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The General Random Ergodic Theorem. II

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§ 1. Introduction

The present paper continues the investigations of "The general random ergodic theorem I", and the following problem is considered:

Suppose $(\Omega, \mathscr{B}, m)$ and (X, Σ, μ) are two σ-finite measure spaces, and $\{f(t,\omega)(x)\}$ and $\{g(t,\omega)(x)\}$ are two (t,ω,x)-measurable families of functions defined on X. Then we can ask under what conditions there exists a function $F^*(\omega,x)$ in $L_1(\Omega \otimes X)$ such that

$$\frac{\int_0^T f(t,\omega)(x)\,dt}{\int_0^T g(t,\omega)(x)\,dt}$$

converges (as $T \to \infty$) to the function $F^*(\omega,x)$ almost everywhere on $\Omega \otimes X$.

The problem analogous to a continuous case arises for discrete families of measurable functions.

In § 2 we state the general random ergodic theorems for a continuous parameter case in connection with this question, of which proofs appear in § 3. In § 4 we shall indicate similar results in a discrete parameter case.

The idea of the proofs, which appeal to the representation into an infinite product space, is due to the author [1] being a slight modification of the method given by Doob [2].

§ 2. The Continuous Case

Let $(\Omega, \mathscr{B}, m)$ and (X, Σ, μ) be two σ-finite measure spaces and $\{S_t\colon t \geqq 0\}$ a measurable non-singular semi-flow on $(\Omega, \mathscr{B}, m)$. We recall that, by virtue of the non-singularity and the group property of $\{S_t\}$, there exists a family $\{\beta(t,\omega)\}$ of positive integrable functions such that, by the Radon-Nikodym theorem,

$$m(S_t A) = \int_A \beta(t,\omega)\,dm$$

for any $A \in \mathscr{B}$, and

$$\beta(t+s,\omega) = \beta(t,\omega)\,\beta(s, S_t\,\omega)$$

for almost all $\omega \in \Omega$, and that if we take $\{S_t\}$ to be measure preserving, the density function $\beta(t,\omega)$ is identically equal to one. Throughout this paper, we shall restrict ourselves to real valued functions, and further, unless otherwise stated, p will be an arbitrary positive integer and each E_k $(k=1,2,\ldots)$ will be a 1-dimensional Borel set.

The following theorem, a continuous analogue of the generalization of Hopf's theorem [3] proved by Halmos [4] and Dowker [5], plays an essential role in our present paper.

Theorem 1. *Let $f(\omega)$ be an integrable function on Ω and $g(\omega)$ a non-negative measurable function on Ω satisfying*

$$\int_0^\infty g(S_t\omega)\,\beta(t,\omega)\,dt=\infty$$

almost everywhere on Ω. Then the limit

$$\lim_{T\to\infty}\frac{\int_0^T f(S_t\omega)\,\beta(t,\omega)\,dt}{\int_0^T g(S_t\omega)\,\beta(t,\omega)\,dt}$$

exists and is finite almost everywhere on Ω.

Now suppose $f=\{f(t,\omega)(x)\colon t\geqq 0\}$ and $g=\{g(t,\omega)(x)\colon t\geqq 0\}$ are two (t,ω,x)-measurable families of functions defined on X. We consider the mapping $H_{(f,g)}$ from $\Omega\otimes X$ to $(\overset{+}{\otimes} R_t)\otimes(\overset{+}{\otimes} R_t)$[1] defined by

$$H_{(f,g)}(\omega,x)=(\theta,\sigma)$$
$$\xi_t(\theta)=f(t,\omega)(x),\qquad \eta_t(\sigma)=g(t,\omega)(x)$$

where $\xi_t(\theta)$ and $\eta_t(\sigma)$ are the t-coordinate functions of θ and σ respectively, and consider the shift transformation semi-group $\{Q_t\colon t\geqq 0\}$ on $(\overset{+}{\otimes} R_t)^2$ defined by

$$Q_t\big((\theta_u,\sigma_u)\colon u\geqq 0\big)=\big((\theta_{u+t},\sigma_{u+t})\colon u\geqq 0\big).$$

By $\mathscr{L}_{(f,g)}$ we stand for the σ-field generated by all finite unions of sets of the form: for every sequence $(t_1,\ldots,t_n)$ with $0\leqq t_1<\cdots<t_n<\infty$,

$$\{(\theta,\sigma)\colon \xi_{t_1}(\theta)\in E_1,\ldots,\xi_{t_p}(\theta)\in E_p,\ \eta_{t_1}(\sigma)\in E_{p+1},\ldots,\eta_{t_p}(\sigma)\in E_{2p}\},$$

and put, for any $\Lambda\in\mathscr{L}_{(f,g)}$,

$$\lambda_{(f,g)}(\Lambda)=m\otimes\mu(H_{(f,g)}^{-1}\Lambda).$$

Obviously $\{Q_t\}$ is $\mathscr{L}_{(f,g)}$-measurable. Moreover, for a function $h(\omega)$ fixed in $L_1(\Omega)$, we consider the mapping Π_h from Ω to $(\overset{+}{\otimes} R_t)$ given by $\Pi_h\omega=\theta$, where $\xi_t(\theta)=h(t,\omega)$, $h(t,\omega)=h(S_t\omega)$ and denote by $\mathscr{A}_h$ the σ-field generated by all finite unions of sets of the form: for any sequence $(t_1,\ldots,t_n)$ of non-negative real numbers with $t_1<\cdots<t_n$,

$$\{\theta\colon \xi_{t_1}(\theta)\in E_1,\ldots,\xi_{t_p}(\theta)\in E_p\}$$

and define $\pi_h(B)=m(\Pi_h^{-1}B)$ for any $B\in\mathscr{A}_h$.

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[1] $\overset{+}{\otimes} R_t=\underset{t\geqq 0}{\otimes} R_t,\ R_t=(-\infty,\infty)$.

We say that the pair (f, g) of two families has the property A if for any $\Lambda \in \mathscr{L}_{(f,g)}$, there exists a positive constant K such that $\lambda_{(f,g)}(Q_t^{-1}\Lambda) \leqq K \cdot \lambda_{(f,g)}(\Lambda)$ for any t with $t \geqq 0$, and that the pair (f, g) has the property B if for any $\Lambda \in \mathscr{L}_{(f,g)}$ with $\lambda_{(f,g)}(\Lambda) > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} \int_0^T \lambda_{(f,g)}(Q_t^{-1}\Lambda)\, dt > 0.$$

Then our main result reads as follows:

Theorem 2. *Let $f = \{f(t,\omega)(x) \colon t \geqq 0\}$ and $g = \{g(t,\omega)(x) \colon t \geqq 0\}$ be two (t,ω,x)-measurable families of functions defined on X satisfying that*

(i) *$f(0,\omega)(x)$ belongs to $L_1(\Omega \otimes X)$,*

(ii) *for almost all $(\omega, x) \in \Omega \otimes X$, $g(t,\omega)(x)$ is positive for any $t \geqq 0$ and $\int_0^\infty g(t,\omega)(x)\, dt = \infty$,*

(iii) *the pair (f, g) has the properties A and B.*

Then there exists a null set N in Ω such that, for any $\omega \in \Omega - N$, the limit

$$\lim_{T\to\infty} \frac{\int_0^T f(t,\omega)(x)\, dt}{\int_0^T g(t,\omega)(x)\, dt}$$

exists and is finite for almost all $x \in X$.

In case the measures m and μ are finite, take $\{S_t \colon -\infty < t < \infty\}$ to be a measurable non-singular flow and consider the product set $(\otimes R_t) \otimes (\otimes R_t)$, where $\otimes R_t = \bigotimes_{-\infty < t < \infty} R_t$, $R_t = (-\infty, \infty)$. Then we have the following

Theorem 3. *Suppose $\{f(t,\omega)(x) \colon -\infty < t < \infty\}$ and $\{g(t,\omega)(x) \colon -\infty < t < \infty\}$ are two (t,ω,x)-measurable families of functions defined on X satisfying that*

(i) *$f(0,\omega)(x)$ belongs to $L_1(\Omega \otimes X)$,*

(ii) *$g(t,\omega)(x)$ is positive for almost all $(\omega, x) \in \Omega \otimes X$ and for any t, and*

(iii) *there exists a positive constant K such that*

$$\begin{aligned}\limsup_{T\to\infty} \frac{1}{T} \int_0^T \mu\{x \colon & f(t_1+t,\omega)(x) \in E_1, \ldots, f(t_p+t,\omega)(x) \in E_p,\\ & g(t_1+t,\omega)(x) \in E_{p+1}, \ldots, g(t_p+t,\omega)(x) \in E_{2p}\}\, \beta(-t,\omega)\, dt\\ \leqq K \cdot \mu\{x \colon & f(t_1,\omega)(x) \in E_1, \ldots, f(t_p,\omega)(x) \in E_p,\\ & g(t_1,\omega)(x) \in E_{p+1}, \ldots, g(t_p,\omega)(x) \in E_{2p}\}\end{aligned}$$

holds almost everywhere on Ω, where $(t_1, \ldots, t_p)$ is an arbitrary sequence of real numbers with $t_1 < \cdots < t_p$. Then, except for a set of $m \otimes \mu$-measure zero, the limit

$$\lim_{T\to\infty} \frac{\int_0^T f(t,\omega)(x)\, dt}{\int_0^T g(t,\omega)(x)\, dt}$$

exists and is finite.

Remark. Note that Theorem 2 extends Theorem 4 obtained by the author in [1] which generalizes Doob's ergodic theorem [2] and that Theorem 3 extends Theorem 1 obtained by the author in the same paper.

The following corollary together with Theorem 1 generalizes Anzai's theorem [6] and Hopf's ergodic theorem [3] which extends Birkhoff-Khintchine's theorem [7].

Corollary 1. *Let* $\{T(t,\omega)x\colon t\geqq 0\}$ *be a* (t,ω,x)*-measurable quasi semi-group of endomorphisms of* X *with respect to* $\{S_t\colon t\geqq 0\}$. *If* m *is finite, then for any* $f\in L_1(X)$ *and a positive measurable function* g *defined on* X *with* $\int_0^\infty g\big(T(t,\omega)x\big)\,dt=\infty$ $m\otimes\mu$*-almost everywhere, the average*

$$\frac{\int_0^T f\big(T(t,\omega)x\big)\,dt}{\int_0^T g\big(T(t,\omega)x\big)\,dt}$$

converges (as $T\to\infty$) *to a finite function almost everywhere on* $\Omega\otimes X$.

§ 3. Proofs of Theorems

The proof of Theorem 1 depends essentially on the following lemmas which are the continuous analogues of those given by Dowker [5].

Lemma 1. *Let* $u(t)$ *be a real valued measurable function defined on* $[0,\infty)$ *and* T *any fixed positive real number. Suppose*

$$\sup_{0<r\leqq T}\int_0^r u(t+s)\,dt\geqq 0$$

for any non-negative real number s. *Then*

$$\int_0^v u(t)\,dt+\int_v^{v+T}\big(u(t)\big)^+\,dt\geqq 0$$

for any positive real number v, *where* $\big(u(t)\big)^+=\max\big(u(t),0\big)$.

Lemma 2. *Let* $f(\omega)$ *be a measurable function defined on* Ω *such that either the positive part or the negative part is integrable. If we put*

$$E(\alpha)=\left\{\omega\colon \sup_{0<v\leqq\alpha}\int_0^v f(S_t\omega)\,\beta(t,\omega)\,dt\geqq 0\right\}$$

for an arbitrary positive rational number α, *then*

$$\int_{E(\alpha)} f(\omega)\,dm\geqq 0.$$

In this time, the proof of Theorem 1 can be easily proved by the same manner as used by Halmos [4].

Proof of Theorem 2. We shall establish the proof by returning to Theorem 1 in which $\{S_t\colon t\geqq 0\}$ is considered a measurable semi-flow.

Let (f, g) be the pair given in the theorem and $h(\omega)$ belong to $L_1(\Omega)$. We define the mapping φ of $\Omega\otimes X$ into $(\overset{+}{\otimes} R_t)\otimes(\overset{+}{\otimes} R_t)\otimes(\overset{+}{\otimes} R_t)$ (simply, $(\overset{+}{\otimes} R_t)^3$) as follows: $\varphi(\omega, x)=(\tau, \theta, \sigma)$, where $\zeta_t(\tau)=h(S_t\omega)$, $\xi_t(\theta)=f(t, \omega)(x)$ and $\eta_t(\sigma)=g(t, \omega)(x)$, and consider the shift transformation semi-group $\{Z_t: t\geqq 0\}$ on $(\overset{+}{\otimes} R_t)^3$ given by

$$Z_t\big((\tau_u, \theta_u, \sigma_u): u\geqq 0\big)=\big((\tau_{u+t}, \theta_{u+t}, \sigma_{u+t}): u\geqq 0\big).$$

Furthermore, we denote by $\mathscr{L}$ the σ-field generated by all finite unions of sets of the form

$$\{(\tau, \theta, \sigma): \zeta_{t_1}(\tau)\in E_1, \ldots, \zeta_{t_p}(\tau)\in E_p, \xi_{t_1}(\theta)\in E_{p+1}, \ldots, \xi_{t_p}(\theta)\in E_{2p}, \\ \eta_{t_1}(\sigma)\in E_{2p+1}, \ldots, \eta_{t_p}(\sigma)\in E_{3p}\},$$

and put $\lambda(\Lambda)=m\otimes\mu(\varphi^{-1}\Lambda)$ for any $\Lambda\in\mathscr{L}$. Clearly $\{Z_t\}$ is $\mathscr{L}$-measurable and λ is a σ-finite measure which is not necessarily invariant under $\{Z_t\}$. The following three lemmas contribute essentially to the proof of the theorem.

Lemma 3. 1°. *$\{Z_t\}$ is non-singular with respect to λ.*

2°. *There exists such a positive constant K as for any $\Lambda\in\mathscr{L}$ with $\lambda(\Lambda)>0$,*

$$0<\limsup_{T\to\infty}\frac{1}{T}\int_0^T \lambda(Z_t^{-1}\Lambda)\,dt\leqq K\cdot\lambda(\Lambda).$$

Proof. Cf. [1], Lemma 4.

From Lemma 3 we have

Lemma 4. *There exists a σ-finite measure ν on $\mathscr{L}$ satisfying*

(i) *$\nu(\Lambda)\leqq K^2\lambda(\Lambda)$ for $\Lambda\in\mathscr{L}$,*

(ii) *ν is equivalent to λ, and*

(iii) *ν is invariant under $\{Z_t\}$.*

Proof. Cf. [1], Lemma 5.

Lemma 5. *If Φ is an arbitrary 3p-dimensional Borel function, then*

$$\int_{(\otimes R_t)^3} \Phi\big(\zeta_{t_1}(\tau), \ldots, \zeta_{t_p}(\tau), \xi_{t_1}(\theta), \ldots, \xi_{t_p}(\theta), \eta_{t_1}(\sigma), \ldots, \eta_{t_p}(\sigma)\big)\,d\lambda(\tau, \theta, \sigma) \\ = \int_{\Omega\otimes X} \Phi\big(h(t_1, \omega), \ldots, h(t_p, \omega), f(t_1, \omega)(x), \ldots, f(t_p, \omega)(x), \\ g(t, \omega)(x), \ldots, g(t, \omega)(x)\big)\,dm\otimes\mu(\omega, x)$$

where $h(t_i, \omega)=\zeta_{t_i}(\tau)$, $f(t_i, \omega)(x)=\xi_{t_i}(\theta)$ and $g(t_i, \omega)(x)=\eta_{t_i}(\sigma)$ $(1\leqq i\leqq p)$.

Proof. Cf. [1], Lemma 2.

Let us continue the proof of the theorem.

Notice that Lemma 4 implies that $\{Z_t\}$ is a measurable semi-flow on $((\overset{+}{\otimes} R_t)^3, \mathscr{L}, \nu)$. In order to apply Theorem 1 to $\{Z_t\}$, we consider the functions $\psi_0^{(1)}$ and $\psi_0^{(2)}$ given by

$$\psi_0^{(1)}(\tau, \theta, \sigma)=\xi_0(\theta) \quad\text{and}\quad \psi_0^{(2)}(\tau, \theta, \sigma)=\eta_0(\sigma).$$

Then, for any $t \geqq 0$,

$$\psi_0^{(1)}(Z_t(\tau, \theta, \sigma)) = \psi_t^{(1)}(\tau, \theta, \sigma)$$

and

$$\psi_0^{(2)}(Z_t(\tau, \theta, \sigma)) = \psi_t^{(2)}(\tau, \theta, \sigma).$$

In view of Lemma 4, Lemma 5 and the conditions (i), (ii) in the theorem, we see that $\psi_0^{(1)}$ is ν-integrable, $\psi_0^{(2)}$ is positive and

$$\int_0^\infty \psi_0^{(2)}(Z_t(\tau, \theta, \sigma))\, dt = \infty \qquad \text{for almost all } (\tau, \theta, \sigma) \in (\overset{+}{\otimes} R_t)^3.$$

Thus, from Theorem 1, it follows that the limit

$$\lim_{T\to\infty} \frac{\int_0^T \psi_0^{(1)}(Z_t(\tau, \theta, \sigma))\, dt}{\int_0^T \psi_0^{(2)}(Z_t(\tau, \theta, \sigma))\, dt}$$

exists and is finite ν-almost everywhere on $(\overset{+}{\otimes} R_t)^3$. Again, by Lemma 4 and Lemma 5, we come to the desired conclusion.

Proof of Theorem 3. In the sequel, we make use of the mapping and the $\mathscr{L}$-measurable shift transformation group $\{Z_t\colon -\infty < t < \infty\}$ as in the proof of the preceding theorem.

The following lemmas stand by the proof of the theorem.

Lemma 6. *For the constant K given in the theorem, it follows that*

$$\begin{aligned}
\limsup_{T\to\infty} \frac{1}{T} \int_0^T m \otimes \mu\{(\omega, x)\colon\ & h(t_1+t, \omega) \in E_1, \ldots, h(t_p+t, \omega) \in E_p, \\
& f(t_1+t, \omega)(x) \in E_{p+1}, \ldots, f(t_p+t, \omega)(x) \in E_{2p}, \\
& g(t_1+t, \omega)(x) \in E_{2p+1}, \ldots, g(t_p+t, \omega)(x) \in E_{3p}\}\, dt \\
\leqq K \cdot m \otimes \mu\{\omega, x)\colon\ & h(t_1, \omega) \in E_1, \ldots, h(t_p, \omega) \in E_p, f(t_1, \omega)(x) \in E_{p+1}, \ldots \\
& f(t_p, \omega)(x) \in E_{2p}, g(t_1, \omega)(x) \in E_{2p+1}, \ldots, g(t_p, \omega)(x) \in E_{3p}\},
\end{aligned}$$

where $(t_1, \ldots, t_p)$ is an arbitrary finite sequence of real numbers with $t_1 < \cdots < t_p$.

Proof. Cf. [1], Lemma 1.

This lemma yields the next

Lemma 7. 1°. *For the constant K in Lemma* 6, *and for any $\Lambda \in \mathscr{L}$,*

$$\limsup_{T\to\infty} \frac{1}{T} \int_0^T \lambda(Z_t^{-1}\Lambda)\, dt \leqq K\lambda(\Lambda).$$

2°. *There exists a finite measure ν on $\mathscr{L}$ such that*

(i) $\nu(\Lambda) \leqq K\lambda(\Lambda)$ *for* $\Lambda \in \mathscr{L}$,

(ii) ν *is invariant under* $\{Z_t\}$, *and*

(iii) $\nu(\Lambda) = \lambda(\Lambda)$

for any $\{Z_t\}$-invariant set $\Lambda \in \mathscr{L}$.

Proof. Cf. [1], Lemma 3.

To establish the proof of the theorem, consider the functions $\psi_0^{(1)}$ and $\psi_0^{(2)}$ given by

$$\psi_0^{(1)}(\tau,\theta,\sigma)=\xi_0(\theta) \quad \text{and} \quad \psi_0^{(2)}(\tau,\theta,\sigma)=\eta_0(\sigma).$$

Then we have, for any t,

$$\psi_0^{(1)}(Z_t(\tau,\theta,\sigma))=\psi_t^{(1)}(\tau,\theta,\sigma)$$

and

$$\psi_0^{(2)}(Z_t(\tau,\theta,\sigma))=\psi_t^{(2)}(\tau,\theta,\sigma).$$

Since Lemma 7 implies that $\{Z_t\}$ is a measurable flow, according to Lemma 5, Lemma 7 and the conditions (i), (ii) in the theorem, one can easily verify that $\psi_0^{(1)}$ is ν-integrable, $\psi_0^{(2)}$ is positive and

$$\int_0^\infty \psi_0^{(2)}(Z_t(\tau,\theta,\sigma))\,dt=\infty$$

for almost all $(\tau,\theta,\sigma)\in(\otimes R_t)^3$ (see Hopf [3]). Accordingly, by Theorem 1, the limit

$$\lim_{T\to\infty}\frac{\int_0^T \psi_0^{(1)}(Z_t(\tau,\theta,\sigma))\,dt}{\int_0^T \psi_0^{(2)}(Z_t(\tau,\theta,\sigma))\,dt}$$

exists and is finite ν-almost everywhere on $(\otimes R_t)^3$, so that the desired conclusion follows from Lemma 5 and Lemma 7.

§ 4. The Discrete Case

In this section, we shall state similar results in a discrete case as in a continuous case.

Let $\{S_n\colon n\geqq 0\}$ be a non-singular discrete semi-flow on $(\Omega,\mathscr{B},m)$. Then, by the non-singularity and the group property of $\{S_n\}$, there exists a family $\{\beta(n,\omega)\}$ of multiplicative density functions with respect to $\{S_n\}$ (see § 2). Here we need the following lemma due to E. Hopf (cf. [3, 4, 5]).

Lemma 8. *Let $f(\omega)$ be an integrable function on Ω and $g(\omega)$ a non-negative measurable function on Ω with $\sum_{k=0}^{\infty} g(S_k\omega)=\infty$ for almost all $\omega\in\Omega$, where $\{S_n\}$ is measure preserving. Then the limit*

$$\lim_{n\to\infty}\frac{\sum_{k=0}^{n-1} f(S_k\omega)}{\sum_{k=0}^{n-1} g(S_k\omega)}$$

exists and is finite almost everywhere on Ω.

Owing to this lemma, Theorem 2 becomes

Theorem 4. *Let $f=\{f(n,\omega)(x)\colon n\geqq 0\}$ and $g=\{g(n,\omega)(x)\colon n\geqq 0\}$ be two (ω,x)-measurable families of functions defined on X satisfying that*

(i) *$f(0,\omega)(x)$ belongs to $L_1(\Omega\otimes X)$,*

(ii) *$g(n,\omega)(x)$ is positive for each $n\geqq 0$ and $\sum_{k=0}^{\infty} g(k,\omega)(x)=\infty$ for almost all $(\omega,x)\in\Omega\otimes X$,*

(iii) *the pair (f,g) has the properties A and B (see § 2).*

Then, except for a set of $m\otimes\mu$-measure zero,

$$\lim_{n\to\infty}\frac{\sum_{k=0}^{n-1} f(k,\omega)(x)}{\sum_{k=0}^{n-1} g(k,\omega)(x)}$$

exists and is finite.

In case m and μ are finite, Theorem 3 becomes

Theorem 5. *Suppose $\{f(n,\omega)(x)\colon n=0,\pm 1,\ldots\}$ and $\{g(n,\omega)(x)\colon n=0,\pm 1,\ldots\}$ are two (ω,x)-measurable families of functions defined on X satisfying*

(i) *$f(0,\omega)(x)$ belongs to $L_1(\Omega\otimes X)$,*

(ii) *$g(n,\omega)(x)$ is positive for each n and for almost all $(\omega,x)\in\Omega\otimes X$,*

(iii) *there exists a positive constant K such that*

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{n}\sum_{R=0}^{n-1}\mu\{x\colon & f(k_1+k,\omega)(x)\in E_1,\ldots,f(k_p+k,\omega)(x)\in E_p,\\ & g(k_1+k,\omega)(x)\in E_{p+1},\ldots,g(k_p+k,\omega)(x)\in E_{2p}\}\cdot\beta(-k,\omega)\\ \leqq K\cdot\mu\{x\colon & f(k_1,\omega)(x)\in E_1,\ldots,f(k_p,\omega)(x)\in E_p,\\ & g(k_1,\omega)(x)\in E_{p+1},\ldots,g(k_p,\omega)(x)\in E_{2p}\}\end{aligned}$$

holds almost everywhere on Ω, where $(k_1,\ldots,k_p)$ is an arbitrary sequence of integers with $k_1<\cdots<k_p$. Then, except for a set of $m\otimes\mu$-measure zero, the limit

$$\lim_{n\to\infty}\frac{\sum_{k=0}^{n-1} f(k,\omega)(x)}{\sum_{k=0}^{n-1} g(k,\omega)(x)}$$

exists and is finite.

In conclusion we note that Theorem 4 and Theorem 5 extend both Gladysz's theorems [8] and Tsurumi's theorem [9] which generalizes Doob's theorem [2] and Hopf's theorem [3].

Corollary 2. *Let $\{T(n,\omega)\,x\colon n\geqq 0\}$ be a (ω,x)-measurable discrete quasi semi-group of endomorphisms of X with respect to $\{S_n\colon n\geqq 0\}$. If m is finite, then for any $f\in L_1(X)$ and a positive measurable function g defined on X with $\sum_{k=0}^{\infty} g\big(T(k,\omega)\,x\big)=\infty$*

for almost all $(\omega, x) \in \Omega \otimes X$, *the limit*

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} f\big(T(k,\omega)\,x\big)}{\sum_{k=0}^{n-1} g\big(T(k,\omega)\,x\big)}$$

exists and is finite almost everywhere on $\Omega \otimes X$.

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