Theorem 2.5 in [3], $E[\Phi(Y 1_A)] \leqq E[\Phi(X 1_A)]$ which implies that

$$\int_A \Phi \circ Y \, dP \leqq \int_A \Phi \circ X \, dP = \int_A E(\Phi \circ X|\mathscr{G}) \, dP \qquad \text{for all } A \in \mathscr{G}$$

and for all convex functions $\Phi: R \to R$ such that $\Phi \circ X \in L^1(\Omega, \mathscr{F}, P)$. Since both $\Phi \circ Y$ and $E(\Phi \circ X|\mathscr{G})$ are $\mathscr{G}$-measurable, inequality (5) follows directly from the last inequality.

If Φ is strictly convex such that $\Phi \circ X$ is integrable, then equality holds in (5) if and only if, for all $A \in \mathscr{G}$, $E[\Phi \circ (E(X|\mathscr{G}) 1_A)] = E[\Phi \circ (X 1_A)]$ which is the case if and only if $E(X|\mathscr{G}) 1_A$ and $X 1_A$ are identically distributed, by Theorem 2.3 in [3].

Finally, using the fact that two integrable random variables Y and Z are identically distributed if and only if $Y \prec Z$ and $Z \prec Y$, it is easily seen that $E(X|\mathscr{G})$ and X are identically distributed if and only if $E(X|\mathscr{G}) 1_A$ and $X 1_A$ are identically distributed for each $A \in \mathscr{G}$.

3. Spectral Orders and Martingales

In what follows, we let N denote the set of natural numbers, i.e., $N = \{1, 2, 3, \ldots\}$, and $\{\mathscr{F}_n : n \in N\}$ an increasing sequence of sub-σ-algebras of $\mathscr{F}$.

Theorem 3. *Suppose $\{X_n, \mathscr{F}_n : n \in N\}$ is a martingale (respectively a submartingale, a supermartingale). Let $\mathscr{F}_\infty$ be the σ-algebra generated by $\mathscr{F}_n$, $n \in N$. Let $\sigma: \Omega \to N \cup \{0\}$ and $\tau: \Omega \to N \cup \{0\}$ be two $\mathscr{F}_\infty$-measurable random variables such that $\sigma \leqq \tau$, $\{\sigma = 0\} = \{\tau = 0\}$, $\{\sigma = n\} \in \mathscr{F}_n$ and $\{\tau = n\} \in \mathscr{F}_n$ for all $n \in N$. Suppose further that one of the following two conditions is satisfied:*

(i) *$\{X_n : n \in N\}$ is uniformly integrable;*

(ii) *σ and τ are essentially bounded.*

If X_0 is any $\mathscr{F}_\infty$-measurable integrable random variable, then $X_\sigma \prec X_\tau$ (respectively $X_\sigma \preccurlyeq X_\tau$, $-X_\sigma \preccurlyeq -X_\tau$).

Proof. Assume that $\{X_n, \mathscr{F}_n : n \in N\}$ is a martingale.

Let $A_n = \{\sigma = n\}$ and $B_n = \{\tau = n\}$, $n \in N \cup \{0\}$. Then the inequality $\sigma \leqq \tau$ implies that $\bigcup_{i=0}^n B_i \subseteq \bigcup_{i=0}^n A_i$ for all $n \in N$. Define $Y_n = \sum_{i=0}^n X_i 1_{B_i} + X_n 1_{C_n} + X_\sigma 1_{D_n}$, where $C_n = \bigcup_{i=0}^n A_i - \bigcup_{i=0}^n B_i$ and $D_n = \Omega - \bigcup_{i=0}^n A_i$, $n \in N$. Clearly $Y_1 = X_\sigma$. We claim that $Y_n \prec Y_{n+1}$ for all $n \in N$. Now, for each $n \in N$, we have $C_n \in \mathscr{F}_n$ and so $X_n 1_{C_n} \prec X_{n+1} 1_{C_n}$, by Theorem 2. Thus, it follows from (2) that

$$
\begin{aligned}
Y_n &= \sum_{i=0}^{n} X_i 1_{B_i} + X_n 1_{C_n} + X_\sigma 1_{D_n} \prec \sum_{i=0}^{n} X_i 1_{B_i} + X_{n+1} 1_{C_n} + X_\sigma 1_{D_n} \\
&= \sum_{i=0}^{n+1} X_i 1_{B_i} + X_{n+1} 1_{C_{n+1}} + X_\sigma 1_{D_{n+1}} = Y_{n+1},
\end{aligned}
$$

where $n \in N$.

Now if (i) is satisfied, then it is easily seen that both X_σ and X_τ are integrable, and so $\{Y_n : n \in N\}$ is uniformly integrable, since $|Y_n| \leqq |X_\tau| + |X_n| + |X_\sigma|$ for all $n \in N$. Moreover, it is clear that $Y_n \to X_\tau$ pointwise everywhere. Thus, $Y_n \to X_\tau$ in L^1 by Theorem T21 in [6, p. 18]. But $X_\sigma = Y_1 \prec Y_n$ whence we conclude by taking limits that $X_\sigma \prec X_\tau$.

Next, if (ii) is satisfied, then it is easily seen that there exists a number $m \in N$ such that $Y_n = X_\tau$ a.s. for all $n \geqq m$. Hence $X_\sigma \prec X_\tau$.

The case that $\{X_n, \mathscr{F}_n : n \in N\}$ is a submartingale or a supermartingale is treated analogously.

It is now easy to derive Doob's Optional Sampling Theorem [4, Theorem 2.2, pp. 302–303].

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