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Spectral Theory of Branching Processes. I*

The Case of Discrete Spectrum

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§ 1. Introduction

A one-dimensional Markov branching process may be characterized as follows. An organism, at the end of its lifetime (of fixed duration), produces a random number ξ of offspring with probability distribution

$$(1) \qquad Pr\{\xi = k\} = a_k \qquad k = 0, 1, 2, \ldots$$

where as usual

$$a_k \geqq 0 \qquad \sum_{k=0}^{\infty} a_k = 1 .$$

All offspring act independently with the same lifetime and distribution of progeny. The population size $X(n)$ at the nth generation is a temporally homogeneous Markov chain whose transition probability matrix is

$$(2) \qquad P_{ij} = Pr\{X(n+1) = j \mid X(n) = i\} = Pr\{\xi_1 + \xi_2 + \cdots + \xi_i = j\}$$

where ξ's are independent observations of a random variable with the probability law (1). An equivalent way to express (2) is through its generating function which is simply

$$(3) \qquad \sum_{j=0}^{\infty} P_{ij} s^j = [f(s)]^i \qquad i = 0, 1, \ldots$$

where

$$f(s) = \sum_{k=0}^{\infty} a_k s^k ,$$

It is a familiar fact that the n-step transition probability matrix

$$P_{ij}^{(n)} = \Pr\{X(n) = j \mid X(0) = i\}$$

possesses the generating function

$$(4) \qquad \sum_{j=0}^{\infty} P_{ij}^{(n)} s^j = [f_n(s)]^i$$

where

$$(5) \qquad f_n(s) = f_{n-1}(f(s))$$

is the nth functional iterate of $f(s)$.

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A finite state Markov transition matrix P_{ij} ordinarily admits a spectral representation of the form

(6) $$P_{ij}^n = \sum_r \lambda_r^n \psi_i(r)\,\theta_j(r)$$

where λ_r $(r = 1, 2, \ldots)$ are the eigenvalues of the matrix P_{ij}, $\psi_i(r)$ denotes the ith component of the rth right eigenvector and $\theta_j(r)$ denotes the jth component of the rth left eigenvector. The system $\{\psi_i(r), \theta_i(r)\}_{r=1}^N$ is chosen to be biorthonormal.

The representation (6) is certainly valid when all eigenvalues are simple, i.e., no elementary divisors arise. For infinite transition probability matrices the possibility of a spectral decomposition like (6) is rare. The existence of eigenvalues is not even assured and indeed, continuous spectrum is usually present.

In the case where P_{ij} is the transition matrix of a one-dimensional random walk (see [7]) then P is a Jacobi matrix and a generalized spectral representation exists. In this case a metric can be introduced such that P becomes self-adjoint and the classical spectral resolution of Hilbert space theory is available. The device of symmetrizing a Markov transition matrix works for diffusion processes on the line and more generally for reversible processes (see [*13*], [*11*]). Unfortunately two or higher dimensional diffusion processes fail usually to be reversible and so the theory of self-adjoint operators in Hilbert space is not applicable.

The transition matrix of a branching process is not symmetrizable. Nevertheless, we will establish, under mild restrictions on the probability generating function $f(s)$, the existence of a spectral representation of P and its iterates.

It is usual in dealing with branching processes to consider three situations according as $m = f'(1)$, the expected number of progeny per individual, is greater than, less than, or equal to 1. The probabilistic nature of the process differs fundamentally for these cases.

When $m > 1$ there is positive probability that the population size becomes infinite $(\Pr\{X(n) \to \infty \mid X(0) = 1\} > 0)$.

When $m \leqq 1$, extinction occurs with certainty, i.e., $\Pr\{X(n) = 0$ for some $n \mid X(0) = 1\} = 1$. However the expected time until extinction is finite or infinite according as $m < 1$ or $m = 1$.

When $m > 1$, the matrix P, although not equivalent to a self-adjoint operator, defines a completely continuous transformation (see Theorem 1 below). Using this fact we develop a spectral decomposition of the form (6) valid for sufficiently large n. By imposing further restrictions on $f(s)$, which still admit most of the important examples, we can guarantee formula (6) for all n. When $f(s)$ is analytic in the neighborhood of 1 and $m < 1$ then P again is completely continuous and the representation (6) holds for large n. For $m = 1$ the operator P is no longer completely continuous. Nevertheless when $f(s)$ generates a Pólya frequency sequence (see below) then P^n admits a spectral representation involving continuous spectrum, of the explicit form

(7) $$P_{ij}^n = \int_0^\infty e^{-n\xi}\, Q_i(\xi)\, d\theta_j(\xi)$$

where $\theta_j(\xi)$ is of bounded variation. The first term of the expansion (6) was recently obtained by KENDALL [*12*] in the case $m > 1$.

In this paper we develop the spectral representation for the cases $m > 1$ generally and for $m < 1$ when $f(s)$ is analytic at 1. We also treat branching processes with immigration.

In the companion publication we elaborate the theory when $m = 1$ which leads to the representation formula (7). In a separate paper we will show how the methods of this paper extend to multi-type branching processes.

The utility of the representation formula in deriving improved local limit theorems, the strong ratio theorem and other probabilistic consequences is also deferred to another publication.

When $f(s)$ is a meromorphic function of the special form

$$f(s) = C \frac{e^{\gamma s} \prod_{i=1}^{\infty} (1 + \alpha_i s)}{\prod_{i=1}^{\infty} (1 - \beta_i s)} = \sum_{k=0}^{\infty} a_k s^k, \tag{8}$$

$$\gamma \geqq 0, \quad \alpha_i \geqq 0, \quad \beta_i \geqq 0, \quad C > 0, \quad \sum (\alpha_i + \beta_i) < \infty$$

i.e., $f(s)$ generates a Pólya frequency sequence then we know several refinements concerning the eigenvectors occurring in (6). The function $\psi_i(r)$ possesses remarkable oscillation properties which we will describe later.

§ 2. The Linear Transformation Associated with a Branching Process

We consider a one-dimensional branching process, as described in Section 1, characterized by the generating function $f(s) = \sum_{k=0}^{\infty} a_k s^k$. In sections 2, 3, 4, and 5 we assume that $a_0 = f(0) > 0$, $a_k \geqq 0$, $f(1) = 1$, and $1 < m \leqq +\infty$ where

$$m = f'(1) = \sum_{k=1}^{\infty} k a_k .$$

The one-step transition matrix $\| P_{i,j} \|$ is defined by

$$\sum_{j=0}^{\infty} P_{i,j} s^j = [f(s)]^i, \quad i = 0, 1, 2, \ldots, \quad |s| < 1 .$$

Under the above assumptions, the equation

$$f(s) = s, \quad 0 < s < 1$$

has a unique solution, $s = q$ $(0 < q < 1)$. We have $f(s) > s$ for $0 \leqq s < q$ and $f(s) < s$ for $q < s < 1$. Since $f(s)$ is a power series with non-negative coefficients we have $|f(s)| \leqq f(|s|)$ and hence

$$|f(s)| < |s| \quad \text{if} \quad q < |s| < 1 . \tag{9}$$

This inequality implies that the mapping $s \to f(s)$ maps each disc $|s| \leqq r$, $q < r < 1$ into a strictly smaller concentric disc of radius $f(r)$.

The value $c = f'(q)$ of the derivative at the fixed point plays an important role in subsequent considerations. Since $f'(s)$ is strictly increasing near $s = q$, and since $f(s) < s$ in an interval to the right of $s = q$, we have

$$0 < c = f'(q) < 1 .$$

Finally we remark that (9), together with Rouché's theorem, implies that $f(s)$ has only the one fixed point, $s = q$, in the interior of the unit circle.

We introduce the Hilbert space $\mathscr{H}$ consisting of all sequences $\xi = \{\xi_i\}_{i=0}^{\infty}$ of complex numbers such that

$$\|\xi\|^2 = \sum_{i=0}^{\infty} |\xi_i|^2 q^i < \infty . \tag{10}$$

The inner product in $\mathscr{H}$ is defined as

$$(\xi, \eta) = \sum_{i=0}^{\infty} \xi_i \bar{\eta}_i q^i \quad \text{for} \quad \xi, \eta \in \mathscr{H} .$$

It is convenient to associate with any $\xi \in \mathscr{H}$ a generating function $\xi(s)$ where

$$\xi(s) = \sum_{i=0}^{\infty} \xi_i s^i . \tag{11}$$

Since $\|\xi\| < \infty$ for $\xi \in \mathscr{H}$ we conclude that $\xi(s)$ is analytic for $|s| < \sqrt{q}$. Furthermore, with the aid of the Schwarz inequality, we derive some simple bounds.

Lemma 1. *If* $\xi \in \mathscr{H}$ *then* $\xi(s) = \sum_{k=0}^{\infty} \xi_k s^k$ *obeys the inequality*

$$|\xi(s)| \leqq \|\xi\| (1 - |s|^2/q)^{-1/2} \quad \text{for} \quad |s| < \sqrt{q} . \tag{12}$$

Proof. We have

$$|\xi(s)| = \Big|\sum_{k=0}^{\infty} \xi_k s^k\Big| \leqq \Big(\sum_{k=0}^{\infty} |\xi_k|^2 q^k\Big)^{1/2} \Big(\sum_{k=0}^{\infty} (|s|^2/q)^k\Big)^{1/2}$$
$$= \|\xi\| (1 - |s|^2/q)^{-1/2}.$$

If $\xi \in \mathscr{H}$ then

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |\xi(re^{i\theta})|^2 d\theta = \sum_{k=0}^{\infty} |\xi_k|^2 r^{2k}, \quad 0 \leqq r < \sqrt{q}$$

which increases with r and $\to \|\xi\|^2$ as $r \to \sqrt{q}$. It follows that the functions $F_r(\theta) = \xi(re^{i\theta})$ converge in norm in $L_2(-\pi, \pi)$ when $r \to \sqrt{q}$ and the limit function $F(\theta)$ satisfies

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |F(\theta)|^2 d\theta = \|\xi\|^2 .$$

(See also [*14*], where it is shown that $F(\theta)$ can be replaced by the pointwise radial limit

$$\lim_{r \to \sqrt{q}} \xi(re^{i\theta})$$

which exists for almost all θ.) If the domain of definition of $\xi(s)$ is extended to include the boundary circle $|s| = \sqrt{q}$ by setting $\xi(\sqrt{q}\, e^{i\theta}) = F(\theta)$ then for $\|\xi\|$ there results the formula

$$\|\xi\|^2 = \frac{1}{2\pi i} \int_{|s| = \sqrt{q}} |\xi(s)|^2 \frac{ds}{s}. \tag{13}$$

Of course if $\xi(s)$ happens to be regular inside some larger circle $|s| = \sqrt{q} + \varepsilon$ then (13) is valid with the ordinary interpretation of $\xi(s)$ for $|s| = \sqrt{q}$.

Let P be the linear operator defined on $\mathscr{H}$ to itself by

$$(\xi P)_j = \sum_{i=0}^{\infty} \xi_i P_{ij}. \tag{14}$$

The next lemma tells us that P is well defined, continuous, and supplies a bound for the norm of P.

Lemma 2. *If $\xi \in \mathscr{H}$ then $\xi P = (\sum_{i=0}^{\infty} \xi_i P_{ij})_{j=0}^{\infty} \in \mathscr{H}$, moreover $(\xi P)(s) = \xi(f(s))$ for $|s| \leqq \sqrt{q}$ and*

$$\|P\| \leqq \left(1 - \left[\frac{f(\sqrt{q})}{\sqrt{q}}\right]^2\right)^{-1/2}. \tag{15}$$

Proof. The inequality (9) shows there is an r such that $\sqrt{q} < r < 1$ and $f(r) < \sqrt{q}$. With this r

$$\sum_{j=0}^{\infty} r^j \sum_{i=0}^{\infty} |\xi_i| P_{ij} = \sum_{j=0}^{\infty} |\xi_i| \sum_{j=0}^{\infty} P_{ij} r^j = \sum_{i=0}^{\infty} |\xi_i| f^i(r) < \infty,$$

because the power series $\xi(s)$ has radius of convergence $\geqq \sqrt{q} > f(r)$. It follows that the sequence $\{(\xi P)_j\}$ defined by (14) is in $\mathscr{H}$ and the power series

$$(\xi P)(s) = \sum_j s^j \sum_i \xi_i P_{ij}$$

has radius of convergence $\geqq r > \sqrt{q}$. Moreover, for $|s| \leqq r$,

$$(\xi P)(s) = \sum_i \xi_i \sum_j P_{ij} s^j = \sum_i \xi_i f^i(s) = \xi(f(s)).$$

From (13) and the inequality (12) we have

$$\begin{aligned} \|\xi P\|^2 &= \frac{1}{2\pi i} \int_{|s|=\sqrt{q}} |\xi(f(s))|^2 \frac{ds}{s} \\ &\leqq \|\xi\|^2 \left(1 - \left[\frac{f(\sqrt{q})}{\sqrt{q}}\right]^2\right)^{-1}, \end{aligned}$$

and (15) follows from this.

In the process of the proof we noted

Corollary 1. *If $\xi \in \mathscr{H}$ then $(\xi P)(s)$ is analytic in $|s| < r$ for some $r > \sqrt{q}$.*

A routine calculation will determine the form of the adjoint operator P^*. Thus, let $\eta \in \mathscr{H}$, then

$$(P^* \eta)_i = \frac{1}{q^i} \sum_{i=0}^{\infty} P_{ij} \eta_j q^j \tag{16}$$

With this definition we have

$$(\xi, P^* \eta) = (\xi P, \eta) \quad \text{for all} \quad \xi, \eta \in \mathscr{H}.$$

It is obvious comparing (14) and (16) that P is not self-adjoint.

From the contraction property (9) of the mapping $s \to f(s)$ we can deduce that

Theorem 1. *The operator P is completely continuous.*

Proof. It suffices to show that for some complete orthonormal sequence $e^{(n)}$, $n = 0, 1, 2, \ldots$ in $\mathscr{H}$, we have

$$\sum_{n=0}^{\infty} \| e^{(n)} P \| < \infty \tag{17}$$

(In other words we show that P is a nuclear operator, i.e., it has a finite trace norm.)
To this end, we choose the complete orthonormal sequence $e^{(n)} = \{\delta_{n,i} q^{-n/2}\}$, so that $e^{(n)}(s) = (s/\sqrt{q})^n$. Then

$$(e^{(n)} P)(s) = \left[\frac{f(s)}{\sqrt{q}}\right]^n$$

and from (13)

$$\| e^{(n)} P \|^2 \leqq \left[\frac{f(\sqrt{q})}{\sqrt{q}}\right]^{2n},$$

so that (17) converges because $f(\sqrt{q}) < \sqrt{q}$ by (9).

§ 3. Eigenvalues and Eigenvectors for the Case $m = f'(1) > 1$

The spectral properties of a completely continuous operator are described in considerable detail in the classical Fredholm theory. The non-zero eigenvalues of P form a finite or countable sequence $\{\lambda_n\}$. The adjoint operator P^* is also completely continuous and has the same sequence of non-zero eigenvalues. Moreover, for each λ_n, the corresponding eigenspaces of P and P^* have the same finite dimension.

We begin by presenting a recursive procedure for determining eigenvalues and eigenvectors of the adjoint operator P^*. Each eigenvalue $\lambda_n \neq 0$ thus determined, is then known to be an eigenvalue of P, so that the equation

$$\xi P = \lambda_n \xi$$

has a non-trivial solution $\xi \in \mathscr{H}$. Hence we deduce that the functional equation

$$\xi(f(s)) = \lambda_n \xi(s)$$

has a non-trivial solution $\xi(s)$ analytic in $|s| < \sqrt{q}$. This method of showing the existence of solutions of the functional equation can be easily circumvented for the simple type of branching process under consideration, but has the advantage that it is readily generalized to certain branching processes of more complex type, especially branching processes with immigration, and multi-type branching processes.

By setting $s = 1$ in the identity

$$\sum_{j=0}^{\infty} q^{-i} P_{ij} q^j s^j = \left[\frac{f(qs)}{q}\right]^i \tag{18}$$

we verify that $e_0 = \{1\} = (1, 1, 1, \ldots, 1, \ldots)$ is an eigenvector of P^* belonging to the eigenvalue $\lambda_0 = 1$. By differentiating (18) with respect to s we obtain

$$\sum_{j=0}^{\infty} q^{-i} P_{ij} q^j j s^{j-1} = i \left[\frac{f(qs)}{q}\right]^{i-1} f'(qs), \tag{19}$$

and now, by setting $s = 1$, we verify that

$$e_1 = \{i\} = (0, 1, 2, \ldots, i, \ldots)$$

is an eigenvector of P^* belonging to the eigenvalue

$$\lambda_1 = c = f'(q).$$

This vector is in the Hilbert space, indeed the vectors $\{i^r\}$, $r = 1, 2, \ldots$ are in the Hilbert space since the series

$$\sum_{i=0}^{\infty} |i|^{2r} q^i$$

are convergent.

By evaluating derivatives of (19) at $s = 1$ we obtain

$$\sum_{j=0}^{\infty} q^{-i} P_{ij} q^j (j)_r = c^r (i)_r + \pi_{r-1}(i), \quad r = 1, 2, 3, \ldots \tag{20}$$

where we use the notation

$$(x)_r = x(x-1)(x-2)\ldots(x-r+1), \quad r = 1, 2, \ldots$$

and where $\pi_{r-1}(x)$ is a polynomial in x of degree $r - 1$ which vanishes at $x = 0$. It will be shown that for $r \geqq 1$ there are monic polynomials $Q_r(x)$ of exact degree r such that $Q_r(0) = 0$ and

$$e_r = \{Q_r(i)\} = (Q_r(0), Q_r(1), Q_r(2), \ldots, Q_r(i), \ldots)$$

is an eigenvector of P^* belonging to the eigenvalue $\lambda_r = c^r = [f'(q)]^r$. We already have $Q_1(x) = x$. Assume $n \geqq 2$ and that $Q_r(x)$ has been determined for $r = 1, 2, \ldots, n-1$. Then

$$Q_n(x) = (x)_n + \sum_{k=1}^{n-1} \alpha_k Q_k(x)$$

where the constants α_k must be chosen so that

$$c^n [(i)_n + \sum_{k=1}^{n-1} \alpha_k Q_k(i)] = c^n (i)_n + \pi_{n-1}(i) + \sum_{k=1}^{n-1} \alpha_k c^k Q_k(i) \tag{21}$$

is satisfied for $i = 0, 1, 2, \ldots$. The known polynomial $\pi_{n-1}(x)$, since it vanishes at $x = 0$, can be expressed as

$$\pi_{n-1}(x) = \sum_{k=1}^{n-1} \beta_k Q_k(x)$$

where the β_k are constants. Since $Q_1(x), \ldots, Q_{n-1}(x)$ are linearly independent we can equate coefficients of $Q_k(i)$ in (21), obtaining

$$c^n \alpha_k = \beta_k + c^k \alpha_k, \qquad k = 1, 2, \ldots, n-1.$$

Since $0 < c < 1$, this has the unique solution

$$\alpha_k = -\beta_k/(c^k - c^n).$$

Therefore $Q_n(x)$ is determined.

Thus we have eigenvalues $\lambda_r = c^r$, $r = 0, 1, 2, \ldots$ of P^*, with eigenvectors

$e_r = \{Q_r(i)\}_{i=0}^{\infty}$ where $Q_r(x)$ is a polynomial of exact degree r, $Q_0(x) = 1$, $Q_1(x) = x$, and $Q_r(0) = 0$ for $r \geqq 1$. It is not difficult to show that polynomial vectors are dense in the Hilbert space, but this fact seems irrelevant to our problem.

We next consider eigenvalues and eigenvectors of P. The results are summarized in the following theorem.

Theorem 2. (i) *The only non-zero eigenvalues of P, and of P^*, are*

$$\lambda_r = c^r, \quad r = 0, 1, 2, \ldots, \quad c = f'(q), \tag{22}$$

and each eigenspace of P, and of P^*, is one-dimensional.

(ii) *The functional equation*

$$A(f(s)) = cA(s), \quad A(q) = 0, \quad A'(q) = 1, \tag{23}$$

has a unique analytic solution $A(s)$ which is regular at $s = q$. The solution is regular in $|s| < 1$.

(iii) *The eigenvectors of P^* are the polynomial eigenvectors*

$$e_r = \{Q_r(i)\}, \qquad r = 0, 1, 2, \ldots \tag{24}$$

constructed previously. The eigenvector $d_r = \{d_i^r\}$ of P belonging to the eigenvalue λ_r has a generating function

$$d_r(s) = \sum_{i=0}^{\infty} d_i^r s^i$$

given by

$$d_r(s) = [A(s)]^r, \qquad r = 0, 1, 2, \ldots. \tag{25}$$

Proof. We already proved in the constructions preceding the theorem that the constants c^r $(r = 0, 1, \ldots)$ are eigenvalues of P^* and hence also of P. Actually the existence of eigenvectors of P^* of the form (24) associated with c^r was established.

Now let $\lambda \neq 0$ be an eigenvalue of P and let $\xi = (\xi_0, \xi_1, \ldots) \in \mathscr{H}$ denote a nontrivial left eigenvector for λ. Thus

$$\sum_{i=0}^{\infty} \xi_i P_{ij} = \lambda \xi_j, \qquad j = 0, 1, 2, \ldots \tag{26}$$

If $\xi(s)$ denotes the generating function of ξ then this is analytic at least for $|s| < \sqrt{q}$ since $\xi \in \mathscr{H}$. We can express (26) equivalently in terms of generating functions as

$$\xi(f(s)) = \lambda \xi(s), \quad |s| < \sqrt{q}. \tag{27}$$

From (9) and (27) we infer that $\xi(s)$ can be analytically continued throughout the unit circle $|s| < 1$.

Now $\xi(s)$ has a zero of order $k \geqq 0$ at $s = q$. Of course, k is finite because by hypothesis $\xi(s)$ is non-trivial. It follows that $\xi(s)$ has an expansion about q of the form

$$\xi(s) = u_k(s-q)^k + u_{k+1}(s-q)^{k+1} + \cdots \qquad u_k \neq 0.$$

Differentiating (27) k times and then substituting $s = q$ yields the equation

$$\lambda u_k k! = c^k u_k k!, \qquad (c = f'(q))$$

which implies $\lambda = c^k$. Thus all eigenvalues of P are given by (22). Moreover, if ξ is an eigenvector of P for the eigenvalue λ_k then the generating function $\xi(s)$ has, at $s = q$, a zero of order k exactly.

We must verify that the eigenspaces are one-dimensional. To this end we show there is at most one analytic function $T(s)$ regular at $s = q$ which satisfies

$$T(f(s)) = c^k T(s) \tag{28}$$

and

$$T(q) = T'(q) = \cdots = T^{(k-1)}(q) = 0, \quad T^{(k)}(q) = k! \tag{29}$$

Differentiation of the functional equation n times yields

$$T^{(n)}(f(s))[f'(s)]^n + R_n(s) = c^k T^{(n)}(s)$$

where $R_n(s)$ is a linear combination of the lower derivatives $T^{(m)}(f(s))$, $m = 1, 2, \ldots, n-1$ whose coefficients depend on $f(s)$ and its derivatives ($R_0(s) \equiv 0$). Substitution of $s = q$ yields

$$(c^k - c^n)\, T^{(n)}(q) = R_n(q). \tag{30}$$

Now if $n \geqq k + 1$ and if we assume that the values $T^{(m)}(q)$, $m < n$ have already been determined, then $R_n(q)$ is completely known, and since $c^k - c^n > 0$, equation (30) determines $T^{(n)}(q)$ uniquely. Hence by induction on n, all the derivatives of $T(s)$ at $s = q$ are uniquely determined, which is what we had to prove. It follows that the eigenvector d_k of P belonging to the eigenvalue $\lambda_k = c^k$ is determined to within a constant factor.

In particular (23) has at most one solution $A(s)$ regular at $s = q$. But P has an eigenvector d_1 belonging to the eigenvalue $\lambda_1 = c$, and after proper normalization its generating function $d_1(s)$ will be a solution of (23). Hence $A(s) = d_1(s)$. As remarked below (27), $d_1(s)$, and so also $A(s)$, is regular in the unit circle.

For $r = 0, 1, 2, \ldots$ let

$$d_r(s) = [A(s)]^r.$$

Then $d_r(s)$ is regular in $|s| < 1$, and hence is the generating function of a vector $d_r \in \mathfrak{H}$. Since $d_r(s) \neq 0$ and $d_r(f(s)) = c^r d_r(s)$, d_r is an eigenvector of P belonging to the eigenvalue $\lambda_r = c^r$. This completes the proof of Theorem 2.

For $r = 0$ we have $d_0(s) \equiv 1$. Thus the eigenvector associated with $\lambda_0 = 1$ is

$$d_0 = (1, 0, 0, \ldots, 0, \ldots).$$

Since $A(q) = 0$, $A'(q) = 1$, the mapping

$$s \to w = A(s)$$

provides a conformal map of some neighborhood of $s = q$ onto a neighborhood of $w = 0$. Therefore there is an inverse function $s = B(w)$ defined by

$$B(A(s)) = s \qquad \text{near} \quad s = q. \tag{31}$$

The function $B(w)$ is regular near $w = 0$ and maps a neighborhood of $w = 0$ onto a neighborhood of $s = q$. From (31) and (23) we obtain

$$f(B(w)) = B(cw), \quad B(0) = q, \quad B'(0) = 1. \tag{32}$$

The next theorem is well known but the method of proof is new.

Theorem 3.

$$\lim_{n\to\infty} \frac{f_n(s)-q}{c^n} = A(s) \tag{33}$$

uniformly in every circle $|s| \leqq r < 1$.

Proof. Iteration of (32) gives, near $w = 0$,

$$f_n(B(w)) = B(c^n w), \tag{34}$$

and then by the substitution $w = A(s)$

$$f_n(s) = B(c^n A(s)) \tag{35}$$

which is valid at first near $s = q$. Assume $|s| \leqq r < 1$ where $r > q$. There is a value n_0 such that for all such s, $n \geqq n_0$ implies $c^n A(s)$ lies inside the circle of convergence of the power series

$$B(w) = q + w + \sum_{k=2}^{\infty} b_k w^k. \tag{36}$$

Then the functional equation (35) is valid for $n \geqq n_0$ and $|s| \leqq r$ by analytic continuation. Hence

$$f_n(s) = B(c^n A(s)) = q + c^n A(s) + 0(|c^n A(s)|^2)$$

and (33) follows.

Corollary 1. $A'(s) > 0$ *for* $0 < s < 1$ *and* $A(s) \to +\infty$ *when* $s \to 1$. *The inverse function* $B(w)$ *is defined and regular in an open set containing the half line* $A(0) < w < \infty$.

Proof. With the aid of (23) and considerations of degree, we see that $A(s)$ cannot be a polynomial. From (33) it follows that $A^{(r)}(0) \geqq 0$ for all $r \geqq 1$, and so

$$A'(s) = \sum_{r=1}^{\infty} A^{(r)}(0) \frac{s^{r-1}}{(r-1)!}$$

shows that $A'(s) > 0$ for $0 < s < 1$. Since $\lim_{s\to 1} A(s) = L$ satisfies $L > 0$, and from (23), $L = cL$, we have $L = +\infty$. It follows that $B(w)$ can be analytically continued from a neighborhood of $w = 0$ along the entire half line $A(0) < w < \infty$.

Corollary 2. *If* $f'(0) > 0$ *then* $B(w)$ *is defined and regular in an open set containing the closed half line* $A(0) \leqq w < \infty$.

Proof. From (23)

$$A'(0) = \frac{1}{c} A'(f(0)) f'(0)$$

which is positive since $A'(f(0)) > 0$ by corollary 1. Hence the analytic continuation of $B(w)$ to some interval $A(0) - \varepsilon < w < \infty$, where $\varepsilon > 0$, is possible.

It is traditional in the theory of branching processes (see [*3*]) to derive the existence of $A(s)$ by proving directly the formula (33) using ad hoc methods. In contrast, we inferred the existence of $A(s)$ by exploiting the property that P is completely continuous coupled with the construction of certain right eigenvectors. This latter technique works in the multi-dimensional branching process case while the classical method does not seem to generalize easily.

§ 4. Spectral Representation of P in the Case $m = f'(1) > 1$

Prior to developing the spectral representation of P it is convenient to relate the eigenvectors of P^* with the function $B(w)$. In fact, let

$$B^j(w) = \sum_{r=0}^{\infty} \psi_j(r)\, w^r, \qquad j = 0, 1, 2, \ldots \tag{37}$$

We claim that $\{\psi_j(r)/q^j\}_{j=0}^{\infty}$ is a right eigenvector for the eigenvalue c^r, in the sense

$$\sum_{j=0}^{\infty} P_{ij}\, \psi_j(r) = c^r\, \psi_i(r)\,. \tag{38}$$

But (38) is equivalent to the double generating function

$$\sum_r \sum_j P_{ij}\, \psi_j(r)\, w^r = \sum_{r=0}^{\infty} c^r\, \psi_i(r)\, w^r = B^i(c\,w)\,.$$

The left hand side is absolutely convergent for small w and equal to

$$\sum_{j=0}^{\infty} P_{ij}\, B^j(w) = f^i(B(w))\,.$$

In other words (38) essentially reduces to the functional relation $f(B(w)) = B(cw)$, and is therefore verified.

We introduce the expansion

$$[A(s)]^r = \sum_{j=0}^{\infty} \theta_j(r)\, s^j\,. \tag{39}$$

Thus, in terms of the notation of Theorem 2, we have

$$d_r = (\theta_0(r)\,, \theta_1(r)\,, \ldots)$$

We can now prepare to state the general spectral representation theorem for P. Refinements will be indicated in Section 5.

Theorem 4. (i) *The operator P admits the spectral representation*

$$P_{ij}^n = \sum_{r=0}^{\infty} (c^r)^n\, \theta_j(r)\, \psi_i(r) \qquad c = f'(q) \tag{40}$$

for all sufficiently large n and all $i, j = 0, 1, \ldots$.

The quantities $\theta_j(r)$ are defined in (39) *and $\psi_i(r)$ in* (37) .

(ii) *If the coefficients $\{\psi_1(r)\}_{r=1}^{\infty}$ in*

$$B(w) = q + \sum_{r=1}^{\infty} \psi_1(r)\, w^r$$

have alternating signs, that is $(-1)^{r-1}\psi_1(r) \geqq 0$, $r \geqq 1$ (or alternating from some r on), then (40) *is valid for all $n \geqq 1$, the series being absolutely convergent. If we also have $f'(0) > 0$ then* (40) *is valid for $n \geqq 0$.*

Remark. Examples can be given in which the coefficients of $B(w)$ are not alternating from any point on. Such examples are most easily constructed using the theory of continuous time branching processes.

Proof. (i) Consider

$$A(f(s)) = c\,A(s)$$

and upon iteration the relation

$$A(f_n(s)) = c^n A(s) \qquad 0 \leqq s < 1 .$$

Inverting by $B(w)$, which is permissible in view of Corollary 1 of Theorem 3 we obtain

$$f_n(s) = B(c^n A(s)) \qquad 0 \leqq s < 1 . \tag{41}$$

Actually (41) is valid for complex s in a neighborhood of the segment $0 \leqq s < 1$. We know that

$$[f_n(s)]^i = \sum_{j=0}^{\infty} P_{ij}^{(n)} s^j$$

and so

$$\text{coefficients } s^j \text{ in } B^i(c^n A(s)) = P_{ij}^{(n)} . \tag{42}$$

Now

$$B^i(w) = \sum_{r=0}^{\infty} \psi_i(r) w^r .$$

Assume for the moment that $c^n A(0)$ lies in the interior of the circle of convergence of $B(w)$. The for $|s|$ sufficiently small, $c^n A(s)$ also lies in the interior and

$$B^i(c^n A(s)) = \sum_{r=0}^{\infty} \psi_i(r) c^{nr} A^r(s) ,$$

Expanding $A^r(s)$ about the origin gives

$$B^i(c^n A(s)) = \sum_{r=0}^{\infty} \psi_i(r) c^{nr} \sum_{j=0}^{\infty} \theta_j(r) s^j$$

the double series converging absolutely for $|s|$ small enough. Picking out the coefficient of s^j in accordance with (42) establishes (40).

Now for n sufficiently large $c^n A(0)$, which tends to zero, falls within the radius of convergence of $B(w)$. We have proved Part (i) of the theorem.

(ii) The argument shows that (40) is valid for all $n \geqq m$ if the radius of convergence of $B(w)$ exceeds $c^m |A(0)|$. Assume $(-1)^{r-1} \psi_1(r) \geqq 0$, $r = 1, 2, \ldots$, and let ϱ be the radius of convergence of $B(w) = q + \sum_{r=1}^{\infty} \psi_1(r) w^r$. Then the Pringsheim theorem (Titchmarsh [*14*], p. 214) assures that $w = -\varrho$ is a singular point of $B(w)$. Since $B(w)$ is regular in a neighborhood of the open half line $A(0) < w < \infty$ by Corollary 1 of Theorem 3, we conclude that $-\varrho \leqq A(0)$. (Note that $A(0) < 0$ since $A(q) = 0$ and $A'(s) > 0$ for $0 < s < 1$.) Therefore $c|A(0)| < \varrho$ and (40) is valid for $n \geqq 1$. If $f'(0) > 0$ then we deduce from Corollary 2 of Theorem 3 that $-\varrho < A(0)$ and hence (40) is valid for $n \geqq 0$.

The proof of Theorem 4 is now finished.

Illustrations and applications of part (ii) will be given in Section 7.

The eigenvectors of P and of P^* are, of course, biorthogonal and if properly normalized

$$\sum_{i=0}^{\infty} \psi_i(r) \theta_i(r') = \delta_{r,r'} . \tag{43a}$$

We can prove (43a) by expanding the left member of

$$A^r(B(w)) = w^r$$

in powers of w.

The dual biorthogonality of the left and right eigenvectors $e_r = \{\psi_i(r)\}_{r=0}^{\infty}$ and $d_j = \{\theta_j(r)\}_{r=0}^{\infty}$, i.e.,

$$\sum_{r=0}^{\infty} \psi_i(r)\, \theta_j(r) = \delta_{ij} \tag{43b}$$

is expressed by the inverse relations

$$B^i(A(s)) = s^i .$$

The identities (43b) are formal and correspond to the case $n = 0$ in (40). However, the convergence is absolute and indeed (43a) is rigorous when the components of $\{\psi_1(r)\}_{r=1}^{\infty}$ alternate is sign. The justification for this assertion is affirmed by adapting a similar argument to that of part (ii) of Theorem 4.

Remark. The spectral representation developed above was derived under the assunption that $m = f'(1) > 1$. It is useful to point out that the spectral representation (40) obtains with no requirements on the order of magnitude of $f'(1)$ whenever $f(1) < 1$ and $f(0) > 0$. In this case there definitely exists a unique fixed point $q = f(q)$, $0 < q < 1$. As previously we introduce the Hilbert space of sequences $\xi = (\xi_0, \xi_1, \ldots)$ of finite norm

$$|\xi| = \sum_{i=0}^{\infty} q^i |\xi_i|^2 < \infty .$$

The complete analysis leading to (40) carries over in toto. The conclusions of Theorem 4 persist which we state as follows:

Theorem 5. *Let* $f(s) = \sum_{k=0}^{\infty} a_k s^k$ *where* $a_0 > 0$, $a_i \geqq 0$ $(i \geqq 1)$ *and* $f(1) < 1$. *We define the matrix* $P = \| P_{ij} \|$ *by the generating function relation* (3). *Then* (40) *holds for all sufficiently large* n.

The right and left eigenvectors of P *are characterized as in Theorem 2.*

If $f(s)$ *generates a Pólya frequency sequence (see* (48)*) then the spectral representation* (40) *prevails for all integers* $n \geqq 0$. *(See Theorem 7 below.)*

Remark. An example can be constructed for which $f(1) < 1$ and the convergence in (40) fails for any finite number of values of n (see [*10*]). It seems likely that for special examples, the convergence in (40) may fail even in the case that $f(1) = 1$. This question remains open.

§ 5. Refinements on the Spectral Representation ($m > 1$)

We pointed out in Theorem 4 that whenever

$$B(w) = q + \sum_{r=1}^{\infty} \psi_1(r)\, w^r \tag{44}$$

has coefficients of alternating signs for $r \geqq 1$ and $f'(0) > 0$ then the spectral representation (40) holds for all integers $n = 0, 1, 2, \ldots$. In this section we investigate conditions which guarantee the property

$$(-1)^{r-1}\psi_1(r) \geqq 0 \qquad r \geqq 1. \tag{45}$$

An example of this phenomena is the following

Theorem 6. *Let* $f(s) = a_0 + a_1 s + a_2 s^2$, $a_0 > 0$, $f'(1) > 1$. *Then the coefficients in* (44) *alternate as in* (45).

Proof. Differentiating $f(B(w)) = B(cw)$ n times and recognizing that $f^{(k)}(s) \equiv 0$ for $k \geqq 3$ we obtain

$$f''(B(w))\, Q_n(w) + f'(B(w))\, B^{(n)}(w) = c^n B^{(n)}(cw)$$

where $Q_n(w)$ is a linear combination of $B^{(i)}(w)\, B^{(n-i)}(w)$, $i = 0, 1, \ldots, n$, with positive coefficients. Substituting $w = 0$ and recalling $B(0) = q$, we obtain

$$b_n = B^{(n)}(0) = \frac{f''(q)\, Q_n(0)}{c^n - c} \text{ for } \quad n > 1. \tag{46}$$

But $f(q) > 0$, $c^n < c$ for $n > 1$ and $b_1 = B'(0) > 0$. We can deduce inductively from the description of Q_n that

$$(-1)^{n-1} b_n \geqq 0 \qquad n \geqq 1 \qquad \text{Q.E.D.} \tag{47}$$

Remark. The conclusion (47) fails already for certain cubic polynomials $f(s) = 2\varepsilon + (1 - 3\varepsilon)\, s + \varepsilon s^3$, $\varepsilon > 0$ if ε is small. We leave the task of checking this fact to the reader. However, see [*8*].

A general class of probability generating functions of vast importance with remarkable regularity properties are of the form

$$f(s) = K e^{\gamma s} \frac{\prod_{i=1}^{\infty} (1 + \alpha_i s)}{\prod_{i=1}^{\infty} (1 - \beta_i s)} = \sum_{k=0}^{\infty} a_k s^k, \tag{48}$$

where the parameters satisfy the conditions

$$\gamma \geqq 0, \ \alpha_i \geqq 0, \ 0 \leqq \beta_i < 1, \ \sum_{i=1}^{\infty} (\alpha_i + \beta_i) < \infty, \ K = e^{-\gamma} \frac{\prod_{i=1}^{\infty} (1 - \beta_i)}{\prod_{i=1}^{\infty} (1 + \alpha_i)}.$$

The generating function (48) generates a Pólya frequency sequence $\{a_k\}_{k=0}^{\infty}$ which exhibits the property that all minors of the matrix $\|a_{k-l}\|_{k,l}$ are non-negative (here $a_{-m} = 0$, $m > 0$). The converse is also true [*1*].

The class of probability generating functions (48) includes the Poisson, the binomial, the negative binomial and numerous other important cases. We record two key properties associated with $f(s)$ of the structure (48).

A. It is proved in [*5*] that the transition probability matrix $\|P_{ij}\|$ of the branching process induced by $f(s)$ of the form (48) is totally positive. This says that all minors of $\|P_{ij}\|$ are non-negative.

Furthermore in the case at hand $P_{0i} \equiv \delta_{i0}$ and for any given minor composed of rows of index $1 \leqq i_1 < i_2 < \cdots < i_r$ and columns of index $1 \leqq j_1 < j_2 < \cdots < j_r$

is strictly positive for n large enough, i.e.,

$$\begin{vmatrix} P^{(n)}_{i_1 j_1}, & P^{(n)}_{i_1,j_2}, \ldots, & P^{(n)}_{i_1,j_r} \\ P^{(n)}_{i_2,j_1}, & P^{(n)}_{i_2,j_2}, \ldots, & P^{(n)}_{i_2,j_r} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ P^{(n)}_{i_r,j_1}, & P^{(n)}_{i_r,j_2}, \ldots, & P^{(n)}_{i_r,j_r} \end{vmatrix} > 0 .$$

In other words there is an iterate P^n which renders a prescribed minor as indicated above strictly positive.

B. Under the conditions satisfied by P above, extending some results of KREIN and GANTMACHER [2], see also [6], we prove in [15] that if $P = \| P_{ij} \|$ is T. P. $(P_{0i} = \delta_{i0}, \ \sum_j P_{ij} \leqq 1, i \geqq 1)$ possessing simple eigenvalues $1 = \lambda_0 > \lambda_1 > \lambda_2 > \cdots > \lambda_n > \cdots$ and associated right eigenvectors

$$\varphi_0(\cdot), \ \varphi_1(\cdot), \ldots, \varphi_n(\cdot), \ldots$$

then $\varphi_0(i) = \delta_{i0}$ and $\varphi_n = (\varphi_n(0), \varphi_n(1), \varphi_n(2), \ldots)$, $n \geqq 1$ exhibits precisely $n-1$ sign changes, where $\varphi_n(0) \equiv 0$, $n \geqq 1$.

Further oscillation properties of $\{\varphi_n\}_{n=0}^{\infty}$ pertaining to interlocking properties of sign changes of the components of successive eigenvectors and zero characteristics of the extended linear interpolation of φ_n are available. We do not elaborate these results here since they will not be needed in the sequel.

With the aid of A. and B. we now prove the following important theorem.

Theorem 7. *Let $f(s)$ be a probability generating function of the form* (48) *and assume $f'(1) = m > 1$. Let $B(w) = q + \sum_{r=1}^{\infty} \psi_1(r) w^r$ be the unique function analytic about the origin and satisfying $f(B(w)) = B(cw)$, $B(0) = q$, $B'(0) = 1$, where $c = f'(q)$. Then*

$$(-1)^{n-1} \psi_1(n) > 0, \qquad n = 1, 2, 3, \ldots \tag{49}$$

Proof. We proved in (38) that the right eigenvectors of P, i.e., the eigenvectors of P^* can be calculated as follows: Let

$$B^k(w) = q^k + \sum_{r=1}^{\infty} \psi_k(r) w^r \qquad k = 0, 1, 2, \ldots \tag{50}$$

The rth eigenvector has components

$$e_r = \left(\psi_0(r), \frac{\psi_1(r)}{q}, \frac{\psi_2(r)}{q^2}, \frac{\psi_3(r)}{q^3}, \ldots\right)$$

where $\psi_0(r) = 0$, $r \geqq 1$. We have for $r \geqq 1$

$$\begin{aligned} \frac{\psi_k(r)}{q^k} &= \text{coefficient } w^r \text{ in} \left[1 + \frac{w}{q} + \sum_{n=2}^{\infty} \psi_1(n) \frac{w^n}{q}\right]^k \\ &= \text{coefficient } w^r \text{ in} \sum_{m=1}^{r} \binom{k}{m} \left[\frac{w}{q} + \sum_{n=2}^{\infty} \frac{\psi_1(n)}{q} w^n\right]^m . \end{aligned} \tag{51}$$

This is clearly a polynomial of degree r in the variable k, in which the coefficient of k^r is $1/(r!\,q^r)$. The polynomial vanishes at $k = 0$, as it should. Therefore

(52) $$\psi_k(r) = q^k \frac{Q_r(k)}{r!\,q^r}, \qquad r \geqq 0,$$

where $Q_r(k)$ is the monic polynomial in (24).

Since $Q_r(k)$ vanishes at $k = 0$ and is positive for k sufficiently large, and exhibits exactly $r - 1$ sign changes (here we use property B decisively), we infer that

$$(-1)^{r-1} Q_r(1) \geqq 0,$$

or what is the same

$$(-1)^{r-1} \psi_1(r) \geqq 0$$

which was desired to be shown.

Corollary 1. *Under the conditions of Theorem 7, the right eigenvector polynomial $Q_r(x)$, $r \geqq 1$, possesses precisely $r - 1$ simple zeros in $(0, \infty)$ and also vanishes at the origin. The zeros on $(0, \infty)$ of $Q_r(x)$ and $Q_{r+1}(x)$ strictly interlace.*

The proof of the last statement will not be given but result from our more refined knowledge concerning eigenvectors of totally positive matrices (see [*15*]). It is not difficult to provide examples where the conclusions of Corollary 1 fail.

Combining Theorems 4 and 7, and noting that $f'(0) > 0$ if $f(s)$ is of the form (48), we obtain the following.

Theorem 8. *Let $f(s)$ be a generating function of the form (48) with $m = f'(1) > 1$. Then the spectral representation*

(53) $$P_{ij}^{(n)} = \sum_{n=0}^{\infty} (c^r)^n \theta_j(r) \psi_i(r)$$

is valid for all integers $n \geqq 0$.

§ 6. Spectral Representation in the Case $m = f'(1) < 1$

We next investigate the existence of a spectral representation when $m < 1$. We assume, as before, that $a_0 = f(0) > 0$. In addition we make the essential simplifying assumption that the analytic function $f(s)$ is regular at $s = 1$. Without such a restriction continuous spectrum is present and the nature of the spectral representation appears to be quite intricate.

Since $f(s)$ is a power series with positive coefficients, regular for $|s| < 1$, and at $s = 1$, the Pringsheim theorem assures that the radius of convergence is actually greater than 1. Since $f'(1) < 1$ we have $f(s) < s$ in an interval to the right of $s = 1$. Choose $a > 1$ such that $f(a) < a$ and such that $f(s)$ is regular in some circle $|s| < a + \varepsilon$ where $\varepsilon > 0$.

We can proceed in two ways. One method is to consider P acting in the Hilbert space of sequences $\xi = \{\xi_i\}$ with

(54) $$\|\xi\|^2 = \sum_{i=0}^{\infty} |\xi_i|^2 a^i < \infty,$$

and imitate the theory of section 3. An alternative method is to reduce the present case to Theorem 5 by considering the generating function

(55) $$h(s) = \frac{1}{a} f(a\,s)$$

for which $h(1) < 1$. We outline the second method.

The unique fixed point of h on (0,1) is $q = 1/a$, and $h'(q) = f'(1) = m < 1$. Associated with h we have a function $A_h(s)$ defined by

$$A_h(s) = \lim_{n\to\infty} \frac{h_n(s) - \frac{1}{a}}{m^n} \tag{56}$$

where the convergence is uniform in every circle $|s| \leqq r < 1$ (in this case, even in a circle of radius > 1). Since $h_n(s) = \frac{1}{a} f_n(as)$ we deduce that

$$A(s) = a A_h\left(\frac{s}{a}\right) = \lim_{n\to\infty} \frac{f_n(s) - 1}{m^n} \tag{57}$$

converges uniformly in every circle $|s| \leqq r < a$. From (57) follows

$$A(f(s)) = m A(s), \quad A(1) = 0, \quad A'(1) = 1, \tag{58}$$

and (58) has only one solution regular at $s = 1$, by the same argument as before (cf. Section 3).

From (58) we obtain

$$A'(s) = \frac{f_n'(s)}{m^n} A'(f_n(s)). \tag{59}$$

But $f_n(a) \to 1$ as $n \to \infty$ if $0 \leqq s \leqq 1$, and $f_n'(s) \geqq [f'(s)]^n > 0$ if $0 < s \leqq 1$. Hence the right member of (59) is positive for large n if $0 < s \leqq 1$, and also for $s = 0$ if $f'(0) > 0$. Thus the inverse function $B(w)$ of $A(s)$, defined so that

$$B(A(s)) = s \quad \text{near} \quad s = 1$$

can be continued analytically along the segment $A(0) < w \leqq 0$, and is even regular in a neighborhood of the closed segment $A(0) \leqq w \leqq 0$ if $f'(0) > 0$. It satisfies

$$f(B(w)) = B(mw), \quad B(0) = 1, \quad B'(0) = 1.$$

We define $\psi_i(r)$, $\theta_j(r)$ by

$$A^r(s) = \sum_{j=0}^{\infty} \theta_j(r) s^j, \quad B^i(w) = \sum_{r=0}^{\infty} \psi_i(r) w^r \tag{60}$$

and then from the functional identity

$$f_n^i(s) = B^i(m^n A(s))$$

we deduce that

$$P_{ij}^n = \sum_{r=0}^{\infty} m^{rn} \psi_i(r) \theta_j(r) \tag{61a}$$

for all sufficiently large n, for $n \geqq 1$ if

$$(-1)^{r-1} \psi_1(r) \geqq 0 \quad \text{for} \quad r \geqq 1, \tag{61b}$$

and also for $n = 0$ if, in addition, $f'(0) > 0$. If $f(s)$ is of the form (48) *then* (62) *is valid.*

From the formula

$$B(w) = 1 + w + \sum_{r=2}^{\infty} \psi_1(r) w^r$$

we can show that $\psi_i(r) = Q_r(i)$ where $Q_r(i)$ is a polynomial in i of degree r, in which the coefficient of i^r is $1/r!$. Since $A(s)$ is analytic in a circle $|s| < a$ where $a > 1$, the coefficients $\theta_j(r)$ converge geometrically to zero as $j \to \infty$, r fixed.

§ 7. Spectral Representation when $f(0) = 0$

First assume $f(0) = 0$, $f'(0) > 0$. We exclude the trivial case $f(s) = s$. It then follows that

$$|f(s)| < |s| \quad \text{if} \quad 0 < |s| < 1 .$$

Only the states $i = 1, 2, 3, \ldots$ are of interest so we use a Hilbert space $\mathscr{H}$ of sequences $\xi = \{\xi_i\}_1^\infty$ with the norm

$$\|\xi\|^2 = \sum_{i=1}^{\infty} |\xi_i|^2 a^i$$

where $0 < a < 1$. The matrix $\| P_{ij} \|$ determined by $f(s)$ acts in $\mathscr{H}$ via the formula

$$(\xi P)_j = \sum_{i=1}^{\infty} \xi_i P_{ij}$$

as a completely continuous operator P. To find eigenvectors of P^* we differentiate the identity

$$\sum_{j=1}^{\infty} a^{-i} P_{ij} a^j s^j = \left[\frac{f(a s)}{a}\right]^i \tag{62}$$

and set $s = 0$. The first derivative yields

$$\sum_{j=1}^{\infty} a^{-i} P_{ij} a^j j \delta_{1,j} = c\, i\, \delta_{1,i}$$

so that $e_1 = (1, 0, 0, \ldots, 0, \ldots)$ is an eigenvector belonging to the eigenvalue $\lambda_1 = c = f'(0)$. Let

$$\begin{aligned} u_r &= \{\delta_{r,i}\}_{i=1}^{\infty}, \quad r = 1, 2, \ldots \\ &= (0, 0, \ldots, 0, 1, 0, \ldots) . \end{aligned}$$

By evaluating higher derivatives of (62) at $s = 0$ we find that P^* maps $r!\, u_r$ into a vector of the form

$$c^r r!\, u_r + \text{linear combination of } u_1, u_2, \ldots, u_{r-1} . \tag{63}$$

It follows that c^r, $r = 1, 2, \ldots$ is an eigenvalue of P^* with an eigenvector of the form u_r + linear combination of $u_1, u_2, \ldots, u_{r-1}$.

In particular c is an eigenvalue of P and we deduce that

$$A(f(s)) = c\, A(s)$$

has a non-trivial solution regular in $|s| < \sqrt{a}$. By familiar arguments we can assume

$$A(0) = 0, \quad A'(0) = 1$$

and the solution is then unique, and regular in $|s| < 1$. The inverse function $B(w)$ is regular near $w = 0$, and here $A(0) = 0$ is definitely within the circle of convergence of $B(w)$ about $w = 0$.

We define $\psi_i(r)$ and $\theta_j(r)$ by

$$B^i(w) = \sum_{r=1}^{\infty} \psi_i(r)\, w^r, \quad A^r(s) = \sum_{j=1}^{\infty} \theta_j(r)\, s^j$$

and conclude that

$$P_{ij}^n = \sum_{r=1}^{\infty} c^{nr}\, \psi_i(r)\, \theta_j(r) \tag{64}$$

where the series converges absolutely for all $i, j \geqq 1$ and $n = 0, 1, 2, \ldots$. Of course, since $\{\psi_i(r)\}_{i=1}^{\infty}$ is a multiple of (63), the series in (64) are actually finite series. Moreover, in this case the matrix P has an inverse P^{-1} and the finite series (64) for $n = -1, -2, \ldots$ determines the corresponding elements of the powers of P^{-1}.

If we assume $f(0) = f'(0) = 0$ then $f(s)$ is of the form

$$f(s) = a_k s^k + \cdots, \quad a_k \neq 0,\ k > 1,$$

and then

$$|f_n(s)| \leqq |s|^{k^n}, \qquad |s| \leqq 1. \tag{*}$$

On the basis of this inequality one shows that the operator P has spectral radius zero. In fact, a simple estimate using (*) and Schwarz's lemma pertaining to analytic functions in the unit circle vanishing at the origin leads to the inequality

$$\| P^n \| \leqq C\, [\sqrt{a}]^{k^n}.$$

Clearly, for $k > 1$, $\lim_{n\to\infty} \| P^n \|^{1/n} = 0$ and thus one cannot expect to find a spectral representation.

§ 8. Examples

It is difficult, inherently so, to obtain explicit expressions for the basic functions $A(s)$ and $B(w)$, in concrete examples. The task of determining $A(s)$ is almost synonomous with that of finding in closed form the iterates $f_n(s)$. We cannot do this even for the simple important cases of $f(s) = a_0 + a_1 s + a_2 s^2$ and $f(s) = e^{\lambda(s-1)}$.

We now treat three examples for which $f_n(s)$ can be displayed in closed form.

Example 1. $f(s) = 1 - \gamma(1-s)^{\beta}$, $0 < \beta < 1$, $0 < \gamma < 1$.
This probability generating function for $\beta = 1/2$ arises in connection with fluctuation theory of coin tossing experiments.

In this example $f(0) = 1 - \gamma$, $f(1) = 1$, $f'(1) = \infty$. The fixed point $q = f(q)$ $(0 < q < 1)$ is

$$q = 1 - \gamma^{1/(1-\beta)} \quad \text{and} \quad c = f'(q) = \beta.$$

By a straightforward calculation

$$\begin{aligned} f_n(s) &= 1 - \gamma^{1+\beta+\cdots+\beta^{n-1}}(1-s)^{\beta^n} \\ &= 1 - \gamma^{(1-\beta^n)/(1-\beta)}(1-s)^{\beta^n} = 1 - (1-q)\left[\frac{1-s}{1-q}\right]^{\beta^n}. \end{aligned}$$

It follows that

$$\lim_{n\to\infty} \frac{f_n(s) - q}{\beta^n} = -(1-q)\log\frac{(1-s)}{1-q} = A(s). \tag{65}$$

Inverting, we obtain

$$B(w) = 1 - (1-q)\, e^{-w/(1-q)}.$$

In the expansion of $B(w)$ the coefficients after the first alternate and therefore the spectral representation (40) for the branching process induced by $f(s) = 1 - \gamma(1-s)^\beta$ holds for all interates $n = 0, 1, 2, \ldots$ as asserted in Theorem 4. Expanding

$$\frac{1}{1 - B(w)\,u} = \sum_{n=0}^{\infty} B^n(w)\, u^n = \frac{1}{1 - u + (1-q)\, u\, e^{-w/(1-q)}}$$

and collecting coefficients leads to the expression

$$\begin{aligned}\psi_n(r) = q^n\, Q_r(n) &= \frac{(-1)^r}{(1-q)^r\, r!} \sum_{k=0}^{n} k^r \binom{n}{k} (-1)^k (1-q)^k \\ &= \frac{(-1)^r}{(1-q)^r\, r!} \left(x \frac{d}{dx}\right)^r \; [1 - (1-q)\, x]^n \big|_{x=1}.\end{aligned}$$

Example 2. $f(s) = \alpha/(1 - \beta s)$ $(\alpha + \beta = 1,\ 1 > \beta > 1/2)$.
This is an example of the class (48). It is readily verified that $q = f(q) = \alpha/\beta$ and $f'(q) = \alpha/\beta$. Applying the elementary theory of linear fractional mappings we get

$$f_n(s) = \frac{q - q^n \left(\dfrac{s-q}{s-1}\right)}{1 - q^n \left(\dfrac{s-q}{s-1}\right)} = q + q^n \left[(q-1)\frac{(s-q)}{(s-1)}\right] + O(q^{2n}). \tag{66}$$

Comparing with (34) we see that

$$A(s) = \frac{(1-q)(s-q)}{1-s} \quad \text{and} \quad B(w) = \frac{w + q(1-q)}{w + (1-q)}. \tag{67}$$

In order to identify the coefficients in the expansion of $B^r(w)$, we introduce the Meixner Polynomials $M_n(x)$ defined by the generating function

$$\sum_{n=0}^{\infty} M_n(x; b, a) \frac{[b]_n\, s^n}{n!} = \frac{\left(1 - \dfrac{s}{a}\right)^x}{(1-s)^{b+x}}, \quad [b]_n = b(b+1)\ldots(b+n-1). \tag{68}$$

Here a and b are parameters subject to the restrictions $0 < a < 1$, $b > 0$. Now

$$\begin{aligned}\sum_{i,r=0}^{\infty} \psi_i(r)\, w^r u^i = \sum_{i=0}^{\infty} B^i(w)\, u^i &= \frac{w + 1 - q}{w(1-u) + (1-q)(1-qu)} \\ &= \frac{1}{1-qu}\left(1 + \frac{w}{1-q}\right) \frac{1}{1 + \left(\dfrac{1-u}{1-qu}\right)\dfrac{w}{1-q}}.\end{aligned}$$

In the expansion of this the coefficient of w^r, $r \geqq 1$, is

$$\left(\frac{-1}{1-q}\right)^{r-1} \frac{(1-u)^{r-1}\, u}{(1-qu)^{r+1}}, \qquad r = 1, 2, \ldots \tag{69}$$

In (68) we replace s by qu, multiply by u, and obtain the identity

$$\begin{aligned}&\left(\frac{-1}{1-q}\right)^{r-1} \sum_{i=0}^{\infty} M_i(r-1; b, a) \frac{[b]_i}{i!} q^i u^{i+1} \\ &= \left(\frac{-1}{1-q}\right)^{r-1} \frac{\left(1 - \dfrac{qu}{a}\right)^{r-1} u}{(1-qu)^{b+r-1}}\end{aligned} \tag{70}$$

When $a = q$ and $b = 2$ the right member of (70) is the same as (69). Hence we conclude that for $r = 1, 2, \ldots$

$$\psi_i(r) = \begin{cases} \left(\frac{-1}{1-q}\right)^{r-1} i\, M_{i-1}(r-1; 2, q)\, q^{i-1}, & i = 1, 2, \ldots \\ 0 \qquad , & i = 0. \end{cases}$$

For $r = 0$, (69) must be replaced by $1/(1 - qu)$. It follows that

$$\psi_i(0) = q^i, \qquad i = 0, 1, 2, \ldots$$

The coefficients $\theta_j(r)$ can be found by a similar calculation using the explicit formula for $A(s)$. We have

$$\sum_{r,j=0}^{\infty} \theta_j(r)\, s^j u^r = \frac{1}{1 - uA(s)} \tag{71}$$

$$= \frac{1}{1 + uq(1-q)} \cdot \frac{1-s}{1 - \frac{1 + u(1-q)}{1 + uq(1-q)} s}$$

from which follows

$$\sum_{r=0}^{\infty} \theta_j(r)\, u^r = \begin{cases} \frac{1}{1 + uq(1-q)}, & j = 0, \\ (1-q)^2 \frac{[1 + u(1-q)]^{j-1} u}{[1 + uq(1-q)]^{j+1}}, & j = 1, 2, \ldots \end{cases} \tag{72}$$

We conclude that

$$\theta_0(r) = (-1)^r q^r (1-q)^r$$

and for $j = 1, 2, \ldots$

$$\theta_j(r) = \begin{cases} 0 & r = 0 \\ (-1)^{r-1} q^{r-1} (1-q)^{r+1} r\, M_{r-1}(j-1; 2, q) & r \geqq 1. \end{cases}$$

Example 3. Let $g(s) = \alpha/(1 - \beta s)$, $h(s) = s^{k_0}$ (k_0 a fixed positive integer) and $f(s) = h^{-1}(g(h(s)))$. It is trivial to verify that $f(s)$ is a probability generating function. The iterates of f can be simply expressed in terms of the iterates of g, viz.

$$f_n(s) = h^{-1}(g_n(h(s))).$$

The iterates of g_n were displayed in explicit terms in (66). In accordance with Theorem 3, let $A(s)$ denote the unique analytic function satisfying

$$A(g(s)) = c\,A(s), \quad A(q) = 0, \; A'(q) = 1 \tag{73}$$

where

$$c = g'(q), \quad g(q) = q \quad 0 < q < 1.$$

We know $A(s)$ in the case at hand. It is formula (67).

Let $A^*(s)$ be the corresponding function associated with $f(s)$, i. e., $A^*(s)$ is analytic in $|s| < 1$ and satisfies

$$A^*(f(s)) = c^* A^*(s), \qquad A^*(q^*) = 0, \frac{dA^*(q^*)}{ds} = 1 \tag{74}$$

where

$$q^* = f(q^*) \qquad 0 < q^* < 1 \qquad c^* = f'(q^*).$$

A direct check shows that $q^* = h^{-1}(q)$ and $c^* = c$. It is easy to relate $A^*(s)$ and $A(s)$. In fact, we claim that

$$A^*(s) = \frac{A(h(s))}{h'(q^*)}. \tag{75}$$

This is proved by verifying that the right side in (75) satisfies (74). Indeed, referring to (73) combined with the fact that $c = c^*$, we get

$$\frac{A(h(f(s)))}{h'(q^*)} = \frac{A(g(h(s)))}{h'(q^*)} = \frac{c\,A(h(s))}{h'(q^*)} = c^* \frac{A(h(s))}{h'(q^*)}.$$

The proof is finished. Now for $h(s) = s^{k_0}$

$$A^{*r}(s) = A^r(h(s))\,d_r = \sum_{n=0}^{\infty} \theta_n(r)\,d_r\,s^{k_0 n} = \sum_{n=0}^{\infty} \theta_n^*(r)\,s^n \qquad (d_r = (k_0\,q^{(k_0-1)/k_0})^{-r})$$

where $\theta_n(r)$ are known explicitly from (72). We can read off the right eigenvectors of the branching process induced by $f(s)$. Thus

$$\theta_n^*(r) = \begin{cases} 0 & \text{if} \quad n \neq k_0 m;\ m = 1, 2, \ldots \\ (-1)^{r-1} c_r M_{r-1}(m-1; 2, q) & n = k_0 m. \end{cases}$$

where $c_r = d_r\,q^{r-1}(1-q)^{r+1}\,r$. Inverting the biorthogonality relations we get, apart from a constant factor f_r

$$\psi_n^*(r) = (-1)^r f_r\,m\,M_{r-1}(m-1; 2, q)\,q^m \qquad n = k_0 m, \quad m = 1, 2, \ldots.$$

Since $(q^*)^{k_0} = q$ and $\psi_n^*(r)/q^n$ is a polynomial of degree r, the values of $\psi_n^*(r)$ for $n \neq k_0 m$ $(m = 1, 2, \ldots)$ are determined by interpolation.

Remark. The methods of Example 2 can be adapted to treat the example $f(s) = [\gamma(\gamma_1 + \beta_1 s)]/(1 - \beta s)$ where $\gamma + \beta = \gamma_1 + \beta_1 = 1$, $0 < \gamma < 1$, $0 \leqq \gamma_1 \leqq 1$.

Example 4. A simple example of the spectral representation for $m < 1$ has $f(s) = \alpha + \beta s$ $(0 < \alpha < 1, \alpha + \beta = 1)$. In a trivial manner we find $A(s) = s - 1$, $B(w) = w + 1$. Then

$$P_{ij}^n = \sum_{r=0}^{\infty} \beta^{nr}\,\theta_j(r)\,\psi_i(r) = \sum_{r=0}^{i} \beta^{nr} \binom{r}{j} (-1)^j \binom{i}{r} (-1)^r.$$

§ 9. Branching Processes with Immigration

Let $f(s)$ be a probability generating function corresponding to the progeny distribution per individual per generation. Let $h(s) = \sum_{k=0}^{\infty} b_k s^k$, $b_k \geqq 0$, $h(1) = 1$ be another probability generating function whose coefficients are the probabilities of the number of new individuals immigrating into the system. Newly arriving individuals undergo growth following the laws of the branching process induced by the generating function $f(s)$.

Then if there are i individuals present in one generation, the probability Q_{ij} that there will be j individuals in the next generation is determined by the generating function

$$\sum_{j=0}^{\infty} Q_{ij}\,s^j = [f(s)]^i\,h(s). \tag{76}$$

The Markov chain with one step transition matrix Q_{ij} is referred to as a *branching process with immigration*. The obvious interpretation are attributed to the factors $f(s)$ and $h(s)$.

Our objective in this section is to develop a spectral representation for the operator $Q = \|Q_{ij}\|$. Note that the nth power of Q has a generating function considerably more complicated than before. Specifically

$$\sum_{j=0}^{\infty} Q_{ij}^{(n)} s^j = [f_n(s)]^i h(f_{n-1}(s))\, h(f_{n-2}(s)) \ldots h(s). \tag{77}$$

We desire to evaluate the coefficient of s^j in an appropriate form.

Case I. $m = f'(1) > 1$. We assume $f(0) > 0$, $f(1) = 1$. Let q be the unique root of $q = f(q)$ in $0 < q < 1$. As earlier we introduce the Hilbert space $\mathscr{H}$ composed of all sequences $\xi = (\xi_0, \xi_1, \ldots)$ of finite norm $\|\xi\|$ where

$$\|\xi\|^2 = \sum_{i=0}^{\infty} |\xi_i|^2 q^i.$$

In $\mathscr{H}$ a linear operator Q is determined by the formula

$$(\xi Q)_j = \sum_{i=0}^{\infty} \xi_i Q_{ij}.$$

We can prove as in Theorem 1 that Q is a completely continuous transformation.

Eigenvalues and eigenvectors of Q^* and Q.

By evaluating derivatives of the identity

$$\sum_{j=0}^{\infty} q^{-i} Q_{ij} q^j s^j = \left[\frac{f(qs)}{q}\right]^i h(s)$$

at $s = q$, we find that $\lambda_r = c^r \gamma$, $r = 0, 1, 2, \ldots$ are eigenvalues of the adjoint operator, Q^* where

$$c = f'(q), \quad \gamma = h(q) > 0.$$

The eigenvector e_r corresponding to λ_r is, as before, related to a polynomial $T_r(x)$ of exact degree r, so that

$$e_r = \{T_r(i)\}_{i=0}^{\infty}. \tag{78}$$

These polynomials can be constructed recursively.

We infer that each number $c^r \gamma$, $r = 0, 1, \ldots$ is an eigenvalue of Q so that the equation

$$\pi_r Q = c^r \gamma \pi_r \tag{79}$$

has a non-trivial solution $\pi_r \in \mathscr{H}$. If λ is an eigenvalue of Q with eigenvector π then the generating function $\pi(s)$ is regular in $|s| < \sqrt{q}$ and satisfies

$$\pi(f(s))\, h(s) = \lambda \pi(s). \tag{80}$$

At $s = q$, $\pi(s)$ has a zero of some finite order $k \geqq 0$, $\pi(s) = u_k (s - q)^k + \cdots$ where $u_k \neq 0$. By differentiating (80) k times with respect to s and setting $s = q$ we get

$$k!\, u_k c^k \gamma = \lambda k!\, u_k$$

so $\lambda = c^k \gamma$. Further differentiation of (80) shows (compare with the argument following (29)) that a solution of (80) which is regular at $s = q$ is unique to within a multiplicative constant. Thus the eigenvalues of Q are $\lambda_r = c^r \gamma$, $r = 0, 1, 2, \ldots$, each eigenspace is one-dimensional, and the generating function $\pi_r(s)$ of an eigenvector associated with λ_r has at $s = q$ a zero of order r exactly.

In particular, it follows that

(81) $$\pi_0(f(s))\, h(s) = \gamma\, \pi_0(s)\,, \quad \pi_0(q) = 1$$

has a unique solution $\pi_0(s)$ regular in $|s| < \sqrt{q}$. By analytic continuation using (81) we see that $\pi_0(s)$ is regular in $|s| < 1$. From (81) we deduce

$$\pi_0(s) = \frac{h(s)}{\gamma} \cdot \frac{h(f(s))}{\gamma} \cdot \cdots \cdot \frac{h(f_{n-1}(s))}{\gamma}\, \pi_0(f_n(s))\,.$$

Since $f_n(s) \to q$, uniformly for $|s| \leqq r < 1$, and $\pi_0(q) = 1$, we have

(82) $$\pi_0(s) = \lim_{n\to\infty} \frac{1}{\gamma^n} \prod_{k=0}^{n-1} h(f_k(s))$$

where the convergence is uniform in every circle $|s| \leqq r < 1$.

The solution of

(83) $$\pi_r(f(s))\, h(s) = c^r \gamma\, \pi_r(s)\,, \quad \pi_r(q) = \pi_r'(q) = \cdots = \pi_r^{(r-1)}(q) = 0\,,$$
$$\pi_r^{(r)}(q) = r!$$

is unique. We can easily verify that if $A(s)$ is the solution of (23) then $[A(s)]^r\, \pi_0(s)$ is a solution of (83). Therefore

(84) $$\pi_r(s) = [A(s)]^r\, \pi_0(s)\,, \qquad r = 0, 1, 2, \ldots$$

We again employ the inverse function $B(w)$ of $A(s)$, and recall that

(85) $$f_n(s) = B(c^n A(s))\,.$$

From (81) we obtain

(86) $$\pi_0(f_n(s))\, h(f_{n-1}(s))\, h(f_{n-2}(s)) \ldots h(s) = \gamma^n\, \pi_0(s)\,.$$

By use of (85) and (86) it is possible to express the generating function (77) in terms of the functions $A(s)$, $B(w)$ and $\pi_0(s)$. Since $\pi_0(q) = 1$ there is a neighborhood of $s = q$ in which π_0 does not vanish and $1/\pi_0(f_n(s))$ is regular at $s = q$. Hence

(87) $$\sum_{j=0}^{\infty} Q_{ij}^{(n)} s^j = B^i(c^n A(s))\, \frac{\gamma^n \pi_0(s)}{\pi_0(B(c^n A(s)))}$$

where the right side is regular near $s = q$.

Since $B(0) = q$ and $1/\pi_0(s)$ is regular at $s = q$, the function $B^i(w)/\pi_0(B(w))$ is regular at $w = 0$ and has an expansion

(88) $$\frac{B^i(w)}{\pi_0(B(w))} = \sum_{r=0}^{\infty} u_i(r)\, w^r$$

with a positive radius of convergence ϱ_0. Hence

(89) $$\sum_{j=0}^{\infty} Q_{ij}^{(n)} s^j = \sum_{r=0}^{\infty} (c^r \gamma)^n\, u_i(r)\, [A(s)]^r\, \pi_0(s)$$

where the series on the right converges absolutely in a neighborhood of $s = q$. If $c^n |A(0)| < \varrho$, which is certainly the case for all sufficiently large n, we conclude that (89) converges absolutely in a neighborhood of the segment $A(0) \leqq s \leqq q$. In this event we can replace $A^r(s)\, \pi_0(s) = \pi_r(s)$ by its series expansion about $s = 0$,

$$A^r(s)\, \pi_0(s) = \pi_r(s) = \sum_{j=0}^{\infty} v_j(r)\, s^j \tag{90}$$

and then invert the order of summation. This leads to the formula

$$Q_{ij}^{(n)} = \sum_{r=0}^{\infty} (c^r \gamma)^n\, u_i(r)\, v_j(r) \tag{91}$$

where the series is absolutely convergent for all i, j and all n such that $c^n |A(0)| < \varrho$.

The coefficients $u_i(r)$ can be related to the polynomials $T_r(i)$ in (78). There is a series expansion

$$\frac{1}{\pi_0(B(w))} = \sum_{p=0}^{\infty} U_p\, w^p . \tag{92}$$

By substituting (92) and (37) in (88) and then appealing to (52) we obtain

$$u_i(r) = q^i \sum_{l=0}^{r} \frac{U_{r-l}}{l!\, q^l}\, Q_l(i) \tag{93}$$

where $Q_l(i)$ is a monic polynomial in the variable i of degree l. It follows that to within a multiplicative constant we have

$$T_r(i) = \sum_{l=0}^{r} \frac{U_{r-l}}{l!\, q^l}\, Q_l(i) \tag{94}$$

and, since $U_0 = 1$, the coefficient of i^r in (94) is $1/(r!\, q^r)$. We summarize the above results as two theorems.

Theorem 9. (i) *If $m = f'(1) > 1$ and $f(0) > 0$ the linear operator Q is completely continuous and its eigenvalues are*

$$\lambda_r = c^r \gamma\,, \qquad r = 0, 1, 2, \ldots$$

where $c = f'(q)$, $\gamma = h(q)$. Each eigenspace of Q is one-dimensional.

(ii) *The functional equation*

$$\pi_0(f(s))\, h(s) = \gamma\, \pi_0(s)\,, \quad \pi_0(q) = 1$$

has a unique solution $\pi_0(s)$ regular at $s = q$. The solution is regular for $|s| < 1$, and is given by (82) „explicitly".

(iii) *The eigenvector e_r of Q^* belonging to the eigenvalue λ_r is a polynomial of degree r, that is*

$$e_r = \{T_r(i)\}_{i=0}^{\infty}$$

where T_r is given by (94). *The corresponding eigenvector π_r of Q has a generating function $\pi_r(s)$ given by* (84) *in which $A(s)$ is the solution of* (23).

Theorem 10. *If $m = f'(1) > 1$ and $f(0) > 0$ then the transition probability matrix $Q_{ij}^{(n)}$ has the spectral representation*

$$Q_{ij}^{(n)} = \gamma^n \sum_{r=0}^{\infty} c^{nr}\, u_i(r)\, v_j(r) \tag{95}$$

where $u_i(r)$, $v_j(r)$ *are defined by* (88) *and* (93). *The series converge absolutely for all* i, j *and all sufficiently large* n.

We next consider conditions which will ensure that (95) converges for all $n \geqq 0$. This will be the case if $|A(0)| < \varrho$ where ϱ is the radius of convergence of the series (88). The following theorem will be proved.

Theorem 11. *If we assume, in addition to the conditions of Theorem* 10, *that* $f(s)$ *and* $h(s)$ *are functions of the form* (48), *then the representation* (95) *holds for all* $n \geqq 0$, *the series being absolutely convergent.*

It is sufficient to show that the radius of convergence of the series (88) exceeds $-A(0)$. We first prove a lemma which does not depend on the special hypothesis that f and h are of the form (48).

Lemma 1. *The functions* $[\pi_0(B(w))]^{-1}$ *and* $[h(B(w))]^{-1}$ *have the same radius of convergence about* $w = 0$.

Proof. From (81) and $f(B(w)) = B(cw)$ we obtain the functional equation

$$[\gamma\,\pi_0(B(w))]^{-1} = [h(B(w))]^{-1}\,[\pi_0(B(cw))]^{-1}.$$

If ϱ is the radius of convergence of the left member then the second factor on the right has radius of convergence $\varrho/c > \varrho$, and the result follows.

The next lemma does not depend on the special hypothesis about h.

Lemma 2. *If* $f(x)$ *is of the form* (48) *then*

$$|B(w) - q| \leqq q \quad \text{for} \quad |w| \leqq -A(0). \tag{96}$$

Proof. Since $f(s)$ is the form (48), $B(w)$ has radius of convergence greater than $-A(0)$ and

$$B(w) - q = \sum_{r=1}^{\infty} \psi_1(r)\, w^r$$

where $(-1)^{r-1}\,\psi_1(r) \geqq 0$ by Theorem 7. Hence in the circle $|w| \leqq -A(0)$, the function $B(w) - q$ has its maximum modulus at $w = A(0)$, and since $B(A(0)) = 0$ the result follows.

Proof of Theorem 11. When $f(s)$ is of the form (48), the radius of convergence of $B^i(w)$ exceeds $-A(0)$ by the results of Section 5. Hence, by Lemma 1 it is sufficient to show that the radius of convergence of $[h(B(w))]^{-1}$ exceeds $-A(0)$. Now since $h(s)$ is of the form (48), $1/h(s)$ is meromorphic, regular at the origin, and all its poles lie on the negative axis. By (96) the range of $B(w)$ in the circle $|w| \leqq A(0)$ lies within a disc entirely contained in the domain of regularity of $1/h(s)$. Hence the radius of convergence of $[h(B(w))]^{-1}$ must exceed $-A(0)$.

Using the same lemmas we can prove the following result.

Theorem 12. *If in addition to the hypotheses of Theorem* 10 *we assume* $f(s)$ *is of the form* (48) *and* $h(s) = s^m h_1(s)$ *where* $m \geqq 1$ *is an integer and* $h_1(s)$ *is of the form* (48) *then the representation* (95) *holds for all* $n \geqq 1$, *the series being absolutely convergent. (Note* $n \geqq 1$ *rather than* $n \geqq 0$.*)*

Case II. We assume $m = f'(1) < 1$, $f(0) > 0$, and $f(s)$ and $h(s)$ analytic in the neighborhood of 1.

We merely record the results since the techniques paraphrase the previous case.

Theorem 13. *Under the condition of Case II the eigenvalues of Q are $1, m, m^2, \ldots$ The corresponding left eigenvectors for m^r has a generating function $\pi_r(s) = \pi_0(s)$ $[A(s)]^r$ which satisfies*

$$\pi_r(f(s))\, h(s) = m^r \pi_r(s) \qquad r = 0, 1, 2, \ldots; \quad \pi_0(1) = 1\,.$$

The generating function $\pi_0(s)$ of the stationary distribution of the process can be explicitly calculated from

$$\pi_0(s) = \lim_{k\to\infty} \prod_{i=0}^{k} h(f_i(s))\,. \tag{97}$$

Furthermore

$$\sum_{j=0}^{\infty} Q_{ij}^{(n)} s^j = \pi_0(s) \frac{B^i(m^n A(s))}{\pi_0(B(m^n A(s)))}$$

The spectral representation

$$Q_{ij}^{(n)} = \sum_{n=0}^{\infty} m^{nr}\, U_i(r)\, V_j(r) \tag{98}$$

is valid for n large where

$$\pi_r(s) = \sum_{i=0}^{\infty} U_i(r)\, s^i \quad \text{and} \quad \frac{B^j(w)}{\pi_0(B(w))} = \sum_{r=0}^{\infty} V_j(r)\, w^r\,.$$

If $f(s)$ and $h(s)$ generate Pólya frequency sequences, i.e., they are of the form (48) *then* (98) *holds for all integer $n \geqq 0$.*

We close this section exhibiting a few simple examples in which $\pi_0(s)$ is calculated.

Example 1. Let $f(s) = (1 + s)/2$, $g(s) = s$ then

$$\pi_0(s) = s \prod_{n=1}^{\infty} \left(1 - \frac{(1-s)}{2^n}\right).$$

This is an elliptic function. As pointed out before (cf. Example 4 of Section 8), $A(s) = s - 1$ and $B(w) = w + 1$.

Example 2. Let $f(s) = (1 + s)/2, g(s) = e^{s-1}$. Then evaluating (82) gives

$$\pi_0(s) = e^{s-1}\,.$$

Example 3. Let $f(s) = \alpha/(1 - \beta s), \alpha + \beta = 1, (1 > \beta > 1/2)$ and $g(s) = e^{s-1}$. Then

$$\pi_0(s) = \exp\left[\sum_{n=1}^{\infty} f_n(s) - 1\right] = \exp\left[-(1-q)\sum_{n=1}^{\infty} \frac{1}{1 - q^n (s-q)/(q-1)}\right]$$

where $q = \alpha/\beta$. This function can be expressed as a contour integral.

In general it is inherently difficult to express $\pi_0(s)$ in terms of elementary functions.

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