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Continuity and Convergence of Some Processes Parameterized by the Compact Convex Sets in R^S *

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1. Counting Random Partitions

Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed $N(0, I)$ random variables in R^s. Let $F_m = \{X_1, \ldots, X_{2^m}\}$ be a random set of points. Denote by $\overline{co}(A)$ the closed convex hull of A. Let K be the set of all closed convex sets in R^s and $K_m = \{\overline{co}(C \cap F_m)\colon C \in K\}$. We give an upper bound on the expected number of sets in K_m.

Lemma 1. *For any integer* $b > 0$

$$E\,\mathrm{card}^b(K_m) \leqq 0(1) \cdot \begin{cases} 2^{2bm}, & s=1 \\ 2^{2^{bm/(b+1)}}, & s \geqq 2. \end{cases}$$

Note 1. The same bound is obtained if we replace $N(0, I)$ by the uniform distribution on the unit ball of R^s.

Note 2. In [4] we showed that if we let

$$F_m = \mathop{\mathsf{X}}_1^s \{j\,2^{-m}\colon j=0, \ldots, 2^m\},$$

then for $s \leqq 2$ the bound of Lemma 1 is obtained for $\mathrm{card}^b K_m$, however in order to get a useful bound for $s > 2$ it is necessary to consider random partitions as in Lemma 1.

Proof of Lemma 1.

The Case $s=1$. This is trivial since

$$E\,\mathrm{card}^b(K_m) = \binom{2^m}{2}^b.$$

The Case $s \geqq 2$. Let Φ denote the normal cumulative distribution $N(0,1)$ in R and φ its density with respect to the Lebesgue measure. Set $a_n^2 = 2s \log n$ and denote by CTE a generic positive constant that may depend on s but not on any free index.

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Now by integrating by parts and then iterating we have,

$$\int_{-\infty}^{a_n} \Phi^n(x)\,\varphi^s(x)\,dx \leqq (CTE/n)\,[\varphi(a_n)+(a_n/n)^{s-1}].$$

Also

$$\int_{a_n}^{\infty} \exp(-x^2/2)\,dx \leqq \exp(-a_n^2/2)$$

and by [8]:

$$P(X_{n+1}\notin \overline{co}(X_1,\dots,X_n)) \leqq CTE\,n^{s-1} \int_{-\infty}^{\infty} \Phi^{n-s}(x)\,\varphi^s(x)\,dx.$$

Therefore,

$$\begin{aligned} P(X_{n+1}\notin \overline{co}(X_1,\dots,X_n)) &\leqq CTE\,n^{s-1}\left[\int_{-\infty}^{a_n} \Phi^{n-s}(x)\,\varphi^s(x)\,dx+\varphi(a_n)\right] \\ &\leqq CTE\,n^{s-1}[(a_n/n)^{s-1}(1/n)+\varphi(a_n)] \leqq CTE\,n^{-\alpha}, \end{aligned}$$

for every $\alpha\in(0,1)$ as soon as $n>n_\alpha$ large.

Let B_{j-1} be the event $\{X_1,\dots,X_{j-1}$ are the extreme points of $\overline{co}(X_1,\dots,X_{j-1})\}$ and B_{j-1}^c denote its complement. Then

$$B_{j-2}=B_{j-2}B_{j-1}^c+B_{j-2}B_{j-1}=B_{j-2}B_{j-1}^c+B_{j-1}.$$

Moreover, given $X_1=x_1,\dots,X_{j-2}=x_{j-2}$, $\overline{co}(x_1,\dots,x_{j-2},X_{j-1})$ is at least as large if B_{j-1} occurs as it is if $B_{j-2}B_{j-1}^c$ occurs. Hence

$$Pr(X_j\in\overline{co}(X_1,\dots,X_{j-1})\,|\,B_{j-1}) \geqq Pr(X_j\in\overline{co}(X_1,\dots,X_{j-1})\,|\,B_{j-2}B_{j-1}^c)$$

so that

$$Pr(X_j\in\overline{co}(X_1,\dots,X_{j-1})\,|\,B_{j-1}) \geqq Pr(X_j\in\overline{co}(X_1,\dots,X_{j-1})\,|\,B_{j-2}),$$

and by proceeding similarly we get

$$Pr(X_j\in\overline{co}(X_1,\dots,X_{j-1})\,|\,B_{j-2}) \geqq Pr(X_j\in\overline{co}(X_1,\dots,X_{j-1})).$$

Hence

$$\begin{aligned} P(B_k) &= \prod_{j=1}^{k} Pr(X_j\notin\overline{co}(X_1,\dots,X_{j-1})\,|\,B_{j-1}) \\ &\leqq \prod_{j=1}^{k} Pr(X_j\notin\overline{co}(X_1,\dots,X_{j-1})) \leqq \prod_{j=1}^{k} CTE\,j^{-\alpha}=(CTE)^k(k!)^{-\alpha}, \end{aligned}$$

so that by Stirling's approximation

$$P(B_k)\leqq O(1)\,(CTE/k)^{\alpha k}.$$

Finally, if we let

$$K_{km}=\{C\in K_m\colon \text{card (extreme points of } C)=k\}.$$

then by symmetry it follows that for any integer $b\geqq 1$,

$$E\,\text{card}^b(K_{km}) \leqq \binom{2^m}{k}^b P(B_k)$$

and hence

$$E\,\mathrm{card}^b(K_m)=E\left[\prod_{k=1}^{2m}\mathrm{card}(K_{km})\right]^b \leqq 2^{bm}\max_{0\leqq k\leqq 2^m}\left[\binom{2^m}{k}^b P(B_k)\right],$$

so that by Stirling's approximation

$$\begin{aligned} E\,\mathrm{card}^b(K_m) &\leqq O(1)\,2^{bm}\max_{0\leqq k\leqq 2^m}[(CTE\,2^m/k)^{bk}(CTE/k)^{\alpha k}] \\ &\leqq O(1)\,2^{bm}\max_{0\leqq\beta\leqq 1}(CTE\,2^m/2^{\beta m})^{b\,2^{\beta m}}(CTE/2^{\beta m})^{\alpha\,2^{\beta m}}\leqq O(1)\,2^{2^{bm/(b+1)}} \end{aligned}$$

2. A Continuity Result

Let K be the space of all compact convex sets in R^s and μ the measure $\mu(A)=\lambda(A\cap S)$ where S denotes the unit ball of R^s and λ the Lebesgue measure on R^s. We endow K with the topology induced by the metric obtained by taking the measures of symmetric differences of sets in K. This metric is equivalent to the Hausdorff metric. We say that a process is path continuous or simply *continuous* if there is a version, having the same finite dimensional distributions, for which almost every path is continuous on K. The following theorem, which gives the continuity of Gaussian processes satisfying a natural Hölder condition, is the central result of this work.

Theorem 1. *Let Z be a Gaussian process parametrized by K and such that for some $c>0$, and for all $A, B\in K$,*

$$E|Z(A)-Z(B)|^2\leqq c\,\mu(A\,\triangle\,B).$$

Then Z is continuous.

Other Related Results. Let F be the space of all nonempty closed subsets of $[0,1]^s$ with the Hausdorff metric d. Then the minimum number of subsets of F of d-diameter less or equal to 2ε needed to cover F,

$$N(F,\varepsilon)=O(1)^{\varepsilon^{-s}}$$

so that Dudley-Strassen's result (Theorem 3.1 in [6]) gives continuity of Gaussian processes Z parameterized by F, under the condition

$$E|Z(A)-Z(B)|^2<c\,d^{2s}(A,B),\qquad \text{for every } A,B\in F,\quad \text{where } c>0.$$

For $s>1$ this condition is too strong to yield the convergence of the series $\sum_m 2^{-m}\log^{1/2}N(F,2^{-m})$ which is the Dudley-Strassen's continuity condition.

A result closer to the one in Theorem 1 is due to X. Fernique. As shown in Dudley [6] this result gives a condition for continuity of Gaussian processes parametrized by the space of all polyhedra in R^s with at most k vertexes.

This is accomplished by imbedding this parameter space into R^{ks} and by using in this space the following type of condition for continuity of Gaussian processes: $E|Z(x)-Z(y)|^2\leqq c|x-y|$.

Strength of the Result. Continuity conditions of the necessary and sufficient type are in general not known. However, in our case, and even for $s=1$, if we remove the condition that Z be Gaussian, then the Poisson process satisfies the Hölder condition but is a jump process, and if we remove the Hölder condition then, as is shown in Berman [1], the paths of Gaussian processes are not only discontinuous but actually extremely irregular.

Proof of Theorem 1. 1. Centering of Z. Given

$$E|Z(A)-Z(B)|^2 \leqq c\,\mu(A \triangle B),$$

then

$$[E[Z(A)-Z(B)]]^2 \leqq E[Z(A)-Z(B)]^2 \leqq c\,\mu(A \triangle B).$$

Hence Z is continuous iff $Z-EZ$ is continuous. Therefore, we are going to assume from now on that $EZ(A)=0$ for every $A \in K$.

2. Bounding the Oscillations of Z. Let P be the probability measure, on a space denoted by Ω, associated to the process Z. Let Q be the product measure, on a space denoted by Ξ, induced by a sequence $X_1, X_2, \ldots$ of independent identically distributed $N(0, I)$ random variables in R^s. Consider the product measure $P \times Q$.

Let $\alpha, \gamma > 0$ and $\beta = \alpha - 2\gamma > 2/3$. As in Lemma 1 let

$$F_m = \{X_1, \ldots, X_{2^m}\} \quad \text{and} \quad K_m = \{\overline{co}(A \cap F_m) \colon A \in K\}.$$

Given $A \in K$ denote by A_m the largest set in K_m contained in A or the empty set if there is none, and denote by A'_m the smallest set in K_m containing A or the unit ball S of R^s if there is none.

Set

$$c_m = \max[\mu(A_m \triangle A'_m) \colon A \in K]$$

and set

$$b'_m = (2^{\beta m} c_m)^{1/2}.$$

It follows from these definitions that

$$(A_m \cap S) \subseteq (A_{m+1} \cap S) \subseteq \cdots \subseteq (A \cap S) \subseteq \cdots \subseteq (A'_{m+1} \cap S) \subseteq (A'_m \cap S)$$

and

$$\mu(A_m \triangle A_{m+1}) \leqq \mu(A_m \triangle A) \leqq \mu(A_m \triangle A'_m) \leqq c_m.$$

Observe that

$$(1/2\pi\sigma^2)^{1/2} \int_{x>b} \exp(-x^2/2\sigma^2)\,dx \leqq (1/2\pi)^{1/2}(\sigma/b)\exp(b^2/2\sigma^2).$$

Hence, given a $\xi_0 \in \Xi$ both

$$\max_{A \in K} P[\omega \in \Omega \colon |Z[A_m(\xi_0)](\omega) - Z[A_{m+1}(\xi_0)](\omega)| > b'_m(\xi_0)]$$

and

$$\max_{A \in K} P[\omega \in \Omega \colon |Z[A_m(\xi_0)](\omega) - Z[A'_m(\xi_0)](\omega)| > b'_m(\xi_0)]$$

are less than or equal to

$$CTE \cdot \exp[-(1/2)[b'_m(\xi_0)/c_m^{1/2}(\xi_0)]^2] = CTE \exp[-(1/2)\,2^{\beta m}]$$

by definition of b'_m.

Now if we denote

$$P_m = P \times Q\left[(\omega,\xi)\in\Omega\times\Xi:\ \max_{A\in K}|Z[A_m(\xi)](\omega) - Z[A_{m+1}(\xi)](\omega)| > b'_m(\xi)\right]$$

and

$$P'_m = P \times Q\left[(\omega,\xi)\in\Omega\times\Xi:\ \max_{A\in K}|Z[A'_m(\xi)](\omega) - Z[A_m(\xi)](\omega)| > b'_m(\xi)\right],$$

then both P_m and P'_m are less than or equal to

$$\begin{aligned} CTE &\int \operatorname{card}^2(K_{m+1}(\xi))\exp\left(-(1/2)\,2^{\beta m}\right)dQ(\xi)\\ &= CTE\exp\left(-(1/2)\,2^{\beta m}\right)E\operatorname{card}^2(K_{m+1})\\ &\leqq O(1)\,2^{-2[\beta-(2/3)]m}, \end{aligned}$$

by Lemma 1. On the other hand, given a $\xi_0\in\Xi$

$$\begin{aligned} \max&\left[P(\omega\in\Omega:\ |Z(A)(\omega)-Z(B)(\omega)|>2^{-\gamma m}):\ A,B\in K_m(\xi_0) \text{ and } \mu(A\triangle B)<2^{-\alpha m}\right]\\ &\leqq CTE\cdot\exp\left[-(1/2)(2^{-\gamma m}/2^{-\alpha m/2})^2\right] \end{aligned}$$

so that if we let

$$\begin{aligned} P''_m = P\times Q\big[&(\omega,\xi)\in\Omega\times\Xi:\ |Z(A)-Z(B)|>2^{-\gamma m} \text{ for some } A,B\in K_m(\xi)\\ &\text{such that } B\subseteq A \text{ and } \mu(A\triangle B)<2^{-\alpha m}\big], \end{aligned}$$

then

$$\begin{aligned} P''_m &\leqq CTE\int \operatorname{card}^2(K_m(\xi))\cdot\exp\left[-(1/2)(2^{-\gamma m}/2^{-\alpha m/2})^2\right]dQ(\xi)\\ &\leqq CTE\exp\left[-(1/2)\,2^{\beta m}\right]E\operatorname{card}^2(K_m)\\ &\leqq O(1)\,2^{-2[\beta-(2/3)]m}, \end{aligned}$$

by Lemma 1. It follows that

$$\sum_m (P_m + P'_m + P''_m) \leqq O(1)\sum_m 2^{-2[\beta-(2/3)]m} < \infty$$

so that if we let $b_m = \max(b'_m, 2^{-\gamma m})$, then by the Borel-Cantelli lemma for almost every (ω,ξ), there is an $m_0(\omega,\xi)$ such that for all $m\geqq m_0(\omega,\xi)$:

$$|Z(A_m) - Z(A_{m+1})| < b'_m < b_m \quad\text{and}\quad |Z(A'_m) - Z(A'_{m+1})| < b'_m < b_m$$

for every $A\in K$. Moreover, $|Z(A)-Z(B)|<2^{-\gamma m}<b_m$, for every $A,B\in K_m$ such that $B\subseteq A$ and $\mu(A\triangle B)<2^{-\alpha m}$.

Now $\gamma>0$ so that $\sum_m 2^{-\gamma m}<\infty$ and hence to show $\sum_m b_m<\infty$ Q-a.s. is equivalent to showing $\sum_m b'_m<\infty$ Q-a.s.

Let $\alpha'\in(0,1)$. Then $Q(c_m>2^{-\alpha' m})\leqq O(1)\,2^{-2(1-\alpha')m}$.

Hence on the one hand given a positive $\alpha''=(\alpha'-\beta)/2$ there is an m_1 sufficiently large so that outside a set of probability

$$Pr\Big[\bigcup_{m>m_1}(b'_m>2^{-\alpha'' m})\Big] \leqq CTE\sum_{m>m_1} 2^{-2(1-\alpha')m}$$

$$\sum_m b'_m \leqq \sum_{m\leqq m_1} b'_m + \sum_{m>m_1} 2^{-\alpha'' m} < \infty.$$

On the other hand,

$$Q[\mu(A \triangle A_m) > 2^{-\alpha' m}] \leqq O(1)\, 2^{-2^{(1-\alpha')m}}.$$

Hence, since $K_m \uparrow \bigcup_m K_m$ this implies that $\bigcup_m K_m$ is dense in K with Q-probability 1. Moreover

$$\begin{aligned} E\,|Z(A)-Z(A_m)|^2 &\leqq E\; CTE\, \mu(A \triangle A_m) \\ &\leqq O(1)[2^{-\alpha' m} + 2^{-2^{(1-\alpha')m}}] \end{aligned}$$

so that $Z(A_m)$ converges to $Z(A)$ in the L_2-sense.

We shall show that on $\bigcup_m K_m$, Z is uniformly continuous with $P \times Q$-probability 1. Its extension to K is then, by Fubini's theorem, a version of Z with continuous sample functions.

3. Uniform Continuity of Z on $\bigcup_m K_m$. Given $\eta, \delta > 0$ let

$$\sum_{m \geqq m_{\eta,\delta}} (P_m + P_m' + P_m'') < \eta$$

and also

$$\sum_{m \geqq m_{\eta,\delta}} b_m < \delta.$$

Given $A \in \bigcup_m K_m$ we have shown that $Z(A_m)$ converges in the L_2-sense, hence in probability, to $Z(A)$. Observe now that outside the set

$$\bigcup_{k \geqq m_{\eta,\delta}} \{\max_{A \in K} |Z(A_k) - Z(A_{k+1})| > b_k\}$$

we have

$$\sum_{k \geqq m_{\eta,\delta}} |Z(A_k) - Z(A_{k+1})| \leqq \sum_{k > m_{\eta,\delta}} b_k < \delta$$

so that outside this set $Z(A_m)$ converges a.s. to $Z(A)$.

Let $A, B \in \bigcup_m K_m$ and assume without loss of generality that $B \subseteq A$. Observe that $\mu(A_m \triangle B) < \varepsilon$ implies $\mu(A_m \triangle (B \cap A_m)_m') < \varepsilon$. The following inequality is satisfied for each m:

$$\begin{aligned} |Z(A)-Z(B)| \leqq\, & |Z(A)-Z(A_{m+1})| + |Z(B)-Z(B_{m+1})| \\ & + |Z(B_{m_{\eta,\delta}}) - Z((B \cap A_{m_{\eta,\delta}})'_{m_{\eta,\delta}})| + |Z(A_{m_{\eta,\delta}}) - Z((B \cap A_{m_{\eta,\delta}})'_{m_{\eta,\delta}})| \\ & + \sum_{k=m_{\eta,\delta}}^{m} |Z(A_k) - Z(A_{k+1})| + \sum_{k=m_{\eta,\delta}}^{m} |Z(B_k) - Z(B_{k+1})|. \end{aligned}$$

Hence outside a set of $P \times Q$-probability less than 6η

$$\sup\left[|Z(A)-Z(B)|:\ A, B \in \bigcup_m K_m \text{ and } \mu(A \triangle B) \leqq 2^{-\alpha m_{\eta,\delta}}\right] \leqq 4\delta.$$

This implies that on $\bigcup_m K_m$, Z is uniformly continuous with $P \times Q$-probability 1 and hence the desired result follows.

3. Applications

In this section we show how the results and methods of Theorem 1 can be used to obtain the convergence of some process parameterized by K. We refer to Billingsley's book [3] for the terminology.

Compactness of Gaussian Processes Parameterized by K. As in Section 2 we let K be the space of all compact convex sets in R^s and μ be the Lebesgue measure on the unit ball of R^s. We denote by $C(K)$ the space of all real valued continuous functions on K.

Corollary 1. *Let $(Z_i)_{i\in I}$ be a family of Gaussian processes parameterized by K and satisfying the conditions:*

$$\lim_{k\to\infty} \sup_{i\in I} Pr(|Z_i(A)|>k)=0, \qquad \text{for some } A\in K;$$

and

$$\sup_{i\in I} E|Z_i(A)-Z_i(B)|^2 \leqq c\,\mu(A\triangle B), \qquad \text{for every } A,B\in K,$$

where $c>0$.

Then $(Z_i)_{i\in I}$ is tight on $C(K)$.

Proof of Corollary 1. We can make K into a compact metric space by metrizing it with the μ-measure of symmetric differences. Now

$$E|Z_i(A)-Z_i(B)|^2 \leqq c\,\mu(A\triangle B) \qquad \text{for every } i\in I$$

so that by Theorem 1 every Z_i is continuous.

Hence by Arzela-Ascoli's and Prohorov's theorems it follows that to obtain the desired result it suffices to present an equicontinuity set R on C(K) which carries almost all of every one of the measures associated to the processes Z_i. This R is obtained from the proof of Theorem 1 by replacing Z by Z_i all throughout the proof.

A Central Limit Theorem. We follow Dudley [7] very closely in the next corollary.

Corollary 2. *Let $X_1, X_2, \ldots$ be mean zero independent identically distributed $C(K)$-valued random variables satisfying*

$$E|X_1(A)-X_1(B)|^k \leqq [c\,\mu(A\triangle B)]^{k/2} \qquad \text{for every } A,B\in K,$$

and for every $k\geqq 2$, where $c>0$. Then $Z_n=(X_1+\cdots+X_n)/n^{1/2}$ converges in distribution on $C(K)$ to a continuous Gaussian process.

Proof of Corollary 2. The finite dimensional distributions of Z_n converge, by the multidimensional central limit theorem, to those of a Gaussian process Z. Moreover

$$EZ(A)=EZ_n(A)=EX_1(A)=0, \qquad EZ(A)\,Z(B)=EX_1(A)\,X_1(B),$$

$$E|Z(A)-Z(B)|^2=E|Z_n(A)-Z_n(B)|^2 = E|X_1(A)-X_1(B)|^2 \leqq c\,\mu(A\triangle B),$$

so that Z is continuous by Theorem 1.

Let

$$Y_j = n^{-1/2}(X_j(A) - X_j(B)), \qquad j = 1, \ldots, n,$$

then

$$EY_j = 0 \quad \text{and} \quad E\,|Y_j|^k \leqq [c\,\mu(A \triangle B)/n]^{k/2} \qquad \text{for every } j = 1, \ldots, n.$$

Now as in Lemma 1 of [7] we have

$$\begin{aligned} E \exp(t\,Y_1) &= 1 + \sum_{k=2}^{\infty} t^k E\,|Y_1|^k/k! \\ &\leqq 1 + \sum_{k=2}^{\infty} t^k [c\,\mu(A \triangle B)/n]^{k/2}/k! \\ &= \exp\left[t[c\,\mu(A \triangle B)/n]^{1/2}\right] - t[c\,\mu(A \triangle B)/n]^{1/2} \\ &\leqq \exp[8\,t^2\,c\,\mu(A \triangle B)/9\,n]. \end{aligned}$$

Hence, for every n

$$\begin{aligned} Pr[|Z_n(A) - Z_n(B)| > \varepsilon] &= Pr\left[\left|\sum_{j=1}^{n} Y_j\right| > \varepsilon\right] \\ &= 2\,Pr \exp\left[\left(t \sum_{j=1}^{n} Y_j\right) > \exp(t\,\varepsilon)\right] \\ &\leqq 2 \exp(-t\,\varepsilon)\,E \exp\left(t \sum_{j=1}^{n} Y_j\right) \\ &\leqq 2 \exp[-t\,\varepsilon + (8\,t^2/9)\,c\,\mu(A \triangle B)] \\ &\leqq 2 \exp[-\varepsilon^2/9\,c\,\mu(A \triangle B)]. \end{aligned}$$

This exponential type of bound is of the same type as the one used for the Gaussian case in Theorem 1. Hence to obtain the convergence of Z_n we simply have to follow the proof of Dudley's central limit theorem [7] using the approach of Theorem 1.

Convergence of Empirical Processes Parameterized by K. The following corollary solves a problem posed by Peter Bickel, and helps to extend some of his results in [2] as well as some results by Dudley [5].

Corollary 3. *Let $X_1, X_2 \ldots$ be independent random variables all uniformly distributed on the unit ball of R^s. Let*

$$Z_n(A) = (1/n)^{1/2} \sum_{j=1}^{n} (1_A(Y_j) - \mu(A)), \qquad A \in K$$

then the empirical processes Z_n converge in distribution to a continuous Gaussian process Z parameterized by K and satisfying

$$EZ(A) = 0, \qquad EZ(A)\,Z(B) = \mu(A \cap B) - \mu(A)\,\mu(B).$$

Proof of Corollary 3.

Let $S(X, \varepsilon)$ be the ball of radius ε and center X in R^s, and $S = S(0, 1)$. Let λ be the normalized Lebesgue measure on the unit ball S of R^s.

Define

$$\partial_\varepsilon A = \{x \in R^s\colon |x-y| \leqq \varepsilon \text{ for some } y \in \text{boundary of } A\},$$
$$C_\varepsilon = \lambda(S(0, \varepsilon)), \quad \text{and} \quad \lambda^\varepsilon(A, X) = C_\varepsilon \lambda(A \cap S(X, \varepsilon)).$$

Then the $X_j(A) = \lambda^\varepsilon(A, Y_j) - \mu(A)$ are independent identically distributed $C(K)$-valued random variables. Moreover for any $A \subseteq S \cap (\partial_\varepsilon S)^c$, by Fubini's theorem,

$$\begin{aligned} E\lambda^\varepsilon(A, Y) &= \int \left(C_\varepsilon \int 1_{A \cap S(y, \varepsilon)}(x)\, d\lambda(x)\right) d\mu(y) \\ &= C_\varepsilon \lambda \times \mu\{(x, y)\colon x \in A \cap X(y, \varepsilon)\} \\ &= C_\varepsilon \lambda \times \mu\{(x, y)\colon y \in A \cap S(x, \varepsilon)\} \\ &= \int \left(C_\varepsilon \int 1_{A \cap S(x, \varepsilon)}(y)\, d\lambda(x)\right) d\mu(y) \\ &= \int 1_A(y) \left(C_\varepsilon \int 1_{S(x, \varepsilon)}(y)\, d\lambda(x)\right) d\mu(y) \\ &= \int 1_A(y)\, d\mu(y) = \mu(A). \end{aligned}$$

Hence in this case $EX_j(A) = 0$.

Now

$$E|X_1(A) - X_1(B)|^k < [4 C_\varepsilon^{-2} \mu(A \triangle B)]^{k/2}.$$

Therefore, by corollary 2, the smoothed empirical processes

$$Z_n^\varepsilon = (X_1 + \cdots + X_n)/n^{1/2}$$

converge in distribution to a continuous Gaussian process Z^ε, with

$$EZ^\varepsilon(A) = EZ_n^\varepsilon(A) = 0,$$

and

$$EZ^\varepsilon(A)\, Z^\varepsilon(B) = EZ_n^\varepsilon(A)\, Z_n^\varepsilon(B) = \lambda^\varepsilon(A, Y_1)\, \lambda^\varepsilon(B, Y_1) - \mu(A)\, \mu(B).$$

On the other hand, by the multidimensional central limit theorem, the finite dimensional distributions of Z_n converge to the finite dimensional distributions of the continuous (by Theorem 1) Gaussian process Z.

Finally, if we can show that both Z_n^ε and Z^ε converge in L_2 as $\varepsilon \to 0$ to Z_n and Z respectively, then the convergence in distribution of Z_n to Z will follow. But

$$\mu(|\lambda^\varepsilon(A, Y_1) - 1_A(Y_1)| > 0) = \mu(Y_1 \in \partial_\varepsilon A) \leqq CTE\, \varepsilon$$

and

$$\begin{aligned} |EZ^\varepsilon(A)\, Z^\varepsilon(B) - EZ(A)\, Z(B)| &= |EZ_n^\varepsilon(A)\, Z_n^\varepsilon(B) - EZ_n(A)\, Z_n(B)| \\ &\leqq |E\lambda^\varepsilon(A, Y_1)\, \lambda^\varepsilon(B, Y_1) - \mu(A \cap B)| \leqq CTE \cdot \varepsilon \end{aligned}$$

hence the corollary follows.

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