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Stochastic Integrals on General Topological Measurable Spaces

Huang Zhiyuan

Mathematics Department, Wuhan University, Wuchang, Hubei, The People's Republic of China

Summary. A general theory of stochastic integral in the abstract topological measurable space is established. The martingale measure is defined as a random set function having some martingale property. All square integrable martingale measures constitute a Hilbert space M^2. For each $\mu \in M^2$, a real valued measure $\langle \mu \rangle$ on the predictable σ-algebra $\mathscr{P}$ is constructed. The stochastic integral of a random function $\mathfrak{h} \in L^2(\langle \mu \rangle)$ with respect to μ is defined and investigated by means of Riesz's theorem and the theory of projections. The stochastic integral operator I_μ is an isometry from $L^2(\langle \mu \rangle)$ to a stable subspace of M^2, its inverse is defined as a random Radon-Nikodym derivative. Some basic formulas in stochastic calculus are obtained. The results are extended to the cases of local martingale and semimartingale measures as well.

The theory of stochastic integrals plays a central role in the field of random analysis. Up to now many kinds of stochastic integrals have been defined. The earliest one, defined by Wiener [15], was only based on the orthogonality of increments of Brownian motions. It could be extended to the case of orthogonal random measure, but the integrands must be restricted to deterministic ones. The pioneering work of K. Itô [9] defined stochastic integrals of random integrands with respect to Brownian motions of which the martingale property had been used. By virtue of Doob-Meyer decomposition theorem for supermartingales, Kunita and Watanabe [11] extended the Itô integral to the case of square integrable martingales. In the last decade, Meyer and Doléans-Dade as well as other authors made a full development in the theory of stochastic integrals with respect to local martingales and semimartingales (cf. [3, 10] and [18]). It was shown (cf. [2]) that semimartingales would be the most suitable integrators of stochastic integrals satisfying the requirements of linearity, continuity and including the usual Stieltjes integrals as special cases.

On the other hand, Skorohod [14] investigated stochastic integrals with respect to Poisson processes, or more generally, to random measures or point processes (also cf. [8]).

Recently, Métivier and Pellaumail [12, 13] considered the case in which both integrands and integrators can take values in Hilbert or Banach spaces. Wong and Zakai [16, 17], Cairoli and Walsh [1] paid attention to those processes with multi-dimensional parameter sets especially stochastic integrals in the plane. In the latter cases, since lack of linearly ordered parameter sets, one must consider martingales with partially ordered parameters. In a recent paper [7], we defined martingale measures and stochastic integrals on separable complete metric spaces. Instead of the Doob-Meyer decomposition theorem, we constructed what we called Doléans measures on the predictable σ-algebra and defined stochastic integrals as linear operators on some Hilbert spaces. The advantages of this approach are that (1) it permits partially ordered parameter sets and, therefore, a greater flexibility and generality than the usual one, and (2) it enables us to reveal the essential feature of a stochastic integral and to simplify the proofs enormously.

In this paper, we try to establish a unified theory of stochastic integrals on an abstract topological measurable space. It will include the usual stochastic integrals with one-dimensional or multi-dimensional parameter sets (with respect to strong martingales) and those with respect to random measures or point processes. The Lebesgue integral and Wiener integral will also be its special cases.

I. Assumptions and Examples

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space, U be a topological space with its Borel σ-algebra $\mathscr{B}$, $\mathscr{C}$ be a sublattice of $\mathscr{B}$ such that $U \in \mathscr{C}$ and $\mathscr{B} = \sigma(\mathscr{C})$. The class $\mathscr{S}$ of all sets of the form $A - B$, where $A, B \in \mathscr{C}$ and $B \subset A$, is a semi-algebra; and the class $\mathscr{A}$ which consists of all finite disjoint unions of sets in $\mathscr{S}$ is an algebra. Clearly, $\mathscr{B} = \sigma(\mathscr{S}) = \sigma(\mathscr{A})$.

Assume that (K): for every $A \in \mathscr{A}$, there exists a sequence of sets $\{A_n\}$ in $\mathscr{A}$ and a sequence of compact sets $\{K_n\}$ such that $A_n \subset K_n \subset A$ for every n and that $A_n \uparrow A$.

Also assume that for every $A \in \mathscr{A}$ a set $t(A) \in \mathscr{C}$ has been assigned such that

(i) $t(A) \cap A = \emptyset$;
(ii) $C \in \mathscr{C},\ C \cap A \neq \emptyset \Rightarrow t(A) \subset C$;
(iii) $A, B \in \mathscr{A},\ A \subset B \Rightarrow t(B) \subset t(A)$.

Let $\{\mathscr{F}_C, C \in \mathscr{C}\}$ be a family of sub-σ-algebras of $\mathscr{F}$ satisfying following conditions:

(F.1) $\mathscr{F}_\emptyset$ contains all $\mathbb{P}$-null sets;
(F.2) $C_1 \subset C_2,\ C_1,\ C_2 \in \mathscr{C} \Rightarrow \mathscr{F}_{C_1} \subset \mathscr{F}_{C_2}$;
(F.3) $C_n \downarrow C,\ \{C_n\} \subset \mathscr{C},\ C \in \mathscr{C} \Rightarrow \mathscr{F}_{C_n} \downarrow \mathscr{F}_C$.

Definition. A square integrable martingale measure is a random set function $\mu = \mu(A, \omega)$ defined on $\mathscr{A} \times \Omega$ with the following properties:

(M.1) $A \in \mathscr{A},\ C \in \mathscr{C},\ A \subset C \Rightarrow \mu(A, \cdot)$ is $\mathscr{F}_C$-measurable;
(M.2) $A \in \mathscr{A},\ C \in \mathscr{C},\ A \cap C = \emptyset \Rightarrow \mathbb{E}(\mu(A) \mid \mathscr{F}_C) = 0$ a.s.;

(M.3)[1] $A, B \in \mathscr{A}$, $A \cap B = \emptyset \Rightarrow \mu(A+B) = \mu(A) + \mu(B)$ a.s.;

(M.4) $\mathbb{E}\mu^2(U) < \infty$;

(M.5) $\{A_n\} \subset \mathscr{A}$, $A_n \downarrow A \in \mathscr{A} \Rightarrow \underset{n\to\infty}{\text{l.i.m.}}\, \mu(A_n) = \mu(A)$ (that is, convergence in $L^2(\Omega, \mathscr{F}, \mathbb{P})$).

The totality of all square integrable martingale measures[2] will be denoted by M^2. To illustrate its intuitive meaning, we consider some examples.

Example 1 (Square integrable martingale in usual sense). Let $U = [0, \infty)$, $\mathscr{C} = \{\emptyset, U, [0, t]: 0 \leqq t < \infty\}$, and $\{X_t, \mathscr{F}_t, t \geqq 0\}$ be a square integrable martingale with right continuous paths. If we take

$$\mathscr{F}_{[0,t]} = \mathscr{F}_t \quad \text{for } t \geqq 0$$

and

$$\mu((s, t]) = X_t - X_s \quad \text{for } t \geqq s \geqq 0,$$

then the random set function μ can be extended to $\mathscr{A}$ preserving additivity. By virtue of the right continuous property and the dominated convergence theorem, we can verify (M.5). Therefore, $\mu \in M^2$.

Example 2 (Square integrable martingale in the plane). Let $U = [0, \infty) \times [0, \infty)$, $\mathscr{C}_0 = \{\emptyset, [0, s] \times [0, t]: 0 \leqq s,\ t \leqq \infty\}$ and $\mathscr{C}$ consists of all finite unions of those sets in $\mathscr{C}_0$. Let $(X_{s,t}, \mathscr{F}_{s,t}, s, t \geqq 0)$ be a square integrable strong martingale in the sense of Wong and Zakai (cf. [17]). Take

$$\mathscr{F}_{[0,s] \times [0,t]} = \mathscr{F}_{s,t} \quad \text{for } 0 \leqq s,\ t \leqq \infty,$$

and if $C = \bigcup_{i=1}^{n} C_i$ where $\{C_i\} \subset \mathscr{C}_0$, we take $\mathscr{F}_C = \bigvee_{i=1}^{n} \mathscr{F}_{C_i}$. For the set $(s_1, s_2] \times (t_1, t_2]$ in $\mathscr{S}$, take

$$\mu((s_1, s_2] \times (t_1, t_2]) = X_{s_2, t_2} - X_{s_2, t_1} - X_{s_1, t_2} + X_{s_1, t_1}.$$

Consequently, we can extend this random set function to the algebra $\mathscr{A}$ in an obvious way and prove that $\mu \in M^2$.

II. The Hilbert Space M^2 and Projections

In this section we will investigate some major properties of square integrable martingale measures.

Theorem 1. *The linear space M^2 with the inner product defined by*

$$(\mu, \nu)_{M^2} = \mathbb{E}\mu(U)\,\nu(U) \quad \mu, \nu \in M^2 \tag{1}$$

is a Hilbert space.

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[1] If the family $\{\mathscr{F}_C, C \in \mathscr{C}\}$ satisfies the condition that $\mathscr{F}_{C_1}$ and $\mathscr{F}_{C_2}$ are conditionally independent given $\mathscr{F}_{C_1 C_2}$ and that $\mathscr{F}_{C_1 \cup C_2} = \mathscr{F}_{C_1} \vee \mathscr{F}_{C_2}$ for all $C_1, C_2 \in \mathscr{C}$, then the property (M.3) can be proved directly.

[2] If almost every sample function of μ coincides with that of ν, then μ and ν are said to be indistinguishable. More precisely, M^2 is the totality of all equivalence classes of indistinguishable square integrable martingale measures.

Proof. Since the proof is similar to that in [7], we only give a sketch here. The map

$$\varphi: M^2 \to L^2(\Omega, \mathscr{F}_U, \mathbb{P}),$$

defined by $\varphi(\mu)=\mu(U)$ for every $\mu \in M^2$, is a linear injection preserving inner product. Suppose that $\{\mu_n\} \subset M^2$ and $\underset{n\to\infty}{\text{l.i.m.}}\, \mu_n(U)=\xi$. If we take

$$\mu(C)=\mathbb{E}(\xi \mid \mathscr{F}_C) \qquad \text{for every} \quad C \in \mathscr{C}$$

and extend it to $\mathscr{A}$ in an obvious way, then we obtain a martingale measure $\mu \in M^2$ such that $\varphi(\mu)=\mu(U)=\xi$. It follows that ξ belongs to the range of φ and this range $Rg(\varphi)$ is closed in the Hilbert space $L^2(\Omega, \mathscr{F}_U, \mathbb{P})$. Since the linear space M^2 is isometrically isomorphic to a closed subspace of a Hilbert space, the theorem follows.

Denote by $\mathscr{R}$ all sets of the form $A \times \Lambda$ where $A \in \mathscr{A}$ and $\Lambda \in \mathscr{F}_{t(A)}$. It is easily checked that $\mathscr{R}$ is a semialgebra. The algebra generated by $\mathscr{R}$ will be denoted by $\mathscr{G}$ and the σ-algebra generated by $\mathscr{G}$ will be said to be predictable σ-algebra and denoted by $\mathscr{P}$.

We may now prove the following

Theorem 2. *If $\mu, \nu \in M^2$, $A, B \in \mathscr{A}$ and $A \cap B=\emptyset$, then $\mathbb{E}\mu(A)\,\nu(B)=0$.*

Proof. Since both A and B are finite disjoint unions of those sets in $\mathscr{S}$, without any loss of generality, we may assume that $A, B \in \mathscr{S}$. Suppose that $A=A_1-A_2$ and $B=B_1-B_2$ where A_1, A_2, B_1, $B_2 \in \mathscr{C}$ and $A_2 \subset A_1$, $B_2 \subset B_1$. Since $AB=\emptyset$ and $A=AB_1+(A-B_1)$, it follows that $AB_1 \subset B_2$. Using (M.1) and (M.2), we have

$$\begin{aligned}\mathbb{E}\mu(AB_1)\,\nu(B) &= \mathbb{E}[\mathbb{E}(\mu(AB_1)\,\nu(B) \mid \mathscr{F}_{B_2})] \\ &= \mathbb{E}[\mu(AB_1)\,\mathbb{E}(\nu(B) \mid \mathscr{F}_{B_2})]=0. \end{aligned} \tag{2}$$

Noting that $B \subset B_1$ and $(A-B_1) \cap B_1=\emptyset$, we also have

$$\begin{aligned}\mathbb{E}\mu(A-B_1)\,\nu(B) &= \mathbb{E}[\mathbb{E}(\mu(A-B_1)\,\nu(B) \mid \mathscr{F}_{B_1})] \\ &= \mathbb{E}[\nu(B)\,\mathbb{E}(\mu(A-B_1) \mid \mathscr{F}_{B_1})]=0. \end{aligned} \tag{3}$$

Combining (2) and (3) implies the desired result.

Corollary 1. *If $\mu, \nu \in M^2$ and $A, B \in \mathscr{A}$, then*

$$\mathbb{E}\mu(A)\,\nu(B)=\mathbb{E}\mu(AB)\,\nu(AB). \tag{4}$$

Proof. The case that $AB=\emptyset$ is exactly the case in Theorem 2, while in the general case we can simply use the additivity of μ and ν.

Corollary 2. *If $\mu \in M^2$, then the set function φ defined by*

$$\varphi(A)=\mathbb{E}\mu^2(A) \qquad \text{for} \quad A \in \mathscr{A} \tag{5}$$

can be uniquely extended to a finite measure on $\mathscr{B}$.

Proof. It is clear from the orthogonality proved in Theorem 2 that φ is finitely additive on $\mathscr{A}$. By (M.5) we know that $\varphi(A_n)\downarrow\varphi(A)$ whenever $A_n\downarrow A$ in $\mathscr{A}$. So the desired extension is followed by the well-known extension theorem of measures.

Corollary 3. *If $\mu\in M^2$, then the random set function μ can be uniquely extended to the σ-algebra $\mathscr{B}$ such that*

$$\mathbb{E}\,\mu(A)\,\mu(B)=\varphi(AB) \quad \text{for } A, B\in\mathscr{B}. \tag{6}$$

Proof. For every $A\in\mathscr{B}$, there exists a sequence $\{A_n\}$ in $\mathscr{A}$ such that $\varphi(A_n \triangle A)\to 0$ as $n\to\infty$. It follows that

$$\mathbb{E}\,|\mu(A_n)-\mu(A_m)|^2=\varphi(A_n\triangle A_m)\to 0$$

as $m, n\to\infty$. This shows that $\{\mu(A_n)\}$ is a Cauchy sequence in L^2. By the completeness of L^2, there exists a limit which only depends on A and does not depend on what sequence $\{A_n\}$ we have chosen. By passage to limit, we can deduce (6) from (4).

Theorem 2 and its corollaries show that a square integrable martingale measure is also an orthogonal random measure.

For every set $A\times\Lambda$ in $\mathscr{R}$, we define an operator $\Pi_{A\times\Lambda}$ on M^2 as follows:

$$\Pi_{A\times\Lambda}\,\mu=\mathbb{1}_\Lambda\,\mu^A, \qquad \mu\in M^2 \tag{7}$$

where $\mathbb{1}_\Lambda$ stands for the indicator of Λ and μ^A is the trace of μ in A, that is,

$$\mu^A(B)=\mu(AB) \qquad \text{for every } B\in\mathscr{A}.$$

Thus we may state and prove the following

Theorem 3. *$\{\Pi_{A\times\Lambda},\ A\times\Lambda\in\mathscr{R}\}$ is a family of projections on the Hilbert space M^2. It can be uniquely extended to a Boolean algebra of projections, $\{\Pi_S, S\in\mathscr{G}\}$, which is isomorphic to the Boolean algebra $\mathscr{G}$.*

Proof. Firstly we show that if $A\times\Lambda\in\mathscr{R}$ and $\mu\in M^2$ then $\Pi_{A\times\Lambda}\,\mu\in M^2$. Actually, if $\mu\in M^2$ and $B\in\mathscr{A}$, then we have $\Pi_{A\times\Lambda}\,\mu(B)=\mathbb{1}_\Lambda\,\mu(AB)$ by definition. Suppose that $B\subset C\in\mathscr{C}$. Clearly, $\mu(AB)$ is $\mathscr{F}_C$-measurable and $\mathbb{1}_\Lambda$ is $\mathscr{F}_{t(A)}$-measurable. If $AB=\emptyset$, then $\mu(AB)=0$ a.s. If $AB\neq\emptyset$, then $AC\neq\emptyset$ and thus $\mathscr{F}_{t(A)}\subset\mathscr{F}_C$. In both case, $\Pi_{A\times\Lambda}\,\mu(B)$ is $\mathscr{F}_C$-measurable. This implies (M.1). Moreover, suppose that $BC=\emptyset$, where $B\in\mathscr{A}$ and $C\in\mathscr{C}$. It follows that

$$\mathbb{E}(\Pi_{A\times\Lambda}\,\mu(B)\,|\,\mathscr{F}_C)=\mathbb{E}(\mathbb{1}_\Lambda\,\mathbb{E}(\mu(AB)\,|\,\mathscr{F}_{C\cup t(A)})|\,\mathscr{F}_C)=0 \text{ a.s.} \tag{8}$$

since AB and $C\cup t(A)$ are disjoint. This implies (M.2). The properties (M.3)$\sim$(M.5) are obviously true. So that $\Pi_{A\times\Lambda}\,\mu\in M^2$ as desired. By the definition and applying Corollary 1 to $\Pi_{A\times\Lambda}\,\mu$ and ν, we have

$$\Pi_{A\times\Lambda}\,\Pi_{A\times\Lambda}=\Pi_{A\times\Lambda} \tag{9}$$

and

$$\begin{aligned}(\Pi_{A\times\Lambda}\,\mu, \nu)_{M^2}&=\mathbb{E}\,\mathbb{1}_\Lambda\,\mu^A(U)\,\nu(U)=\mathbb{E}\,\mathbb{1}_\Lambda\,\mu(A)\,\nu(A)\\&=\mathbb{E}\,\mathbb{1}_\Lambda\,\nu^A(U)\,\mu(U)=(\mu, \Pi_{A\times\Lambda}\,\nu)_{M^2}.\end{aligned} \tag{10}$$

It follows that $\Pi_{A\times\Lambda}$ is a projection on M^2 whenever $A\times\Lambda\in\mathscr{R}$. The subspace which corresponds to the projection $\Pi_{A\times\Lambda}$ will be denoted by $M^2(A\times\Lambda)$. According to Theorem 2, subspaces $M^2(A_1\times\Lambda_1)$ and $M^2(A_2\times\Lambda_2)$ are orthogonal whenever the sets $A_1\times\Lambda_1$ and $A_2\times\Lambda_2$ are disjoint. In that case, the sum of these two projections is still a projection and, moreover, we have

$$\Pi_{A_1\times\Lambda_1}+\Pi_{A_2\times\Lambda_2}=\Pi_{(A_1\times\Lambda_1)\cup(A_2\times\Lambda_2)} \tag{11}$$

provided $(A_1\times\Lambda_1)\cup(A_2\times\Lambda_2)\in\mathscr{R}$. Consequently, the family of projections $\{\Pi_{A\times\Lambda}, A\times\Lambda\in\mathscr{R}\}$ can be extended to the larger one $\{\Pi_S, S\in\mathscr{G}\}$ in an obvious way. Moreover, since these projections are commutative, it follows that the product

$$\Pi_{S_1}\Pi_{S_2}=\Pi_{S_2}\Pi_{S_1}=\Pi_{S_1\cap S_2} \tag{12}$$

is also a projection for any $S_1, S_2\in\mathscr{G}$. Hence, we can define

$$\Pi_{S_1}\vee\Pi_{S_2}=\Pi_{S_1\cup S_2} \tag{13}$$

and

$$\Pi_{S_1}\wedge\Pi_{S_2}=\Pi_{S_1\cap S_2} \tag{14}$$

for S_1, $S_2\in\mathscr{G}$. It is easy to verify that the distributive law also holds and for any $S\in\mathscr{G}$ we have

$$I-\Pi_S=\Pi_{S^c}. \tag{15}$$

Clearly, $\Pi_\emptyset=0$ and $\Pi_{U\times\Omega}=I$. Therefore, $\{\Pi_S, S\in\mathscr{G}\}$ forms a Boolean algebra of operators which is isomorphic to the Boolean algebra $\mathscr{G}$ and the proof is complete.

We will now extend the family of operators to a Boolean σ-algebra. To do this, we need the following

Theorem 4. *For every $\mu\in M^2$, there exists a finite measure $\langle\mu\rangle$ on $\mathscr{P}$ such that*

$$\mathbb{E}\mathbb{1}_\Lambda\,\mu(A)\,\mu(B)=\langle\mu\rangle(AB\times\Lambda) \tag{16}$$

holds for every $A\times\Lambda\in\mathscr{R}$ and every $B\in\mathscr{A}$.

For each pair μ, ν in M^2, there exists a finite signed measure $\langle\mu,\nu\rangle$ on $\mathscr{P}$ such that

$$\mathbb{E}\mathbb{1}_\Lambda\,\mu(A)\,\nu(B)=\langle\mu,\nu\rangle(AB\times\Lambda) \tag{17}$$

holds for every $A\times\Lambda\in\mathscr{R}$ and every $B\in\mathscr{A}$.

Proof.[3] Since $\Pi_{A\times\Lambda}\mu\in M^2$, applying Corollary 1 to $\Pi_{A\times\Lambda}\mu$ and ν, we have

$$\mathbb{E}\mathbb{1}_\Lambda\,\mu(A)\,\nu(B)=\mathbb{E}\mathbb{1}_\Lambda\,\mu(AB)\,\nu(AB). \tag{18}$$

In the case that $\mu=\nu$ and $A=B$, it reduces to $\mathbb{E}\mathbb{1}_\Lambda\,\mu^2(A)$. Noting that

$$\mathbb{E}\mathbb{1}_\Lambda\,\mu^2(A)=\|\Pi_{A\times\Lambda}\mu\|^2_{M^2},$$

3 This proof was inspired by an analogous theorem due to C. Doléans-Dade (cf. also [12, 13]).

we may define

$$\langle\mu\rangle(A\times\Lambda)=\|\Pi_{A\times\Lambda}\mu\|^2_{M^2} \tag{19}$$

for $A\times\Lambda\in\mathscr{R}$. Since disjoint sets in $\mathscr{G}$ correspond to orthogonal subspaces of M^2, it follows that the set function $\langle\mu\rangle$ is additive on $\mathscr{R}$ and can be extended to $\mathscr{G}$ preserving additivity.

Suppose that $\{S_n\}\subset\mathscr{G}$, $S_n\downarrow\emptyset$ and

$$S_n=\sum_{k=1}^{m_n}(A_k^{(n)}\times\Lambda_k^{(n)})$$

where $A_k^{(n)}\times\Lambda_k^{(n)}\in\mathscr{R}$ $(n=1,2,\ldots;\, k=1,2,\ldots,m_n)$. Using condition (K) and property (M.5), for any $\varepsilon>0$ and for every n and k, we can choose a compact set $C_k^{(n)}$ and a set $B_k^{(n)}$ in $\mathscr{A}$ such that $B_k^{(n)}\subset C_k^{(n)}\subset A_k^{(n)}$ and

$$\mathbb{E}\mu^2(A_k^{(n)}-B_k^{(n)})<\varepsilon/2^n m_n \qquad (n=1,2,\ldots;\, k=1,2,\ldots,m_n). \tag{20}$$

Take

$$V_n'=\sum_{k=1}^{m_n}(B_k^{(n)}\times\Lambda_k^{(n)}),\qquad E_n'=\bigcap_{k=1}^{n}V_k' \qquad (n=1,2,\ldots)$$

and

$$V_n=\sum_{k=1}^{m_n}(C_k^{(n)}\times\Lambda_k^{(n)}),\qquad E_n=\bigcap_{k=1}^{n}V_k \qquad (n=1,2\ldots).$$

Clearly, we have V_n', $E_n'\in\mathscr{G}$ and $E_n'\subset E_n\subset V_n\subset S_n$ for every n. In view of (20), we have

$$\langle\mu\rangle(S_n-V_n')\leqq\sum_{k=1}^{m_n}\mathbb{E}\,\mathbb{1}_{\Lambda_k^{(n)}}\,\mu^2(A_k^{(n)}-B_k^{(n)})<\frac{\varepsilon}{2^n},$$

hence

$$\langle\mu\rangle(S_n-E_n')\leqq\sum_{k=1}^{n}\langle\mu\rangle(S_k-V_k')<\varepsilon \qquad (n=1,2,\ldots). \tag{21}$$

Since for every $n\geqq 1$ and $\omega\in\Omega$, the section $(E_n)_\omega$ of E_n at ω is a compact set in U and since $E_n\downarrow\emptyset$, it follows that for each $\omega\in\Omega$ there exists a positive integer $n_0=n_0(\omega)$ such that $(E_n)_\omega=\emptyset$ and, therefore, $(E_n')_\omega=\emptyset$ holds for $n>n_0$. Let G_n be the projection of E_n' in Ω, i.e.

$$G_n=\{\omega\in\Omega\colon (E_n')_\omega\neq\emptyset\} \qquad (n=1,2,\ldots).$$

Since $G_n\downarrow\emptyset$, it follows that

$$\langle\mu\rangle(E_n')\leqq\mathbb{E}\,\mathbb{1}_{G_n}\mu^2(U)\to 0$$

as $n\to\infty$. Therefore, in view of (21), we have

$$\lim_{n\to\infty}\langle\mu\rangle(S_n)=0. \tag{22}$$

By the extension theorem of measures, we can uniquely extend the set function $\langle\mu\rangle$ to a finite measure on $\mathscr{P}$.

Moreover, for $\mu, \nu \in M^2$ and $A \times \Lambda \in \mathscr{R}$, if we take

$$\langle \mu, \nu \rangle (A \times \Lambda) = \mathbb{E}\, \mathbb{1}_{\Lambda}\, \mu(A)\, \nu(A), \tag{23}$$

then we have

$$\langle \mu, \nu \rangle = \tfrac{1}{4} [\langle \mu + \nu \rangle - \langle \mu - \nu \rangle] \tag{24}$$

on $\mathscr{R}$. Since $\langle \mu + \nu \rangle$ and $\langle \mu - \nu \rangle$ can be extended to finite measures on $\mathscr{P}$ respectively, it follows that $\langle \mu, \nu \rangle$ can be uniquely extended to a finite signed measure on $\mathscr{P}$. The remaining part of this theorem is obvious.

If $\mu, \nu \in M^2$ and $\langle \mu, \nu \rangle = 0$, then μ and ν are said to be strongly orthogonal and denoted by $\mu \perp\!\!\!\perp \nu$. For any $\mu \in M^2$, we can complete the σ-algebra $\mathscr{P}$ with respect to measure $\langle \mu \rangle$ and denote its completion by $\mathscr{P}_\mu$. The Hilbert space $L^2(U \times \Omega, \mathscr{P}_\mu, \langle \mu \rangle)$ will be denoted by H^2_μ as in [7].

Remark 1. In the case of Example 1, if the martingale measure μ is constructed from a square integrable martingale with continuous paths, then the σ-algebra $\mathscr{P}_\mu$ will contain so-called optional σ-algebra; if the corresponding measure $\langle \mu \rangle$ is absolutely continuous with respect to $\lambda^1 \times \mathbb{P}$ (where λ^1 stands for Lebesgue measure), then $\mathscr{P}_\mu$ will contain so-called progressive σ-algebra (cf. [12]).

Remark 2. For an orthogonal random measure μ, we can only construct a measure $\langle \mu \rangle$ on the σ-algebra $\mathscr{B} \times \mathscr{F}_\emptyset$. It follows that H^2_μ only consists of deterministic functions provided $\mathscr{F}_\emptyset$ is the trivial σ-algebra. This explains the reason why one should restrict himself to the deterministic integrands when an orthogonal random measure is used to be an integrator. Thus, the Wiener integral, as we have indicated in the introduction part, is a special case of ours.

The most important result in this paper is the following

Theorem 5. *The family of projections on M^2, $\{\Pi_S, S \in \mathscr{G}\}$, can be extended to a Boolean σ-algebra, $\{\Pi_S, S \in \mathscr{P}\}$, which is isomorphic to $\mathscr{P}$. It is also a spectral measure*[4] *in $U \times \Omega$. Moreover, for $\mu, \nu \in M^2$ and $S \in \mathscr{P}$, we have*

$$\|\Pi_S\, \mu\|^2_{M^2} = \langle \mu \rangle (S) \tag{25}$$

and

$$(\Pi_S\, \mu, \nu)_{M^2} = \langle \mu, \nu \rangle (S). \tag{26}$$

Proof. For any fixed set $S \in \mathscr{P}$, the functional defined by

$$\varphi(\mu, \nu) = \langle \mu, \nu \rangle (S) \qquad \mu, \nu \in M^2 \tag{27}$$

being a bounded, symmetric and bilinear functional on M^2, uniquely determines a selfconjugate operator Π_S on M^2 such that

$$(\Pi_S\, \mu, \nu)_{M^2} = \langle \mu, \nu \rangle (S). \tag{28}$$

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[4] A projection-valued set function $\Pi(B)$ defined on a σ-algebra $\mathscr{B}$ of a measurable space $(E, \mathscr{B})$ is said to be a spectral measure in E if (i) $\Pi(E) = I$ and (ii) for every sequence of disjoint sets $\{B_n\}$ in $\mathscr{B}$, $\Pi(\bigcup_n B_n) = \sum_n \Pi(B_n)$ holds in the sense of strong convergence of operators (cf. [6], § 36 or [4]).

For $S \in \mathscr{G}$, the operator Π_S coincides with the projection defined in Theorem 3. The family $\{\Pi_S, S \in \mathscr{P}\}$ thus defined is a family of projections. In fact, suppose that $\{S_n\} \subset \mathscr{P}$, $S \in \mathscr{P}$ and $S_n \downarrow S$, if $\{\Pi_{S_n}\}$ are projections, then

$$\|\Pi_{S_n} \mu - \Pi_S \mu\|_{M^2}^2 = \langle \mu \rangle (S_n - S) \to 0$$

as $n \to \infty$ for every $\mu \in M^2$. This means that $\{\Pi_{S_n}\}$ strongly converges to the operator Π_S. It follows that Π_S is also a projection. The proof for increasing sequences is similar. Hence, the totality of all sets S in $\mathscr{P}$, of which the corresponding operators Π_S are projections on M^2, is a monotone class. Since this monotone class contains the algebra $\mathscr{G}$, so it also contains the σ-algebra $\mathscr{P}$.

If we regard $\{\Pi_S, S \in \mathscr{P}\}$ as a projection-valued set function, then we have $\Pi_{U \times \Omega} = I$. And if $\{S_n\}$ is a sequence of disjoint sets in $\mathscr{P}$, of which the union is S, then

$$\left\|\Pi_S \mu - \sum_{k=1}^{n} \Pi_{S_k} \mu\right\|_{M^2}^2 = \langle \mu \rangle \left(S - \sum_{k=1}^{n} S_k\right) \to 0$$

as $n \to \infty$ for every μ in M^2. Accordingly, $\{\Pi_S, S \in \mathscr{P}\}$ is a spectral measure in $U \times \Omega$.

By virtue of correspondance between projections on M^2 and sets in $\mathscr{P}$, it can be shown easily that $\{\Pi_S, S \in \mathscr{P}\}$ is a Boolean σ-algebra of operators and isomorphic to $\mathscr{P}$. This completes the proof.

Remark 3. This kind of projections is an extension of the concept of stopped processes. Actually, for the case of Example 1, if τ is a stopping time, then the random set $S = [[0, \tau]]$ (cf. [10]) is a predictable set and $\Pi_S \mu$ is exactly the martingale stopped at τ. It is remarkable that, by means of extensions of projections, we have easily proved the stopped martingale theorem for square integrable martingales.

Corollary 4. *If* $\mu, \nu \in M^2$ *and* $S \in \mathscr{P}$, *then*

$$\langle \Pi_S \mu, \nu \rangle = \langle \Pi_S \mu, \Pi_S \nu \rangle = \langle \mu, \Pi_S \nu \rangle. \tag{29}$$

Proof. In view of (26), taking account of the subspace $M^2(S)$ which corresponds to the projection Π_S, we see that equations (29) are nothing but elementary properties of projections.

III. Stochastic Integrals and Random Radon-Nikodym Derivatives

Now we proceed to define the stochastic integral. To do this, we need the following

Theorem 6. *For every* $\mu \in M^2$ *and every* $\mathfrak{h} \in H_\mu^2$, *there exists a unique element* $\lambda \in M^2$ *such that*

$$(\lambda, \nu)_{M^2} = \int_{U \times \Omega} \mathfrak{h} \, d\langle \mu, \nu \rangle \tag{30}$$

holds for every $\nu \in M^2$.

Proof. The proof is just the same as in [7], where the Kunita-Watanabe inequality

$$\int_{U\times\Omega} |\mathfrak{h}\mathfrak{g}|\,|d\langle\mu,\nu\rangle| \leqq \Big(\int_{U\times\Omega} \mathfrak{h}^2\, d\langle\mu\rangle\Big)^{1/2} \Big(\int_{U\times\Omega} \mathfrak{g}^2\, d\langle\nu\rangle\Big)^{1/2} \tag{31}$$

and the Riesz representation theorem had been used.

Definition. The unique element λ in M^2 mentioned in Theorem 6 is called the stochastic integral of $\mathfrak{h}$ with respect to μ and denoted by

$$\lambda = \int \mathfrak{h}\, d\mu = I^{\mathfrak{h}}\mu = I_\mu \mathfrak{h}. \tag{32}$$

Remark 4. For every $\mu\in M^2$, I_μ is a bounded linear operator from H_μ^2 into M^2. In the case of $\mathfrak{h}$ being a bounded $\mathscr{P}$-measurable function, we can define the spectral integral of $\mathfrak{h}$ with respect to spectral measure $\Pi(S)$: $I^{\mathfrak{h}} = \int \mathfrak{h}\,\Pi(dS)$ (cf. [4] or [6]). But by our definition, the class of integrands is larger than the class of bounded $\mathscr{P}$-measurable functions.

Suppose that $\mu\in M^2$ and $S\in\mathscr{P}$. It will cause no confusion to denote the projection on Hilbert space H_μ^2 by the same symbol Π_S:

$$\Pi_S \mathfrak{h} = \mathfrak{h}\,\mathbb{1}_S \qquad \mathfrak{h}\in H_\mu^2. \tag{33}$$

Thus we have

Theorem 7. *The projections $\Pi_S(S\in\mathscr{P})$ and the stochastic integral operators $I_\mu(\mu\in M^2)$ or $I^{\mathfrak{h}}(\mathfrak{h}\in H_\mu^2)$ commute, i.e.*

$$\Pi_S I_\mu = I_\mu \Pi_S \tag{34}$$

or

$$\Pi_S I^{\mathfrak{h}} = I^{\mathfrak{h}} \Pi_S \tag{35}$$

holds.

Proof. By the definition of stochastic integral, for every $\nu\in M^2$, we have

$$(I_\mu \Pi_S \mathfrak{h}, \nu)_{M^2} = \int_{U\times\Omega} \Pi_S \mathfrak{h}\, d\langle\mu,\nu\rangle = \int_S \mathfrak{h}\, d\langle\mu,\nu\rangle$$

and

$$(\Pi_S I_\mu \mathfrak{h}, \nu)_{M^2} = (I_\mu \mathfrak{h}, \Pi_S \nu)_{M^2} = \int_{U\times\Omega} \mathfrak{h}\, d\langle\mu, \Pi_S\nu\rangle = \int_S \mathfrak{h}\, d\langle\mu,\nu\rangle.$$

Combining these two equations, we assert that

$$I_\mu \Pi_S \mathfrak{h} = \Pi_S I_\mu \mathfrak{h}$$

holds for every $\mathfrak{h}\in H_\mu^2$. This implies (34).

Similarly, for every $\nu\in M^2$, we have

$$(I^{\mathfrak{h}} \Pi_S \mu, \nu)_{M^2} = \int_{U\times\Omega} \mathfrak{h}\, d\langle\Pi_S \mu, \nu\rangle = \int_S \mathfrak{h}\, d\langle\mu,\nu\rangle = (\Pi_S I^{\mathfrak{h}} \mu, \nu)_{M^2}.$$

Consequently, (35) follows and the proof is complete.

A closed subspace Q of M^2 is said to be stable if it is invariant under all projections $\Pi_S(S\in\mathscr{P})$, i.e.

$$\mu\in Q,\, S\in\mathscr{P} \Rightarrow \Pi_S\,\mu\in Q.$$

Two simple necessary and sufficient conditions for stability are in the following

Theorem 8. *Let Q be a closed subspace of M^2. The following conditions are equivalent:*

1° *Q is a stable subspace of M^2;*

2° $\nu\in M^2,\quad \nu\perp Q \Rightarrow \nu\perp\!\!\!\perp Q$;

3° $\mu\in Q,\quad \mathfrak{h}\in H^2_\mu \Rightarrow I_\mu\,\mathfrak{h}\in Q.$

Proof. 1°$\Rightarrow$2°: Suppose that Q is stable and $\nu\perp Q$, that is, $(\mu,\nu)_{M^2}=0$ holds for every $\mu\in Q$. But by the definition of stability, for every $S\in\mathscr{P}$, we have $\Pi_S\,\mu\in Q$ provided $\mu\in Q$. Therefore

$$\langle\mu,\nu\rangle(S)=(\Pi_S\,\mu,\nu)_{M^2}=0.$$

In other words, $\mu\perp\!\!\!\perp\nu$ for every $\mu\in Q$.

2°$\Rightarrow$1°: Suppose that condition 2° is satisfied and $\mu\in Q$, $S\in\mathscr{P}$. Let

$$M^2=Q\oplus Q^\perp$$

be an orthogonal decomposition of M^2. If $\nu\in Q^\perp$, then condition 2° implies that

$$(\Pi_S\,\mu,\nu)_{M^2}=\langle\mu,\nu\rangle(S)=0.$$

Hence, $\Pi_S\,\mu\perp\nu$ and $\Pi_S\,\mu\in Q$.

3°$\Rightarrow$1°: Simply taking $\mathfrak{h}=\mathbb{1}_S$ for every $S\in\mathscr{P}$, we see that 1° is a special case of 3°.

1°$\Rightarrow$3°: Suppose that Q is stable. In other words, 3° is satisfied for $\mathfrak{h}=\mathbb{1}_S(S\in\mathscr{P})$. Since the class of all linear combinations in family $\{\mathbb{1}_S, S\in\mathscr{P}\}$ is dense in H^2_μ, by virtue of linearity and continuity of operator I_μ, we conclude that 3° is also satisfied for every $\mathfrak{h}\in H^2_\mu$.

Concerning the inverse of the stochastic integral operator I_μ, we have the following

Theorem 9. *For every $\mu\in M^2$, the operator I_μ is an isometry from H^2_μ to a closed subspace (denoted by $Rg(I_\mu)$) of M^2. Moreover, if $\lambda\in Rg(I_\mu)$, then*

$$I_\mu^{-1}\,\lambda=\frac{d\,\langle\mu,\lambda\rangle}{d\,\langle\mu\rangle}\qquad a.e.\ \langle\mu\rangle, \tag{36}$$

that is, the Radon-Nidokym derivative of $\langle\mu,\lambda\rangle$ with respect to $\langle\mu\rangle$.

Proof. Let $\mu\in M^2$, $\mathfrak{h}\in H^2_\mu$ and $\lambda=I_\mu\,\mathfrak{h}$. By definition we have

$$(\lambda,\nu)_{M^2}=\int\limits_{U\times\Omega}\mathfrak{h}\,d\,\langle\mu,\nu\rangle$$

for every $\nu \in M^2$. If we take $\nu = \Pi_S \mu$ for every $S \in \mathscr{P}$, then we obtain

$$\langle \lambda, \mu \rangle (S) = (\lambda, \Pi_S \mu)_{M^2} = \int_{U \times \Omega} \mathfrak{h} \, d\langle \mu, \Pi_S \mu \rangle = \int_S \mathfrak{h} \, d\langle \mu \rangle \qquad (S \in \mathscr{P}).$$

Therefore,

$$\mathfrak{h} = \frac{d\langle \mu, \lambda \rangle}{d\langle \mu \rangle} \quad \text{a.e. } \langle \mu \rangle.$$

It follows from the uniqueness of Radon-Nikodym derivative that the inverse I_μ^{-1} of I_μ is well-defined on the range of I_μ, $Rg(I_\mu)$, and equation (36) holds.

Moreover, if $\mathfrak{g} \in H_\mu^2$, then $I_\mu \mathfrak{g} \in Rg(I_\mu)$ and

$$(I_\mu \mathfrak{h}, I_\mu \mathfrak{g})_{M^2} = \int_{U \times \Omega} \mathfrak{g} \, d\langle \mu, I_\mu \mathfrak{h} \rangle = \int_{U \times \Omega} \mathfrak{g} \mathfrak{h} \, d\langle \mu \rangle = (\mathfrak{h}, \mathfrak{g})_{H_\mu^2} \tag{37}$$

which means that the operator I_μ preserves inner product and is, therefore, an isometry between H_μ^2 and $Rg(I_\mu)$. The theorem is established.

Corollary 5. *For any $\mu \in M^2$, the subspace $Rg(I_\mu)$ is stable.*

Proof. Suppose that $\mu \in M^2$, $\mathfrak{h} \in H_\mu^2$, $\lambda = I_\mu \mathfrak{h} \in Rg(I_\mu)$ and $S \in \mathscr{P}$. By Theorem 7 we have

$$\Pi_S \lambda = \Pi_S I_\mu \mathfrak{h} = I_\mu \Pi_S \mathfrak{h} \in Rg(I_\mu)$$

which implies the stability of $Rg(I_\mu)$.

Corollary 6. *If $\mu \in M^2$ and $\mathfrak{h} \in H_\mu^2$, then $\langle I_\mu \mathfrak{h} \rangle$ is absolutely continuous with respect to $\langle \mu \rangle$ and*

$$\mathfrak{h}^2 = \frac{d\langle I_\mu \mathfrak{h} \rangle}{d\langle \mu \rangle} \quad \text{a.e. } \langle \mu \rangle. \tag{38}$$

Proof. Taking $\mathfrak{g} = \mathfrak{h}$ in (37), we obtain

$$\|I_\mu \mathfrak{h}\|_{M^2}^2 = \int_{U \times \Omega} \mathfrak{h}^2 \, d\langle \mu \rangle = \|\mathfrak{h}\|_{H_\mu^2}^2.$$

Replacing $\mathfrak{h}$ by $\Pi_S \mathfrak{h}$ for every $S \in \mathscr{P}$, we can obtain

$$\langle I_\mu \mathfrak{h} \rangle (S) = \int_S \mathfrak{h}^2 \, d\langle \mu \rangle$$

which implies (38).

Theorem 10. *If $\mu \in M^2$, then for every $\lambda \in M^2$ there exists a unique element $\mathfrak{h} \in H_\mu^2$ and a unique element $\nu \in M^2$, such that $\nu \perp\!\!\!\perp \mu$ and*

$$\lambda = \int \mathfrak{h} \, d\mu + \nu. \tag{39}$$

Proof. Since $Rg(I_\mu)$ is closed in M^2, by the orthogonal decomposition of M^2:

$$M^2 = Rg(I_\mu) \oplus Rg(I_\mu)^\perp,$$

we know there exists $\mathfrak{h} \in H_\mu^2$ and $\nu \in Rg(I_\mu)^\perp$ such that (39) holds. Since $Rg(I_\mu)$ is also stable and $v \perp Rg(I_\mu)$, it follows from Theorem 8 that $v \perp\!\!\!\perp Rg(I_\mu)$. The uniqueness follows from that of orthogonal decomposition.

Definition. Suppose that $\lambda, \mu \in M^2$, λ is said to be absolutely continuous with respect to μ and denoted by $\lambda \ll \mu$ if $\lambda \in Rg(I_\mu)$. In that case, the $\mathscr{P}_\mu$-measurable function

$$\frac{d\lambda}{d\mu} = \frac{d\langle \mu, \lambda \rangle}{d\langle \mu \rangle} \tag{40}$$

is said to be the random Radon-Nikodym derivative of λ with respect to μ.

Now we can prove some basic formulas in stochastic calculus.

Theorem 11. *Suppose that $\lambda, \mu, v \in M^2$. If $\lambda \ll \mu$ and $\mu \ll v$, then $\lambda \ll v$ and*

$$\frac{d\lambda}{dv} = \frac{d\lambda}{d\mu} \cdot \frac{d\mu}{dv} \quad \text{a.e. } \langle v \rangle. \tag{41}$$

Proof. Since $\mu \in Rg(I_v)$ and $Rg(I_v)$ is stable, it follows from Theorem 8 that $\lambda \in Rg(I_v)$. If we denote $\frac{d\lambda}{d\mu}$ by $\mathfrak{h}$ and $\frac{d\mu}{dv}$ by $\mathfrak{g}$, then

$$(\lambda, v)_{M^2} = \int_{U \times \Omega} \mathfrak{h}\, d\langle \mu, v \rangle = \int_{U \times \Omega} \mathfrak{h}\mathfrak{g}\, d\langle v \rangle.$$

Replacing v by $\Pi_S v$ for every $S \in \mathscr{P}$, we obtain

$$\langle \lambda, v \rangle (S) = \int_S \mathfrak{h}\mathfrak{g}\, d\langle v \rangle.$$

Consequently

$$\frac{d\lambda}{dv} = \frac{d\langle v, \lambda \rangle}{d\langle v \rangle} = \mathfrak{h}\mathfrak{g} = \frac{d\lambda}{d\mu} \cdot \frac{d\mu}{dv} \quad \text{a.e. } \langle v \rangle$$

as desired.

Corollary 7. *If $\lambda \ll v$ and $\mu \ll v$, then $\langle \lambda, \mu \rangle \ll \langle v \rangle$ and*

$$\frac{d\lambda}{dv} \cdot \frac{d\mu}{dv} = \frac{d\langle \lambda, \mu \rangle}{d\langle v \rangle} \quad \text{a.e. } \langle v \rangle. \tag{42}$$

In the case $\lambda = \mu$, it reduces to Corollary 6.

Corollary 8. *If $\lambda \ll \mu$, then for every $v \in M^2$ we have $\langle \lambda, v \rangle \ll \langle \mu, v \rangle$ and*

$$\frac{d\lambda}{d\mu} = \frac{d\langle \lambda, v \rangle}{d\langle \mu, v \rangle} \quad \text{a.e. } \langle \mu \rangle. \tag{43}$$

In the case $v = \mu$, it reduces to (36). In view of above formulas, we can formally replace $d\langle \lambda, \mu \rangle$ by $d\lambda \cdot d\mu$ in the calculation whenever both sides make sense.

Corollary 9. *If $\mathfrak{h}$ and $\mathfrak{g}$ are bounded $\mathscr{P}$-measurable functions, then*

$$I^{\mathfrak{h}} I^{\mathfrak{g}} = I^{\mathfrak{g}} I^{\mathfrak{h}} = I^{\mathfrak{h}\mathfrak{g}}. \tag{44}$$

That is, the Banach algebra (denoted by $\mathscr{H}$) consists of all bounded $\mathscr{P}$-measurable functions is isomorphic to the Banach algebra $\{I^{\mathfrak{h}}, \mathfrak{h} \in \mathscr{H}\}$ of commutative self-conjugate operators on M^2 (cf. [4]).

Theorem 12. *If $\mu \in M^2$ and for almost all $\omega \in \Omega\, \mu(\cdot, \omega)$ is a finite signed measure on $\mathscr{B}$, and if $\mathfrak{h}$ is a bounded $\mathscr{P}$-measurable function, then the stochastic integral $\int \mathfrak{h}\, d\mu$ coincides with the sample Lebesgue integral.*

Proof. For $\mathfrak{h} = \mathbb{1}_{A \times \Lambda}$, where $A \times \Lambda \in \mathscr{R}$, the two kinds of integrals are equal to $\Pi_{A \times \Lambda}\, \mu = \mathbb{1}_\Lambda\, \mu_A$. By virtue of linearity and continuity of the two kinds of integrals, we can show that the class of $\mathscr{P}$-measurable functions of which the two kinds of integrals coincide with each other contains all bounded $\mathscr{P}$-measurable functions. This completes the proof.

IV. Extensions

In the above paragraphs, we have concentrated our attention on square integrable martingale measures. But it is not difficult to extend to the case of local martingales or semi-martingales. Now we give an outline of such extensions.

For any sequence $\mathscr{Q} = \{S_n\}$ of sets in $\mathscr{P}$ increasing to $U \times \Omega$, define

$$\|\mu\|_n = \|\Pi_{S_n}\, \mu\|_{M^2} = (\langle\mu\rangle(S_n))^{1/2}, \qquad \mu \in M^2,\ n \geqq 1.$$

Clearly, $\{\|\cdot\|_n, n \geqq 1\}$ is a family of seminorms and

$$\|\mu\|_{\mathscr{Q}} = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|\mu\|_n}{1 + \|\mu\|_n}, \qquad \mu \in M^2$$

defines a quasi-norm $\|\cdot\|_{\mathscr{Q}}$ which is weaker than $\|\cdot\|_{M^2}$. If we take the standard completion of M^2 with respect to $\|\cdot\|_{\mathscr{Q}}$ and identify every Cauchy sequence with its limit, then we obtain a Fréchet space $M^2_{\mathscr{Q}}$. Extending the operators $\{\Pi_{S_n}\}$ to $M^2_{\mathscr{Q}}$ by continuity, we see that $\Pi_{S_n}\, \mu \in M^2$ for any $\mu \in M^2_{\mathscr{Q}}$ and $S_n \in \mathscr{Q}$.

Definition. Define

$$M^2_{\text{loc}} = \bigcup_{\mathscr{Q}} M^2_{\mathscr{Q}}$$

where the union is taken over all the sequences $\mathscr{Q}$'s described above. Any element in M^2_{loc} is said to be a locally square integrable martingale measure. If $\mu \in M^2_{\mathscr{Q}}$ for some sequence $\mathscr{Q}$, then the sequence $\mathscr{Q}$ is called a localized sequence of μ.

It is easy to see that for every $\mu \in M^2_{\text{loc}}$, there exists a unique σ-finite measure $\langle\mu\rangle$ on $\mathscr{P}$ such that

$$\mathbb{E}\, \mathbb{1}_\Lambda\, \mu(A)\, \mu(B) = \langle\mu\rangle(AB \times \Lambda), \qquad \text{for } A \times \Lambda \in \mathscr{R},\ B \in \mathscr{A}.$$

Let $\mu \in M^2_{\text{loc}}$ and $\mathscr{Q} = \{S_n\}$ be a localized sequence of μ. Denote by $H^2_{\mu,\text{loc}}$ the totality of all random functions $\mathfrak{h}$ defined on $U \times \Omega$ such that

$$\mathfrak{h}\, \mathbb{1}_{S_n} \in H^2_{\mu_n} \quad \text{for every } n \geqq 1,$$

where

$$\mu_n = \Pi_{S_n}\, \mu \in M^2.$$

Clearly, $H^2_{\mu,\mathrm{loc}}$ is a Fréchet space with quasi-norm

$$\|\mathfrak{h}\|_\mu = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|\mathfrak{h}\|_n}{1+\|\mathfrak{h}\|_n}$$

where

$$\|\mathfrak{h}\|_n = \|\mathfrak{h}\,\mathbb{1}_{S_n}\|_{H^2_{\mu_n}} = \Big(\int_{S_n} \mathfrak{h}^2\, d\langle\mu_n\rangle\Big)^{1/2}, \qquad n \geqq 1.$$

Denote $\mathfrak{h}\,\mathbb{1}_{S_n}$ by $\mathfrak{h}_n$. The stochastic integrals

$$\lambda_n = \int \mathfrak{h}_n\, d\mu_n \qquad n \geqq 1$$

are well defined since $\mu_n \in M^2$ and $\mathfrak{h}_n \in H^2_{\mu_n}$ for every $n \geqq 1$. It is easy to verify that

$$\Pi_{S_n}\, \lambda_{n+1} = \lambda_n \qquad \text{for } n \geqq 1.$$

Hence, there exists a unique element $\lambda \in M^2_{\mathrm{loc}}$ such that

$$\Pi_{S_n}\, \lambda = \lambda_n \qquad \text{for } n \geqq 1.$$

The unique element λ denoted by $\int \mathfrak{h}\, d\mu$ is also said to be the stochastic integral of $\mathfrak{h}$ with respect to μ. Since the properties of such kind of stochastic integrals can be easily deduced from above obtained results and the properties of projections, we will not give details here.

Definition. A random set function $v = v(A, \omega)$ defined on $\mathscr{A} \times \Omega$ is said to be an adaptive random signed measure if for almost all $\omega \in \Omega$, $v(\cdot, \omega)$ can be extended to a σ-finite signed measure on $\mathscr{B}$ and $v(A, \cdot)$ is $\mathscr{F}_C$-measureable whenever $A \subset C$, $A \in \mathscr{A}$ and $C \in \mathscr{C}$. The totality of all adaptive random signed measures will be denoted by M^σ. A random function $\mathfrak{h} = \mathfrak{h}(x, \omega)$ defined on $U \times \Omega$ is said to be a locally bounded predictable random function if there exists a sequence of sets $\{S_n\}$ in $\mathscr{P}$ such that $S_n \uparrow U \times \Omega$ and for every $n \geqq 1$, $\mathfrak{h}\,\mathbb{1}_{S_n}$ is a bounded $\mathscr{P}$-measurable function. The totality of such functions will be denoted by H^b_{loc}. A random set function $\lambda = \lambda(A, \omega)$ defined on $\mathscr{A} \times \Omega$ is said to be a semi-martingale measure if there exists $\mu \in M^2_{\mathrm{loc}}$ and $v \in M^\sigma$ such that $\lambda = \mu + v$. The totality of all semi-martingale measures will be denoted by M^s.

For $\lambda \in M^s$ and $\mathfrak{h} \in H^b_{\mathrm{loc}}$, we can define the stochastic integral as follows:

$$\int \mathfrak{h}\, d\lambda = \int \mathfrak{h}\, d\mu + \int \mathfrak{h}\, dv,$$

where $\lambda = \mu + v$, $\mu \in M^2_{\mathrm{loc}}$ and $v \in M^\sigma$. The first integral on the right side is a stochastic integral, while the second one is a sample Lebesgue integral. It follows from Theorem 12 that this definition doesn't depend on the decompositions of λ and the integral $\int \mathfrak{h}\, d\lambda$ itself is also a semi-martingale measure.

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References

1. Cairoli, R., Walsh, J.B.: Stochastic integrals in the plane. Acta Math. **134**, 111–183 (1975)
2. Dellacherie, C.: Un survol de la theorie de l'integrale stochastique. Stoc. Proc. Appl. **10**, 115–144 (1980)
3. Doléans-Dade, C., Meyer, P.A.: Intégrales stochastiques par rapport aux martingales locales. Sém. Probab. IV. Lect. Notes in Math. **124**, 77–107. Berlin-Heidelberg-New York: Springer 1970
4. Dunford, N., Schwartz, J.T.: Linear Operator (II). New York: Wiley-Interscience 1963
5. Halmos, P.R.: Measure Theorey. Princeton, New Jersey: Van Nostrand-Reinhold 1950
6. Halmos, P.R.: Introduction to Hilbert Space and the Theory of Spectral Multiplicity. New York: Chelsea 1951
7. Huang, Z.: Martingale measures and stochastic integrals on metric spaces. Wuhan Univ. J. special issue of Math. **(I)**, 89–101 (1981)
8. Ikeda, N., Watanabe, S.: Stochastic Differential Equations and Diffusion Processes. Oxford New York Tokyo: North-Holland/Kodansha 1981
9. Itô, K.: Stochastic integral. Proc. Imp. Acad. Tokyo **20**, 519–524 (1944)
10. Jacod, J.: Calcul Stochastique et Problèmes de Martingales. Lect. Notes in Math. **714**. Berlin-Heidelberg-New York: Springer 1979
11. Kunita, H., Watanabe, S.: On square integrable martingales. Nagoya Math. J. **30**, 209–245 (1967)
12. Métivier, M.: Reele und vektorwertige Quasimartingale und die Theorie der Stochastischen Integration. Lect. Notes in Math. **607**. Berlin-Heidelberg-New York: Springer 1977
13. Métivier, M., Pellaumail, J.: Mesures stochastiques à valeurs dans des espace L_0. Z. Wahrscheinlichkeitstheorie verw. Gebiete **40**, 101–114 (1977)
14. Skorohod, A.V.: Studies in the theory of random processes. Reading, Massachusetts: Addison-Wesley 1965
15. Wiener, N.: Differential space. J. Math. Phys. **58**, 131–174 (1923)
16. Wong, E., Zakai, M.: Martingales and stochastic integrals for processes with a multidimensional parameter. Z. Wahrscheinlichkeitstheorie verw. Gebiete **29**, 109–122 (1974)
17. Wong, E., Zakai, M.: Weak martingales and stochastic integrals in the plane. Ann. of Probability **4**, 570–586 (1976)
18. Yen, K.A.: An introduction to the theory of martingale and stochastic integral. [In Chinese] Shanghai (1981)

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