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Renewal Sets and Random Cutouts

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1. Introduction, Definitions, Summary, and Applications

The zeros of Brownian motion and the zeros of other recurrent processes with independent stable increments are well known to constitute stochastic generalizations of Cantor's triadic set. These and other "random Cantor-like sets" have found applications in mathematics ([6, 7]) and in physics (theories of turbulence and of noise [9–11]), but in both instances the original definitions appear excessively indirect and cumbersome. Alternative definitions closer to Cantor's classical construction are desirable. For an important class of sets that includes the preceding examples, the present paper will provide such a construction. With no additional effort, Cantor-like constructions will be provided for all "cutout renewal sets", which are generalized renewal sets satisfying a certain "cutout condition". A generalized renewal set is defined as the closure of values of a process having stationary, nonnegative, independent and infinitely divisible increments[1]. Conversely, if a generalized renewal set is a cutout renewal set, it is obtainable by a Cantor-like construction.

As is well known, Cantor had proceeded by successive steps, where each step consisted in cutting off the open middle third of each of a finite number of closed intervals. The initial interval – one with which the first step starts – is $[0, 1]$, so the second step starts with the two intervals $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$, etc. Partly stochastic variants of this construction have been considered by Salem [8] and Dvoretzky [3], but the construction studied in the present paper is different; I think it is simpler and more natural for application to both mathematics and physics, and in addition it will help improve the solutions of Dvoretzky's problem (see [12]). Cantor's initial interval $[0, 1]$ is replaced by the whole real line R; the cutouts remain almost surely denumerable but are made entirely random, mutually independent and stationary according to the following definition.

Definition 1.1. A cutout of starting point t and duration z will be an open interval $]t, t+z[$ with $-\infty < t < \infty$ and $z > 0$. To obtain random cutouts, one maps each cutout into the point of the open upper half plane H of coordinates t and z, and one selects the points (t, z) at *random*. (The "chance variable" ω will be omitted when possible.)

In this paper, the following rules will be used. The number of points (t, z) in a rectangle interior to H will be *almost surely* (a.s.) *finite*, so the total number of

[1] This last definition should perhaps be motivated: A classical renewal set (see, e.g., [2]) is a discrete ordered set $T(\omega, h)$ such that the renewal times $T(\omega, h) - T(\omega, h-1) = U_h$ are nonnegative, independent and identically distributed variables. Hence a classical renewal set can be considered the set of values of the nondecreasing random sequences with increments U_h. The generalization used consists in replacing this sequence by the more general process postulated.

cutouts will be a.s. denumerable. Different cutouts will be *statistically independent*, meaning that, given any finite collection of non-overlapping rectangles of H whose sides are parallel to the axes, the numbers of points (t, z) in the rectangles are independent. The distribution of cutouts will be *stationary*, meaning that the distribution of the number of points (t, z) in a rectangle of H is unchanged when this rectangle is shifted along the t-axis.

Proposition 1.2. *If the cutouts are a.s. denumerable, statistically independent and stationary, the number of points* (t, z) *within an interior rectangle* $]t', t''[\times[z', z''[$ *of* H *is a Poisson random variable of expectation* $(t''-t')[F(z'')-F(z')]$, *where* $F(z)$ *is a finite, non-decreasing and left continuous function of* z *defined for* $0<z<\infty$.

Proof. Count the points (t, z) in the rectangle $]0, t[\times[z_0, z[$. Considered as a random function of t—with z_0 and z fixed—this number has independent and stationary increments and its jumps are all equal to 1, so it must be a Poisson process of expectation proportional to t. Next—z_0 and t being fixed—consider this number of points as a function of z; it is integer-valued and its increments are independent but no longer necessarily stationary, so it must be a Poisson process of expectation proportional to $F(z)-F(z_0)$, with F as characterized in Proposition 1.2.

Obviously, the function $F(z)+\Delta$ (Δ real) is equivalent to $F(z)$.

Values of ω such that the points (t, z) are non-denumerable have zero probability and will be discarded, so the cutouts can be arbitrarily ordered by an index n and designated by $[t_n(\omega), t_n(\omega)+z_n(\omega)]$.

Definition 1.3. The *uncovered set* of R will be the random set

$$S(\omega)=R-\bigcup_{n=1}^{\infty}\,]t_n(\omega), t_n(\omega)+z_n(\omega)[\,.$$

Definition 1.4. The *centered uncovered set* $S_t(\omega)$ will be defined as the uncovered set $S(\omega)$ conditioned to contain t:

$$S_t(\omega)=\{S(\omega)\,|\,t\in S(\omega)\}\,.$$

Comment. The set $S_0(\omega)$ can be obtained by conditioning the point (t, z) to lie outside of the domain defined by the inequalities $t+z>0$ and $t<0$. Conversely, the set $S(\omega)$ can be obtained from the collection of $S_t(\omega)$ by attributing to t a uniform (Lebesgue) measure over R.

Digression. This last construction characterizes $S(\omega)$ as an element of a measure space which is shift invariant and whose total measure is infinite. The quotient of this space by the class of equivalence of $S(\omega)$ may happen to have either a finite or an infinite measure. In the former case, $S(\omega)$ is an ordinary random set; in the latter case, it is a "sporadic set" according to the definitions in Mandelbrot [9]. Contitioning sets of the form $\beta(t', t'')=\{\omega\colon S(\omega)\cap[t', t'']\neq\emptyset\}$ have finite measures, and conditioned sets of the form $\{S(\omega)|\omega\in\beta(t', t'')\}$ are ordinary (finite measure) random sets invariant under some but not all shifts [9, p. 159].

Definition 1.5. The maximal open intervals of $R-S(\omega)$, which are non-overlapping, will be called *intermissions* of $S(\omega)$. They are denumerable and will be ordered arbitrarily, their end points being designated by $T_m'(\omega)$ and $T_m''(\omega)$ and their durations by $U_m(\omega)$. Each intermission is a union of cutouts. When $S(\omega)$ includes closed intervals, these will be called *acts* and the duration of the act preceding intermission number m will be designated by $Y_m(\omega)$.

Problem 1.6. The *principal problem* raised and solved in this paper is to describe the structure of the random set $S_0(\omega)$ (Sections 2 and 3). *Subsidiary problems:* to determine under which conditions R is a.s. covered by cutouts (Section 4), and to solve Dvoretzky's problem better [12]. The solution of the principal problem is not quite definitive. It is summarized by the following definition and theorem.

Definition 1.7. Cutouts are said to satisfy the *renewal condition* if the function $F(z)$ satisfies the following conditions:

a) $F(\infty)<\infty$; if so, we shall write $F(\infty)=0$.

b) $H<\infty$, where $H=\int_0^1 \exp\left[-\int_s^1 F(z)\,dz\right] ds$.

c) K exists and $K<\infty$, where

$$K=\lim_{m\to 0}\exp\left[\int_m^1 z\,dF(z)\right][|F(m)|^{-1}]\{1-\exp[-m|F(m)|]\}.$$

Comment. For c) to hold, sufficient conditions are either c′) or c″):

c′) $\lim_{m\to 0}\left[\int_m^1 z\,dF(z)-\log|F(m)|\right]=-\infty.$

c″) $\lim_{m\to 0}\left[\int_m^1 z\,dF(z)+\log m\right]=-\infty.$

Condition c) leaves me uncomfortable; I suspect it could be simplified, but I don't know how. In any event, the present form of Theorem 1.8 is adequate for the applications in view[2].

Theorem 1.8. *If the renewal condition holds, then* $S_0(\omega)$ *is identical in distribution to the closure of the values of the process*

$$T(\omega,x)=x\exp\left[-\int_0^1 z\,dF(z)\right]+J(\omega,z),$$

with the process J defined as follows: $J(\omega,0)=0$, *increments of* $J(\omega,x)$ *are independent and nonnegative, and the logarithmic generating function* (l.g.f.) *of* $J(\omega,x)-J(\omega,0)$ *is*

$$-\log\int_0^\infty e^{-bu}\,d\,Pr\{J(\omega,x)-J(\omega,0)<u\}$$
$$=\frac{x}{\exp[F(1)]\int_0^\infty \exp\left[-bs-\int_s^1 F(z)\,dz\right]ds+K}.$$

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[2] *Note Added in Proof.* A full solution of the subsidiary problem above has since been obtained by L.A. Shepp, in a paper titled "Covering the line with random intervals", to be submitted to this Journal. As I had suspected, condition c) above is not needed. I suspect that it is not needed either for the principal problem, implying that every reference to K below will disappear from a more successful study of the problem.

If either $F(\infty)=\infty$, or $H=\infty$, or $K=\infty$, $S_0(\omega)$ is a.s. degenerate, in the sense that for all $t>0$, $S_0(\omega)\cap]0,t]$ is a.s. empty.

Definition 1.9. If a process $T(\omega, x)$ is such that its l.g.f. can be written in the form in Theorem 1.8 with F a finite non-decreasing function satisfying $H<\infty$, $K<\infty$ and $F(\infty)<\infty$, $T(\omega, x)$ will be said to satisfy the *cutout condition.*

Converse Theorem 1.10. *If a process satisfies the cutout condition, the closure of its values can be constructed as the set left uncovered by a process of random cutouts.*

Application to Statistically Self Similar Cutouts, with $F(z)=-Q/z$

Definition 1.11. Random cutouts will be called *statistically self similar* if the distribution of the points (t, z) is left unchanged by any similarity whose apex is an arbitrary point of the t axis and whose ratio r is positive.

Proposition 1.12. *Random cutouts are self similar iff $F(z)=-Q/z+F_0$ with $Q>0$ and F_0 finite.*

Proof. As a preliminary, we prove that $F(\infty)<\infty$. Let $r>1$, and consider in H the two rectangles $]t, t+dt[\times[z, rz[$ and $]rt, rt+r\,dt[\times[rz, r^2 z[$. They are similar with apex 0 and ratio r, so by hypothesis the numbers of points (t, z) they contain have identical distributions and — in particular — equal expectations

$$[F(rz)-F(z)]\,dt=[F(r^2 z)-F(rz)]\,r\,dt.$$

By iteration, for every integer k, one has $F(r^{k+1} z)-F(r^k z)=r^{-k}[F(rz)-F(z)]$ and — by summation —

$$F(r^{k+1} z)-F(z)=(1+r^{-1}+\cdots+r^{-k})[F(rz)-F(z)].$$

As $k\to\infty$, $F(\infty)-F(z)=r(r-1)^{-1}[F(rz)-F(z)]<\infty$, so $F(\infty)<\infty$ as announced.

To prove that $F(z)=-Q/z$, normalize $F(z)$ so that $F(\infty)=0$ and consider the two strips $]t, t+dt[\times[z, \infty[$ and $]rt, rt+r\,dt[\times[rz, \infty[$. By the same argument as above, $F(z)=rF(rz)$ for all r und z. The only monotone solution of this question is known to be $F(z)=-Q/z$.

Applying Theorem 1.7 to the self similar case, we obtain

Proposition 1.13. *For self similar cutouts with $Q\geq 1$, one has $H=K=\infty$, so that $S_0(\omega)\cap]0,t]$ is a.s. empty. For self similar cutouts with $0<Q<1$, the renewal condition holds. In fact, $T(\omega, x)=J(\omega, x)$, with $J(\omega, x)$ the process of random independent increments such that for $x>0$*

$$E\exp\{-b[J(\omega, x)-J(\omega, 0)]\}=\exp[-x\,b^{1-Q}\,e^Q/\Gamma(1-Q)].$$

This process $J(\omega, x)$ is called *stable subordinator* or stable non-decreasing process of Lévy exponent equal to $\alpha=1-Q$. The set of its values is called *Lévy self similar perfect set.* In the special case $Q=0.5$, it is identical in distribution to the set of zeros of Brownian motion starting from $t=0$; in the special cases $0.5\leq Q<1$, to the set of zeros of a symmetric stable process of exponent $1/Q$. The self similarity of S follows from the self similarity of the cutouts, but it is also easy to verify directly that the set S_0 and all sets rS_0 obtained by similarity of apex 0 and of arbitrary ratio r are identical in distribution. The Hausdorff dimension [1], Fourier dimension [6, 7], "capacitary" dimension, and "similarity" dimension of $S(\omega)$ are a.s. equal to $1-Q$.

Self similar perfect sets – as has been said – have begun to play important roles in the study of noises and other turbulence-like fluctuations, and in the theory of trigonometric series. In both contexts, the original definition was awkward because it was based upon the values of $T(\omega, x)$, that is, upon intermissions that are non-overlapping so they cannot be chosen independently. The Cantor-like construction proposed in the present paper appears more "natural".

Corollary 1.14. *Multidimensional Brownian motion is nonrecurrent a.s.*

Proof. This fact is well known, but it may be of interest to show it can be rederived as Corollary of Proposition 1.13. The N dimensional Brownian motion $B(t)$ is a random vector whose N coordinates $B_k(t)$ are independent, identically distributed one dimensional Brownian motions. A zero of $B(t)$ is a zero common to the N functions $B_k(t)$. Since the set of zeros of $B_k(t)$ is identical in distribution to the uncovered set corresponding to random cutouts with $F(z)=-(2z)^{-1}$, the set of zeros of $B(t)$ is identical in distribution to the uncovered set with $F(z)=-Q/z$ and $Q=N/2$. For all $N \geqq 2$, $Q \geqq 1$, so the set of zeros reduces a.s. to 0 and it is a.s. that the multidimensional Brownian motion never returns to the origin.

2. Structure of the Random Set S_0 when Cutout Lengths Are Bounded away from Zero and Infinity

Throughout this section, it will be assumed that z is bounded by a *positive minimum* m, also called *internal* or *inner scale*, and a *finite maximum* M, also called *external* or *outer scale*. Then acts and intermissions alternate and the number of acts and intermissions intersecting $[0, 1]$ is finite, so it is possible to order intermissions chronologically and designate their end points by T_h and T_h'. In this section, every random variable will have parameters m and M, for example $S(m, M)$. Save for possibly ambiguous cases, m and/or M and/or ω can be omitted.

Notation 2.1. *We shall write*

$$A=F(M)-F(m); \quad B=\exp\left[-\int_m^M z\,dF(z)\right]; \quad C=\exp\left[-\int_m^1 z\,dF(z)\right].$$

Proposition 2.2. *The instants t_n constitute a Poisson process of mean recurrence time equal to $1/A$. Hence*

$$Pr\{\min_n \{t_n \mid t_n>0\} \geqq y\}=Pr\{Y \geqq y\}=\exp\{-y[F(M)-F(m)]\}=\exp\{-yA\}.$$

In particular, $E[Y(m, M)]=1/A$.

Proposition 2.3. *The probability that an instant t belongs to S is equal to*

$$Pr\{t \in S\}=Pr\{0 \in S\}=B.$$

Proof. The event $\{0 \in S(m, M)\}$ is the intersection of the mutually independent events $\{0 \in S(z, z+dz)\}$ with $m<z<z+dz<M$. Thus,

$$\log Pr\{0 \in S(m, M)\} = \int_m^M \log Pr\{0 \in S(z, z+dz)\}$$

$$= \int_m^M \log Pr\{\text{there is no point } (t, z) \text{ in the strip }]-z, 0[\times[z, z+dz[\}$$

$$= \int_m^M \log\{\exp[-z\,dF(z)]\}\,dz = -\int_m^M z\,dF(z) = \log B.$$

Proposition 2.4. *The probability that an instant $t>0$ belongs to S_0 is equal to*

$$Pr\{t \in S_0\} = Pr\{t \in S \mid 0 \in S\} = \exp\left[-\int_m^M \min(z, t)\,dF(z)\right].$$

Proof. The joint event $\{t \in S(m, M)$ and $0 \in S(m, M)\}$ is the intersection of the mutually independent joint events $\{t \in S(z, z+dz)$ and $0 \in S(z+z+dz)\}$, carried over z and dz satisfying $m<z<z+dz<M$. Hence

$$\log Pr\{t \in S(m, M) \text{ and } 0 \in S(m, M)\} = \int_m^M \log Pr\{t \in S(z, z+dz) \text{ and } 0 \in S(z, z+dz)\}$$

$$= \int_m^M \log Pr\{\text{there is no point } (t, z) \text{ in either of the strips}$$

$$]-z, 0[\times[z, z+dz[\text{ and }]t-z, t[\times[z, z+dz[\}$$

$$= \int_m^M \log Pr\{\text{there is no point } (t, z) \text{ in the strip}$$

$$]0, \min(t+z, 2z)[\times[z, z+dz[\}$$

$$= -\int_m^M \min(t+z, 2z)\,dF(z).$$

Finally,

$$\log Pr\{t \in S(m, M) \mid 0 \in S(m, M)\} = -\int_m^M \min(t+z, 2z)\,dF(z) + \int_m^M z\,dF(z)$$

$$= -\int_m^M \min(t, z)\,dF(z),$$

as announced.

Proposition 2.5. *The event that the last point of an act of S_0 lies between $t>0$ and $t+dt$ has the probability density*

$$A \exp\left\{-\int_m^M \min(t, z)\,dF(z)\right\}.$$

Notation 2.6. *The g.f.'s of the variables U and Y, and of the density of Proposition 2.5 will be designated by*

$$U^*(b\,|\,m, M) = \int_0^\infty e^{-bu}\, dP\{U(m, M) < u\},$$

$$Y^*(b\,|\,m, M) = \int_0^\infty e^{-bu}\, dPr\{Y(m, M) < u\} = \frac{F(M) - F(m)}{b + F(M) - F(m)} = \frac{A}{b + A},$$

$$W^*(b\,|\,m, M) = [F(M) - F(m)] \int_0^\infty \exp\left[-bs - \int_m^M \min(s, z)\, dF(z)\right] ds.$$

Theorem 2.7. $U^*(b\,|\,m, M) = \dfrac{1}{Y^*(b\,|\,m, M)} - \dfrac{1}{W^*(b\,|\,m, M)}.$

Proof. T_0', defined as the first point T_h' to the right of 0, is identical to

$$\min_n \{t_n | t_n > 0\}$$

and its g.f. is $Y^*(b|m, M)$. The position of T_h' is obtained by adding to the position of T_0' the sum of the following h independent intermissions and h independent acts. Therefore, the g.f. of T_h' is

$$Y^*(b|m, M)\, [Y^*(b|m, M)\, U^*(b|m, M)]^h.$$

Further, the g.f. of the event $\{t < T_h' < t + dt$ for some $h\}$ is equal to

$$Y^* + Y^{*2}\, U^* + Y^{*3}\, U^{*2} + \cdots = \frac{Y^*(b\,|\,m, M)}{1 - U^*(b\,|\,m, M)\, Y^*(b\,|\,m, M)}.$$

By a known result of renewal theory [3, 4] this is also equal to the g.f. of the density evaluated in Proposition 2.5, namely $W^*(b\,|\,m, M)$. Solving for $U^*(b\,|\,m, M)$, we obtain the result claimed.

The following definition introduces a process $X(\omega, t\,|\,m, M)$ such that $S(\omega|m, M)$ is the closure of the values of X. The reason for this definition is that it makes the limit behavior of $S_0(\omega)$ for $m \to 0$ and $M \to \infty$ reducible to classical theorems of probability—which do not concern sets but processes

Definition 2.8. $X(\omega, t\,|\,m, M)$ will be defined as equal to $1/C$ times the total length of the intervals in $S(\omega\,|\,m, M) \cap [0, t]$. The function $T(\omega, x\,|\,m, M)$ will be defined as $\max[t\,|\,X(\omega, t\,|\,m, M) < x]$, that is, as the inverse function of $X(\omega, t\,|\,m, M)$, "filled-in" so as to be defined for all x and to be left continuous.

Proposition 2.9.

$$EX(\omega, t\,|\,m, M) = C^{-1} E\,|S(m, M) \cap [0, t]| = t\, C^{-1}\, Pr\{0 \in S(m, M)\} = tB/C.$$

The derivative $X'(\omega, t|m, M)$ of $X(\omega, t|m, M)$ exists for all instants t in the interior of S or of $R - S$, and satisfies

$$X'(\omega, t\,|\,m, M) = \begin{cases} 1/C & \text{if } t \text{ is in the interior of } S, \\ 0 & \text{if } t \text{ is in the interior of } R - S. \end{cases}$$

Clearly, each intermission of $S(\omega\,|\,m, M)$ corresponds to a jump of T, and each act corresponds to a zone in which the derivative of $T(\omega, x\,|\,m, M)$ is C. The jumps and the linear contributions can be considered separately by writing

$$T(\omega, x|m, M) = J(\omega, x|m, M) + xC.$$

Theorem 2.10. *$J(\omega, x|m, M)$ is a process with independent increments for which the Lévy measure function of the interval (u, ∞) is*

$$L(u|m, M) = AC\, Pr\{U \geqq u\}.$$

Proof. As a first consequence of its definition, J is a process of independent increments whose Lévy measure is proportional to $Pr\{U \geqq u\}$, say $\delta\, Pr\{U \geqq u\}$. To determine the value of this constant δ, note that the interval of x between successive jumps is an exponential random variable of expectation $1/\delta$. During such an interval, the linear function xC increases by an exponential variable of expectation $\delta^{-1}C$. Since that variable must be identical to Y in distribution, $\delta^{-1}C = 1/A$, and

$$\delta\, Pr\{U \geqq u\} = AC\, Pr\{U \geqq u\}$$

as asserted.

Theorem 2.11. *The g.f. of $T(\omega, x|m, M)$ is equal to*

$$T^*(b, x|m, M) = \exp[-xCA/W^*(b|m, M)].$$

Proof. As is well known, the g.f. of $J(\omega, x|m, M)$ is

$$J^*(b, x|m, M) = \exp\left[-x\int_0^\infty (e^{-bu} - 1)\, dL(u|m, M)\right]$$

and therefore the g.f. of T is

$$T^*(b, x|m, M) = J^*(b, x|m, M)\exp(-bxC).$$

By Theorems 2.7 and 2.10,

$$\begin{aligned}-\int_0^\infty (e^{-bu} - 1)\, dL(u|m, M) &= AC[U^*(b|m, M) - 1] \\ &= AC(-1 + 1/Y^*) + AC/W^*.\end{aligned}$$

In addition (Notation 2.6), $-1 + 1/Y^* = b/A$, so $AC(-1 + 1/Y^*) = bC$. Plugging into T^*, we obtain the result claimed.

Proposition 2.12. *One can write*

$$W^*/CA = W_1 + W_2 + W_3$$

with the following definitions

$$\begin{aligned} W_1(b|m, M) &= \exp\left[\int_m^1 z\, dF(z)\right][b + F(M) - F(m)]^{-1} \\ &\quad \cdot \{1 - \exp[-mb - mF(M) + mF(m)]\} \\ W_2(b|m, M) &= \int_m^M \exp\left[-bs - \int_s^1 F(z)\, dz + F(1) - sF(M)\right] ds \\ W_3(b|m, M) &= b^{-1}\exp\left[-bM - \int_1^M z\, dF(z)\right]. \end{aligned}$$

Proof. By definition,

$$W^*/AC = \int_0^\infty \exp\left[-bs - \int_m^M \min(s,z)\,dF(z) + \int_m^1 z\,dF(z)\right] ds.$$

Divide the interval of integration into the spans $(0,m)$, (m,M) and (M,∞), and designate the corresponding partial integrals by $W_1(b|m,M)$, $W_2(b|m,M)$, and $W_3(b|m,M)$. We have

$$\begin{aligned}
W_1(b|m,M) &= \int_0^m \exp\left[-bs - s\int_m^M dF(z) + \int_m^1 z\,dF(z)\right] ds \\
&= \int_0^m \exp\left\{-s[b+F(M)-F(m)] + \int_m^1 z\,dF(z)\right\} ds \\
&= \exp\left[\int_m^1 z\,dF(z)\right][b+F(M)-F(m)]^{-1} \\
&\quad \cdot \{1-\exp[-mb - mF(M) + mF(m)]\}, \\
W_2(b|m,M) &= \int_m^M \exp\left[-bs - \int_m^s z\,dF(z) - \int_s^M s\,dF(z) + \int_m^1 z\,dF(z)\right] ds \\
&= \int_m^M \exp\left\{-bs + \int_s^1 z\,dF(z) - s[F(M)-F(s)]\right\} ds \\
&= \int_m^M \exp\left[-bs - \int_s^1 F(z)\,dz + F(1) - sF(M)\right] ds, \\
W_3(b|m,M) &= \int_m^\infty \exp\left[-bs - \int_m^M z\,dF(z) + \int_m^1 z\,dF(z)\right] ds \\
&= \int_m^\infty \exp\left[-bs - \int_1^M z\,dF(z)\right] ds \\
&= b^{-1}\exp\left[-bM - \int_1^M z\,dF(z)\right].
\end{aligned}$$

The proof is complete.

3. Structure of the Random Set S_0 when Cutout Length Is not Bounded away from Zero and/or Infinity

In this section we consider a general finite nondecreasing and left continuous function $F(z)$ defined for $0<z<\infty$. If $F(z)$ is truncated to $]m,M]$, that is, made constant for $0<z\leqq m$ and for $M<z<\infty$, the results of Section 2 are applicable. One can treat the general case by first truncating z to $]m,M]$ and then letting $m\to 0$ and $M\to\infty$.

Passage to the Limit $M\to\infty$. From Proposition 2.5, it follows that if $F(\infty)=\infty$, then there is a.s. no point of S_0 in $]0,t]$ $(0<t<\infty)$. Therefore, a necessary condition of non degeneracy of S_0 is that $F(\infty)<\infty$. There is no loss of generality in writing $F(\infty)=0$. This is condition a) of Theorem 1.8.

When $M \to \infty$, the term $W_3(b|m, M)$ of Proposition 2.12 tends to zero for all b $(0 \leqq b \leqq \infty)$.

Passage to the Limit $m \to 0$. Observe that a sufficient condition for the convergence of $S_0(\omega)$ to a nondegenerate limit random set is that

$$\Delta T = T(\omega, x|m, M) - T(\omega, 0|m, M)$$

converge to a nondegenerate limit r.f., while the a.s. convergence of ΔT to 0 is sufficient for $S_0(\omega) \cap]0, t]$ to be a.s. empty. We shall be content to explore these sufficient conditions. (I wonder wheter one could obtain a generalization by studying the limits for $m \to 0$ of all r.f.s $\Delta T(\omega, \varphi(x, m)\, m, M)$ where $\varphi(x)$ is monotone increasing; I have ascertained that with $\varphi(x, m) = \psi(m)\, x$ one gains nothing, but I have not gone beyond. This is one question I shall leave open.)

The limit behavior of ΔT depends on the limit behavior of T^* and thus on the limit behavior of $W_1 + W_2$. The limit behavior of W_2 depends on the value of

$$H(F) = \int_0^1 \exp\left[\int_s^1 |F(z)|\, dz\right] ds,$$

to be denoted by H where there is no ambiguity. Either $H < \infty$ or $H = \infty$. When $H < \infty$, then W converges monotonically to a continuous function of b $(0 \leqq b < \infty)$ that tends to zero as $b \to \infty$ and is finite for $0 < b < \infty$. On the contrary, when $H = \infty$, then for $0 \leqq b < \infty$, $\lim_{m \to 0} W_2 = \infty$.

Next, the limit behavior of W_1 depends on the behavior of

$$K(m, F) = \exp\left[\int_m^1 z\, dF(z)\right] [|F(m)|]^{-1} \{1 - \exp[-m|F(m)|]\}$$

again denoted by $K(m)$ when there is no ambiguity. $K(m)$ may either have a limit K or oscillate without bound. According to its behavior and to the value of H, the function F can be classified as follows.

When $H = \infty$ and/or $K = \infty$, then for every $x > 0$,

$$Pr\{\lim_{m \to 0} \Delta T(x|m, M) = 0\} = \lim_{b \to \infty} \lim_{m \to 0} T^*(b, x|m, M) = 1.$$

Therefore, for every $t > 0$, $Pr\{S_0(\omega) \cap]0, t] = \emptyset\} = 1$, which proves the assertions concerning H and K in the second part of Theorem 1.8. Example: $F(z) = -Q/z$, with $Q \geqq 1$.

When $H < \infty$ and $K < \infty$, then the first part of Theorem 1.8 is proved. Also,

$$Pr\{\lim_{m \to 0} \Delta T(x|m, M) = 0\} = \lim_{b \to \infty} \lim_{m \to 0} T^*(b, x|m, M) = \exp(-x/K) < 1.$$

In particular, when $K = 0$, then $\lim_{m \to 0} \Delta T > 0$ a.s.

Example: $K = 0$ when $F(z) = -Q/z$ with $Q < 1$. No example when $H < \infty$ and $K > 0$ is known to me. The possibility of such example is a second question I shall leave open. This finishes the proof of Theorem 1.8.

The case when $H < \infty$ and $K(m)$ has no limit will be left open.

4. Almost Sure Covering of R by Cutouts

Definition 4.1. When $Pr\{S \cap [t', t''] = \emptyset\} = 1$ for all finite t', t'', the cutouts are said to a.s. *cover* R. When the cutouts a.s. cover R but cease to a.s. cover R when $F(z)$ is replaced by $F(1)$ for all $z < 1$, one has a *high frequency* a.s. *cover*. When the cutouts a.s. cover R but cease to a.s. cover R when $F(z)$ is replaced by $F(1)$ for all $z > 1$, one has a *low frequency* a.s. *cover*.

Proposition 4.2. *R is a.s. covered iff it is low frequency and/or high frequency a.s. covered.*

Proof. High frequency and low frequency cutouts are independent.

Proposition 4.3. *A necessary and sufficient condition for low frequency a.s. covering is* $\lim_{M \to \infty} B(m, M) = 0$.

Proof. Necessity: By Proposition 2.3, $Pr\{t' \in S(m, M) = B(m, M)$. When the condition in this proposition fails, t' itself is not a.s. covered, and a fortiori $[t', t'']$ is not a.s. low frequency covered. Sufficiency: The probability of the event that $[0, t]$ is not covered by any single cutout is readily seen to be

$$\exp\left\{ -\int_t^\infty (z-t)\, dF(z) \right\}.$$

When the condition in this proposition holds, this probability vanishes for all t. Hence, every $[t', t'']$ is a.s. low frequency covered.

Remark 4.4. A.s. low frequency covering is equivalent to a.s. covering by a single cutout.

Lemma 4.5.

$$Pr\{[0, t] \subset R - S(m, M) | 0 \notin S(m, M)\} = \int_t^\infty Pr\{U(m, M) \geqq u\}\, du / E[U(m, M)]$$

$$= \frac{\int_t^\infty Pr[U(m, M) \geqq u]\, du}{\int_0^\infty Pr[U(m, M) \geqq u]\, du}.$$

Proof. This probability is unchanged if the scale of time is collapsed in such a way that the acts are reduced to single points while the intermissions remain unchanged. By such a transformation, $S(m, M)$ is made into an ordinary renewal process, for which the result is classical (see [2]).

Lemma 4.6. *There exists some* $\varepsilon(m, M)$ *between* 0 *and* 1 *such that*

$$Pr\{[0, t] \not\subset R - S(m, M)\} = A(m, M) B(m, M) \int_0^t Pr\{U(m, M) \geqq u\}\, du + B(m, M) \varepsilon(m, M)$$

$$= \exp\left[-\int_1^M z\, dF(z) \right] \int_0^t L(u | m, M)\, du + B(m, M)\, \varepsilon(m, M).$$

Proof. The denominator in the last expression of Lemma 4.5 is known to be equal to $EU(m, M)$. Further,

$$\frac{EU(m, M)}{Pr\{0 \notin S(m, M)\}} = \frac{EY(m, M)}{Pr\{0 \in S(m, M)\}}.$$

By Proposition 2.2, $EY = 1/A$, and therefore

$$EU(m, M) = [1 - B(m, M)]/A(m, M)\, B(m, M).$$

On the other hand,

$$Pr\{[0, t] \not\subset R - S(m, M)\} = Pr\{[0, t] \not\subset R - S(m, M) | 0 \notin S(m, M)\}\, Pr\{0 \notin S(m, M)\}$$
$$+ Pr\{[0, t] \not\subset R - S(m, M) | 0 \in S(m, M)\}\, Pr\{0 \in S(m, M)\}.$$

The first addend is evaluated by Lemma 4.5, after inserting the above obtained expression for EU. Then this addend is evaluated by Theorem 2.10. The second addend lies between 0 and 1.

Theorem 4.7. *A sufficient condition for a.s. high frequency covering is that the limit for $m \to 0$ of $T(\omega, x|m, M)$ is degenerate, more specially that $H = \infty$ and/or $K = \infty$. A sufficient condition for high frequency covering to have a probability less than 1 is that the limit for $m \to 0$ of $T(\omega, x|m, M)$ is non degenerate, namely that $H < \infty$ and $K < \infty$.*

Proof. When $T(\omega, x|m, M)$ is asymptotically degenerate, $\lim_{m \to 0} B(m, M) = 0$, and

$$\int_0^t L(u|m, M)\, du = 0, \quad \textit{so} \;\; Pr\{[0, t] \not\subset R - S(m, M)\} \to 0.$$

When $T(\omega, x|m, M)$ is asymptotically non degenerate,

$$\lim_{m \to 0} \int_0^t L(u|m, M)\, du > 0 \quad \text{so} \;\; Pr\{[0, t] \not\subset R - S(m, M)\}$$

remains bounded away from 0 for all t. The same applies to any other interval $[t', t'']$, which proves the theorem.

Corollary 4.8. *When $F(z) = -Q/z$, R is a.s. high frequency covered when $Q \geqq 1$, and is high frequency covered with probability less than 1 when $Q < 1$.*

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