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Pointwise Convergence in Terms of Expectations

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Introduction

This paper is concerned with the connection between almost sure convergence of a sequence of random variables and convergence of certain related expectations. Theorems of the kind we are interested in were proved by Meyer [7, p. 232] and Mertens [6, p. 47] in the continuous-parameter case, and by Baxter [1] in the discrete-parameter case. For example, Baxter's theorem is the following: Let $(X_n)_{n\geqq 1}$ be a sequence of random variables with values in a compact metric space S, and let the set Γ of bounded stopping times be directed by the obvious ordering. Then $(X_n)_{n\geqq 1}$ converges almost surely if and only if the generalized sequence $(\int \phi(X_\tau))_{\tau\in\Gamma}$ of expectations converges for every real-valued continuous function ϕ on S.

In the present paper we generalize this theorem in two ways: we replace S by an arbitrary complete separable metric space, and we use as few test functions ϕ as possible. If S is the real line, the single test function $\phi(x)=x$ suffices (Theorem 2); for any complete separable metric space, a countable set of functions suffices (Theorem 3); and for a separable Banach space, there is a countable set of convex functions which suffices (Theorem 4). We have included a different proof of the key step in Baxter's proof (Corollary 1), in order to make the present paper self-contained.

We wish to thank T. Figiel for simpler proofs of two of our theorems.

§ 1. Notation and Terminology

Throughout this paper $(\Omega, \mathscr{F}, P)$ is a probability space. We recall that a real random variable is a mapping $X: \Omega\to R$ which is $\mathscr{F}$-measurable. If S is a Polish space (i.e. S is a complete separable metric space), an S-valued random variable is a mapping $X: \Omega\to S$ which is measurable as a mapping of $(\Omega, \mathscr{F})$ into $(S, \mathscr{B}(S))$, where $\mathscr{B}(S)$ is the σ-algebra of Borel sets of S.

If $(X_i)_{i\in I}$ is any family of real (or S-valued) random variables, we denote by $\sigma((X_i)_{i\in I})$ the smallest sub-σ-algebra of $\mathscr{F}$ with respect to which every X_i, $i\in I$, is measurable.

Let $(\mathscr{F}_n)_{n\geqq 1}$ be an increasing sequence of sub-σ-algebras of $\mathscr{F}$. We recall that a mapping $\tau: \Omega\to N^*\cup\{+\infty\}=\{1,2,3,\dots,+\infty\}$ is called a *stopping time* (with respect to $(\mathscr{F}_n)_{n\geqq 1}$) if $\{\omega\,|\,\tau(\omega)=n\}\in\mathscr{F}_n$ for each $n\in N^*$.

In what follows, whenever $(X_n)_{n\geqq 1}$ is a sequence of real (or S-valued) random variables, we let $\mathscr{F}_n=\sigma(X_1,\dots,X_n)$ for each $n\geqq 1$, and we denote by Γ *the set of all bounded stopping times* (with respect to $(\mathscr{F}_n)_{n\geqq 1}$).

§ 2. The Case of Real-Valued Random Variables

We begin with the following elementary but extremely useful result:

Lemma 1. *Let* $(X_n)_{n\geqq 1}$ *be a sequence of real random variables. Let* $\mathscr{F}_n = \sigma(X_1, \ldots, X_n)$ *for each integer* $n \geqq 1$ *and assume that* $\mathscr{F} = \sigma\big(\bigcup_{n\geqq 1} \mathscr{F}_n\big)$. *Let* Y *be a real random variable with the following property: For each* $\omega \in \Omega$, $Y(\omega)$ *is a cluster value of the sequence* $(X_n(\omega))_{n\geqq 1}$. *Then given any* $\varepsilon > 0$, $\delta > 0$ *and integer* $m \geqq 1$, *there is a bounded stopping time* τ *such that* $\tau \geqq m$ *and*

$$P(\{\omega \mid |X_{\tau(\omega)}(\omega) - Y(\omega)| > \delta\}) \leqq \varepsilon.$$

Proof. Since $\mathscr{F} = \sigma\big(\bigcup_{n\geqq 1} \mathscr{F}_n\big)$, there is an integer $N \geqq m$ and a random variable Z, measurable with respect to $\mathscr{F}_N$, such that

$$P\left(\left\{\omega \,\Big|\, |Y(\omega) - Z(\omega)| < \frac{\delta}{2}\right\}\right) > 1 - \frac{\varepsilon}{2}. \tag{1}$$

But $\left\{\omega \,\Big|\, |Y(\omega) - Z(\omega)| < \frac{\delta}{2}\right\} \subset \left\{\omega \,\Big|\, |X_n(\omega) - Z(\omega)| < \frac{\delta}{2} \text{ for some } n \geqq N\right\}$. Thus there is an integer $N' > N$ such that

$$P\left(\left\{\omega \,\Big|\, |X_n(\omega) - Z(\omega)| < \frac{\delta}{2} \text{ for some } n \text{ with } N \leqq n \leqq N'\right\}\right) > 1 - \frac{\varepsilon}{2}.$$ Define the bounded stopping time τ as follows. Given $\omega \in \Omega$, if there is an integer n such that $N \leqq n \leqq N'$ and $|X_n(\omega) - Z(\omega)| \leqq \frac{\delta}{2}$, then let $\tau(\omega)$ be the smallest such integer; if there is no such integer, then let $\tau(\omega) = N'$. Thus

$$P\left(\left\{\omega \,\Big|\, |X_{\tau(\omega)}(\omega) - Z(\omega)| \leqq \frac{\delta}{2}\right\}\right) > 1 - \frac{\varepsilon}{2}, \tag{2}$$

so that, by (1) and (2),

$$P(\{\omega \mid |X_{\tau(\omega)}(\omega) - Y(\omega)| \leqq \delta\}) > 1 - \varepsilon.$$

Thus Lemma 1 is proved.

From Lemma 1 we easily obtain the following

Theorem 1. *Let* $(X_n)_{n\geqq 1}$ *be a sequence of real random variables. For each integer* $n \geqq 1$, *let* $\mathscr{F}_n = \sigma(X_1, \ldots, X_n)$ *and assume that* $\mathscr{F} = \sigma\big(\bigcup_{n\geqq 1} \mathscr{F}_n\big)$. *Let*

$$Y\colon \Omega \to \bar{R} = [-\infty, \infty]$$

be $\mathscr{F}$*-measurable and such that: For each* $\omega \in \Omega$, $Y(\omega)$ *is a cluster value of the sequence* $(X_n(\omega))_{n\geqq 1}$. *Then we have:*

i) *There is a strictly increasing sequence* $(\tau(n))_{n\geqq 1}$ *of bounded stopping times such that*

$$\lim_n X_{\tau(n)}(\omega) = Y(\omega), \quad P\text{-almost surely.}$$

ii) *If, in addition, there is a constant* $C > 0$ *such that* $\int |X_\sigma| \leqq C$ *for each bounded stopping time* σ, *then* Y *is integrable.*

iii) *In particular, if there is a random variable* $g \geqq 0$, *g integrable such that* $|X_n| \leqq g$ *for all n, then* $(X_{\tau(n)})_{n\geqq 1}$ *converges to Y in the* L^1*-mean and hence* $(\int X_{\tau(n)})_{n\geqq 1}$ *converges to* $\int Y$.

Proof. Since ii) follows from i) and Fatou's lemma, and iii) follows from i) and the Lebesgue dominated convergence theorem, we need only prove i).

We consider first the case when Y is real-valued, $Y: \Omega \to R$. In this case the existence of the sequence $(\tau(n))_{n\geqq 1}$ is obtained by applying Lemma 1 inductively.

We pass now to the general case, $Y: \Omega \to \bar{R}$. Let $f: \bar{R} \to [-1, 1]$ be the homeomorphism given by

$$f(x) = \frac{x}{1+|x|} \quad \text{for } x \in R,\ f(-\infty) = -1, \quad f(+\infty) = 1$$

and let g be the inverse homeomorphism. We can reduce this case to the previous case by considering

$$X'_n = f(X_n), \quad \text{and} \quad Y' = f(Y)$$

and then composing back with g. Thus Theorem 1 is proved.

In the case when the function g below reduces to a constant $C \geqq 0$, the result of Corollary 1 is due to Baxter (see [1]); for the continuous parameter case see [7], Prop. 6(a), p. 232.

Corollary 1. *Let* $(X_n)_{n\geqq 1}$ *be a sequence of random variables and suppose there is an integrable random variable* $g \geqq 0$ *such that* $|X_n| \leqq g$ *for all n. The following assertions are then equivalent:*

1) $\lim_n X_n(\omega)$ *exists P-almost surely.*

2) *The generalized sequence* $(\int X_\tau)_{\tau\in\Gamma}$ *is convergent.*

Proof. The implication 1)⇒2) is trivial (a version of the Lebesgue dominated convergence theorem).

2)⇒1). By Theorem 1, there exist strictly increasing sequences $(\tau(n))_{n\geqq 1}$ and $(\sigma(n))_{n\geqq 1}$ of bounded stopping times such that

$$\lim_n X_{\tau(n)}(\omega) = \limsup_n X_n(\omega), \qquad P\text{-almost surely},$$

$$\lim_n X_{\sigma(n)}(\omega) = \liminf_n X_n(\omega), \qquad P\text{-almost surely}.$$

Then

$$\int (\limsup_n X_n - \liminf_n X_n) = \lim_n \int (X_{\tau(n)}) = 0$$

so that $\limsup_n X_n = \liminf_n X_n$, P-almost surely. Thus the corollary is proved.

Remarks. 1) It is easy to construct an example of a sequence $(X_n)_{n\geqq 1}$ of integrable random variables such that $X_n \geqq 0$, $\int X_\tau \leqq 1$, $\lim_n X_n(\omega)$ exists P-almost surely, and yet for which the generalized sequence $(\int X_\tau)_{\tau\in\Gamma}$ fails to converge. (Take for $(\Omega, \mathscr{F}, P)$ the Lebesgue measure space on $[0, 1]$ and define $X_{2n}(\omega) = 0$ and

$$X_{2n+1}(\omega) = \begin{cases} 2^n & \text{for } \omega \in (0, 2^{-n}) \\ 0 & \text{for } \omega \notin (0, 2^{-n}) \end{cases}.$$

2) Let $(X_n)_{n\geqq 1}$ be a sequence of integrable real random variables. It is easily seen that the generalized sequence $(\int X_\tau)_{\tau\in\Gamma}$ is monotone increasing (that is, the

relations $\tau_1, \tau_2 \in \Gamma$, $\tau_1 \leqq \tau_2$ imply $\int X_{\tau_1} \leqq \int X_{\tau_2}$) if and only if $(X_n)_{n \geqq 1}$ is a submartingale.

Lemma 2. *Let $(X_n)_{n \geqq 1}$ be a sequence of integrable real random variables and suppose that*

$$\sup_n \int |X_n| < \infty .$$

Then the following assertions are equivalent:

1) *The generalized sequence $(\int X_\tau)_{\tau \in \Gamma}$ converges.*
2) *The generalized sequences $(\int X_\tau^+)_{\tau \in \Gamma}$ and $(\int X_\tau^-)_{\tau \in \Gamma}$ both converge.*[1]

Proof: Since 2)⇒1) clearly, we only have to prove the implication 1)⇒2).

Assume 1). We divide the proof into parts:

I) We show first that under our assumptions

$$\sup_{\tau \in \Gamma} \int X_\tau^+ < \infty, \quad \sup_{\tau \in \Gamma} \int X_\tau^- < \infty.$$

Since the generalized sequence $(\int X_\tau)_{\tau \in \Gamma}$ converges, it is eventually bounded, and hence to prove our assertion it is enough to show that $(\int X_\tau^+)_{\tau \in \Gamma}$ is bounded above. Let then $\tau \in \Gamma$; choose $n \geqq \tau$ and define the stopping time σ by

$$\sigma(\omega) = \begin{cases} \tau(\omega) & \text{on } \{X_\tau \geqq 0\} \\ n & \text{on } \{X_\tau < 0\}. \end{cases}$$

Then $X_\tau^+ \leqq X_\sigma + |X_n|$ and so $\int X_\tau^+ \leqq \int X_\sigma + \int |X_n|$. This proves our assertion.

II) We now show that the generalized sequence $(\int X_\tau^+)_{\tau \in \Gamma}$ converges (this of course will imply that the generalized sequence $(\int X_\tau^-)_{\tau \in \Gamma}$ converges also). The device used below is borrowed from Baxter's paper.

Given $\varepsilon > 0$, choose an integer n_0 such that the relations $\sigma \in \Gamma$, $\tau \in \Gamma$, $\sigma \geqq n_0$, $\tau \geqq n_0$ imply

$$|\int X_\sigma - \int X_\tau| \leqq \varepsilon. \tag{1}$$

Next choose $\tau_0 \in \Gamma$, $\tau_0 \geqq n_0$ such that for any $\sigma \in \Gamma$, $\sigma \geqq \tau_0$ we have

$$\int X_\sigma^+ \leqq \int X_{\tau_0}^+ + \varepsilon. \tag{2}$$

Let now $\sigma \in \Gamma$, $\sigma \geqq \tau_0$ and define the new stopping time σ_1 by

$$\sigma_1(\omega) = \begin{cases} \sigma(\omega) & \text{on } \{X_{\tau_0} \geqq 0\} \\ \tau_0(\omega) & \text{on } \{X_{\tau_0} < 0\}. \end{cases}$$

Then, on the one hand by (1) we have

$$|\int X_{\sigma_1} - \int X_{\tau_0}| \leqq \varepsilon,$$

and on the other hand

$$\int X_{\tau_0} = \int_{\{X_{\tau_0} \geqq 0\}} X_{\tau_0}^+ + \int_{\{X_{\tau_0} < 0\}} X_{\tau_0}$$

$$\int X_{\sigma_1} = \int_{\{X_{\tau_0} \geqq 0\}} X_\sigma + \int_{\{X_{\tau_0} < 0\}} X_{\tau_0}.$$

[1] Lemma 2 and Theorem 2 remain valid if for instance we replace the condition

$$\sup_n \int |X_n| < \infty$$

by the condition

$$\sup_n \int X_n^+ < \infty.$$

We deduce

$$\int X_{\tau_0}^+ \leqq \int_{\{X_{\tau_0} \geqq 0\}} X_\sigma + \varepsilon \leqq \int X_\sigma^+ + \varepsilon. \tag{3}$$

Combining (2) and (3) we obtain for any $\sigma \in \Gamma$, $\sigma \geqq \tau_0$:

$$\left|\int X_\sigma^+ - \int X_{\tau_0}^+\right| \leqq \varepsilon$$

and thus Lemma 2 is proved.

We may now state our main theorem in the real-valued case:

Theorem 2. *Let $(X_n)_{n \geqq 1}$ be a sequence of integrable real random variables. Suppose that*

$$\sup_n \int |X_n| < \infty.$$

Consider the following assertions:

1) *The generalized sequence $(\int X_\tau)_{\tau \in \Gamma}$ converges.*
2) *The sequence $(X_n)_{n \geqq 1}$ converges P-almost surely.*

Then 1)$\Rightarrow$2).

Proof. We assume that 1) holds. By Lemma 2, it follows that the generalized sequences

$$\left(\int X_\tau^+\right)_{\tau \in \Gamma}, \quad \left(\int X_\tau^-\right)_{\tau \in \Gamma}$$

converge. Hence in proving the implication 1)$\Rightarrow$2) of our theorem we may assume without loss of generality that $X_n \geqq 0$ for all n.

We now prove the implication 1)$\Rightarrow$2) by proving the contrapositive. Suppose then that the sequence $(X_n)_{n \geqq 1}$ does not converge P-almost surely. There are then real numbers $\alpha < \beta$ such that $P(A) > 0$, where

$$A = \{\omega \mid \liminf X_n(\omega) < \alpha < \beta < \limsup X_n(\omega)\}.$$

We will show that the generalized sequence $(\int X_\tau)_{\tau \in \Gamma}$ is not Cauchy. For

$$\varepsilon = \frac{(\beta - \alpha)\, P(A)}{2}$$

we show that given any integer $M \geqq 1$ there exist bounded stopping times $\tau_1 \geqq M$, $\tau_2 \geqq M$ with $\int X_{\tau_2} - \int X_{\tau_1} \geqq \varepsilon$.

Let $\delta = \varepsilon / 2\beta$ and let $M \geqq 1$ be any integer. There exists a set B and an integer $N \geqq M$ such that $B \in \mathscr{F}_N$ and $P(A \Delta B) \leqq \delta$. There exist integers $N'' > N' > N$ such that if:

$$\Omega_0 = \{\omega \mid \inf_{N \leqq n \leqq N'} X_n(\omega) < \alpha < \beta < \sup_{N' \leqq n \leqq N''} X_n(\omega)\}$$

then $P(A - \Omega_0) \leqq \delta$. Define now

$$C_1 = \{\omega \in B \mid \inf_{N \leqq n \leqq N'} X_n(\omega) < \alpha\},$$
$$C_2 = \{\omega \in C_1 \mid \sup_{N' \leqq n \leqq N''} X_n(\omega) > \beta\}.$$

Then

$$\begin{aligned} &C_1 \in \mathscr{F}_{N'}, \quad C_2 \in \mathscr{F}_{N''}, \quad C_2 \subset C_1 \subset B \\ &P(C_2) \geqq P(A) - 2\delta, \quad P(C_1 - C_2) \leqq 2\delta. \end{aligned} \tag{1}$$

Define now stopping times τ_1, τ_2 by:

$$\tau_1(\omega) = \begin{cases} N' & \omega \notin C_1 \\ \inf\{n \mid N \leqq n \leqq N', \quad X_n(\omega) < \alpha\} & \omega \in C_1, \end{cases}$$

$$\tau_2(\omega) = \begin{cases} N' & \omega \notin C_1 \\ N'' & \omega \in C_1 - C_2 \\ \inf\{n \mid N' \leqq n \leqq N'', \quad X_n(\omega) > \beta\} & \omega \in C_2. \end{cases}$$

Then $M \leqq N \leqq \tau_1 \leqq \tau_2$ and by (1) we have

$$\begin{aligned} \int X_{\tau_2} - \int X_{\tau_1} &= \int (X_{\tau_2} - X_{\tau_1}) = \int_{C_1} (X_{\tau_2} - X_{\tau_1}) \\ &= \int_{C_2} (X_{\tau_2} - X_{\tau_1}) + \int_{C_1 - C_2} X_{\tau_2} - \int_{C_1 - C_2} X_{\tau_1} \\ &\geqq (\beta - \alpha)\, P(C_2) + 0 - \alpha P(C_1 - C_2) \\ &\geqq (\beta - \alpha)\big(P(A) - 2\delta\big) - 2\delta\alpha \\ &= (\beta - \alpha)\, P(A) - 2\delta\beta = 2\varepsilon - \varepsilon = \varepsilon. \end{aligned}$$

This completes the proof of Theorem 2.

Corollary 2 (Submartingale Convergence Theorem, see [2, p. 324], [8, p. 131], or [4, p. 146]). *Let $(X_n)_{n \geqq 1}$ be a sequence of integrable random variables such that:*

i) $E(X_{n+1} | \mathscr{F}_n) \geqq X_n$ *for all* $n \geqq 1$,

ii) $\sup_n \int |X_n| < \infty$.

Then $(X_n)_{n \geqq 1}$ converges to a limit P-almost surely.

Proof. Since the generalized sequence $(\int X_\tau)_{\tau \in \Gamma}$ is monotone increasing and bounded, this is an immediate application of Theorem 2.

Remarks. 1) The proof of Theorem 2 is completely elementary and makes use only of Lemma 2.

2) The implication 2)$\Rightarrow$1) in Theorem 2 is in general false, as Remark 1) at the end of Corollary 1 shows.

3) Without the assumption of L^1-boundedness on the sequence $(X_n)_{n \geqq 1}$, Theorem 2 is in general false, as the following simple example illustrates: Let $(u_n)_{n \geqq 1}$ be a sequence of independent identically distributed random variables with $u_n = 1$ or -1 with probability 1/2 and set $X_n = u_1 + \cdots + u_n$, for each $n \geqq 1$. This seems to contradict Remark 2) on p. 50 of [6].

§ 3. The Case of Abstract-Valued Random Variables

In this section S is a Polish space (that is, a complete separable metric space) and we consider S-valued random variables. We denote by $C_R(S)$ the set of all real-valued continuous functions on S.

We begin by introducing the following

Definition. We say that an at most countable set $\{\phi_j | j \in J\}$, where $\phi_j \in C_R(S)$ for each $j \in J$, is a *determining set for* S if:

The mapping Φ: $x \to (\phi_j(x))_{j \in J}$ of S into $R^J = \prod_{j \in J} R_j$ is a *homeomorphism* of S onto a *closed* subset of R^J.

Remarks. 1) Let $\mathscr{K} \subset C_R(S)$ be an at most countable set. It is easily seen that the following are equivalent assertions: i) $\mathscr{K}$ is a determining set for S; ii) whenever $(x_n)_{n \geq 1}$ is a sequence of points in S such that $\lim_n \phi(x_n)$ exists, for each $\phi \in \mathscr{K}$, then there is $x \in S$ such that $\lim_n x_n = x$.

2) For $S = R$ the following are examples of determining sets:

a) $\{\phi\}$, where $\phi(x) = x$ for $x \in R$;
b) $\{\phi_1, \phi_2\}$, where $\phi_1(x) = x^+$, $\phi_2(x) = x^-$ for $x \in R$;
c) $\{\psi_1, \psi_2\}$, where $\psi_1(x) = |x|$, $\psi_2(x) = |x+1|$ for $x \in R$.

3) Suppose that S is a Polish space and that $\mathscr{K} \subset C_R(S)$ is a determining set for S. Then the set

$$\{|\phi| \,|\, \phi \in \mathscr{K}\} \cup \{|\phi + 1| \,|\, \phi \in \mathscr{K}\}$$

is a determining set $\subset C_R^+(S)$.

4) Let S be compact metric. Let $\mathscr{K} \subset C_R(S)$ be *any* (at most) countable set separating points of S. Then $\mathscr{K}$ is a determining set for S.

We now establish the existence of a determining set for an arbitrary Polish space:

Theorem 3. *Let S be an arbitrary Polish space. There exists then a determining set for S.*

Proof. We are indebted to T. Figiel for the proof below, which is much simpler than our original proof.

The existence of a determining set follows from the following classical facts of topology (for these we refer the interested reader to [5]):

I) There is a homeomorphic embedding ϕ: $S \to R^N$ (here $N = \{0, 1, 2, \ldots\}$).

II) S is an absolute G_δ, therefore $R^N - \phi(S) = \bigcup_{n \in N} F_n$, where each F_n is a closed subset of R^N.

III) The mapping

$$x \to \left(\phi(x), \left(\frac{1}{\operatorname{dist}_{R^N}(\phi(x), F_n)}\right)_{n \in N}\right)$$

is a homeomorphic embedding of S onto a *closed* subset of $R^N \times R^N$.

This completes the proof of Theorem 3.

We next show how to construct "nice" determining sets in separable Banach spaces. The construction below was shown to us by Figiel. It considerably simplifies and generalizes our original construction. The construction is based on the following Lemma:

Lemma 3. *Let X be a separable Banach space. Let $f: X \to [0, +\infty]$ be a convex function, finite on a dense subset of X and such that $f^{-1}([0, a])$ is compact for all $a \in R^+ = [0, +\infty)$. For each integer $n \geqq 1$* define

$$f_n(x) = \inf\left\{f(y) \,\middle|\, \|y-x\| \leqq \frac{1}{n}\right\}, \quad \textit{for } x \in X;$$

then f_n is finite-valued, continuous and convex.

Proof. Since f is finite on a dense subset of X, it is clear that $f_n(x) < \infty$ for each $x \in X$.

To prove continuity of f_n it is enough to establish that $f_n^{-1}([0, a])$ is closed for all $a \in R^+$, and $f_n^{-1}([0, a))$ is open for all $a \in R^+$. The first assertion follows from the identity:

$$f_n^{-1}([0, a]) = f^{-1}([0, a]) + \left\{z \in X \,\middle|\, \|z\| \leqq \frac{1}{n}\right\}$$

which represents $f_n^{-1}([0, a])$ as the algebraic sum of a compact set and a closed set.

To prove the second assertion we note first that, since f is convex, we also have:

$$f_n(x) = \inf\left\{f(y) \,\middle|\, \|y-x\| < \frac{1}{n}\right\}, \quad \text{for } x \in X.$$

It follows that we have the representation

$$f_n^{-1}([0, a)) = f^{-1}([0, a)) + \left\{z \in X \,\middle|\, \|z\| < \frac{1}{n}\right\}$$

and thus $f_n^{-1}([0, a))$ is open. Hence the continuity of f_n is proved.

We next turn to the proof of convexity. Let $x \in X$, $y \in X$, $\alpha \in R$ with $0 \leqq \alpha \leqq 1$ and let $z = \alpha x + (1-\alpha) y$. Let $\varepsilon > 0$. Choose $x' \in X$, $y' \in X$ such that

$$\|x'-x\| \leqq \frac{1}{n}, \quad f(x') \leqq f_n(x) + \varepsilon,$$

$$\|y'-y\| \leqq \frac{1}{n}, \quad f(y') \leqq f_n(y) + \varepsilon.$$

Let $z' = \alpha x' + (1-\alpha) y'$. Then we have:

$$\|z'-z\| = \|\alpha(x'-x) + (1-\alpha)(y'-y)\| \leqq \alpha \|x'-x\| + (1-\alpha)\|y'-y\| \leqq \frac{1}{n}$$

and hence

$$\begin{aligned} f_n(z) &\leqq f(z') \leqq \alpha f(x') + (1-\alpha) f(y') \\ &\leqq \alpha\big(f_n(x)+\varepsilon\big) + (1-\alpha)\big(f_n(y)+\varepsilon\big) \\ &= \alpha f_n(x) + (1-\alpha) f_n(y) + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the convexity of f_n is proved. This completes the proof of Lemma 3.

The following remark will be used in the proof of Theorem 4 below:

Remark. Let X be a Banach space, $E \subset X$. Suppose that for each $n \geqq 1$ there is a compact set $K_n \subset X$ with the property

$$x \in E \Rightarrow d(x, K_n) \leqq \frac{1}{n}.$$

Then E is (strongly) relatively compact.

Theorem 4. *Let X be an arbitrary separable Banach space. There exists then a determining set for X consisting of convex functions.*

Proof. Let $(y_n)_{n \geqq 1}$ be a sequence of points in X such that $\lim_n \|y_n\| = 0$ and such that the linear space spanned by $A = \{y_1, y_2, \ldots, y_n, \ldots\}$ is dense in X. The set $A \cup \{0\}$ is compact. Let K be the closed convex symmetric hull of $A \cup \{0\}$; then K is also compact. Let f be the gauge function of K:

$$f(x) = \begin{cases} \inf\left\{\lambda \mid \lambda > 0, \dfrac{x}{\lambda} \in K\right\}, & \text{if this set is non-void} \\ +\infty & \text{otherwise.} \end{cases}$$

It is clear that $f\colon X \to [0, +\infty]$ and that (see for instance [3], p. 411):

$$f(a\,x) = a\,f(x), \qquad \text{for} \qquad x \in X,\ a \in R^+,$$

$$f(x+y) \leqq f(x) + f(y), \qquad \text{for any } x \in X,\ y \in X.$$

Furthermore, it is easily seen that $f^{-1}([0, a]) = a\,K$ for every $a \in R^+$. Thus f satisfies the assumptions of Lemma 3. For each integer $n \geqq 1$ let f_n be defined from f as in Lemma 3.

Let X' be the dual of the Banach space X and let $\{x_1', x_2', \ldots, x_n', \ldots\} \subset X'$ be a countable *total* set (see [3, p. 418]); this means that the relations $x \in X$ and $x_n'(x) = 0$ for all $n \geqq 1$, imply $x = 0$.

We now define $\mathscr{K}$ as follows:

$$\mathscr{K} = \{f_1, f_2, \ldots, f_n, \ldots\} \cup \{x_1', x_2', \ldots, x_n', \ldots\}.$$

We show that $\mathscr{K}$ is a determining set for X. It is enough to show that if $(x_k)_{k \geqq 1}$ is any sequence of points in X such that $\lim_k \phi(x_k)$ exists for each $\phi \in \mathscr{K}$, then $\lim_k x_k$ exists (strongly). For each $n \geqq 1$, let $a_n \in R$ such that $a_n > \sup_k f_n(x_k)$, and let $K_n = f^{-1}([0, a_n])$. Then K_n is compact and it is easily seen that:

For each

$$k \geqq 1, \qquad d(x_k, K_n) \leqq \frac{1}{n}.$$

By the Remark preceding Theorem 4, we deduce that the set $\{x_1, x_2, \ldots, x_k, \ldots\}$ is (strongly) relatively compact. Let $(x_{k(p)})_{p \geqq 1}$ be a convergent subsequence of $(x_k)_{k \geqq 1}$ and let $x = \lim_p x_{k(p)}$. We show that $(x_k)_{k \geqq 1}$ itself must converge to x. In fact, otherwise there would exist another subsequence of $(x_k)_{k \geqq 1}$ converging to some element $y \neq x$. Since $\lim_k x'_n(x_k)$ exists, we deduce $x'_n(x) = x'_n(y)$, for each $n \geqq 1$. Since $\{x'_1, x'_2, \ldots, x'_n, \ldots\}$ is total, it follows that $x = y$. Hence $\lim_k x_k = x$ and Theorem 4 is proved.

Remarks. 1) Let (S, d) be a Polish space. Let $A \subset S$ be a countable dense set and for each $a \in A$, let

$$\phi_a(x) = d(x, a), \quad \text{for } x \in S.$$

We have:

i) The mapping $\Phi: x \to (\phi_a(x))_{a \in A}$ of (S, d) into $R^A = \prod_{a \in A} R_a$ is a homeomorphism of (S, d) onto $\Phi(S) \subset R^A$.

ii) Suppose the distance d is bounded. Then $\{\phi_a | a \in A\}$ is a determining set for (S, d) if and only if (S, d) is compact.

2) Let X be a separable infinite-dimensional Banach space, let $A \subset X$ be a countable dense set and for each $a \in A$, let

$$\phi_a(x) = \|x - a\|, \quad \text{for } x \in X.$$

Then $\{\phi_a | a \in A\}$ is *not* a determining set for X.

We may now state the abstract version of Theorem 2:

Theorem 5. *Let S be a Polish space and let $(X_n)_{n \geqq 1}$ be a sequence of S-valued random variables. Suppose there is $\mathscr{K} \subset C_R(S)$, a determining set for S, with the following property: For each $\phi \in \mathscr{K}$, we have:*

i) $\sup_n \int |\phi(X_n)| < \infty$;

ii) *The generalized sequence $\left(\int \phi(X_\tau)\right)_{\tau \in \Gamma}$ converges.*

Then the sequence $(X_n)_{n \geqq 1}$ converges to a limit P-almost surely.

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