Z. Wahrscheinlichkeitstheorie verw. Gebiete
34, 205–223 (1976)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Limit Theorems for the Motion of a Poisson System of Independent Markovian Particles with High Density

Anders Martin-Löf
Department of Mathematics, Uppsala University
Sysslomansgatan 8, S-75223 Uppsala, Sweden

Introduction

In this paper we want to relate two different aspects of diffusion theory, the probabilistic and the physical, deterministic aspect. In both the diffusion or forward equation $\frac{d\mu}{dt}=A^*\mu$ for the density μ is basic. (A is the generator and its adjoint A^* the Fokker-Planck operator.) In the probabilistic theory one studies the random motion of an individual particle and μ is interpreted as the probability density for its position in space. In the physical theory one studies a "gas" of particles and μ is interpreted as the actual density of particles in space. The former is a non-random quantity, but if the particles of the gas move randomly the latter is a random quantity, and one can ask in what sense they are related. The physicists explanation is that in physics there are in general many particles even in an "infinitesimal" volume, so in some sense the density is large and by some law of large numbers its fluctuations ought to be small, so it can be well approximated by its average which is given by the probability density.

Here we consider the simplest possible system where this question can be rigorously studied, namely the gas consists of non-interacting particles and the motion of each individual one is Markovian and defined by the generator A. At any time the particles form a Poisson system in space with a density determined by the forward equation. In order to obtain a situation where the density is large we take the density as $\rho\cdot\mu$, where μ is fixed and $\rho\to\infty$. It is then straightforwardly shown that the actual random particle density divided by ρ converges in probability to the non-random density μ which is given by the forward equation. This is the law of large numbers for the system. We can also go further and study the fluctuations around equilibrium suitably scaled by $\sqrt{\rho}$ in the limit $\rho\to\infty$. A central limit is proved saying that these fluctuations are in the limit distributed as a certain Gaussian random density field.

The physical theory of fluctuations in continuous media gives a prescription for how this random field $g(t,x)$ can be constructed as follows [2, 5]: Since the random particle density evolves in a Markovian way, and the average motion is that determined by the forward equation one makes the Ansatz that $g(t,x)$ is a Gauss-Markov process determined by the generalized Langevin equation $\frac{dg}{dt}=A^* g+w$ obtained by adding in the forward equation a white noise $w(t,x)$. The correlations of this noise are uniquely determined by the requirement that the correlations of g at any fixed time should have values which are obtained directly from the limit of the Poisson distribution in space at any time:

$$E(g(x,t)\,g(y,t))=\mu(x)\,\delta(x-y)\equiv Q^g(x,y).$$

The relation between the correlations of g, Q^g, and those of w, Q^w, is obtained by solving the equation for g:

$$g(t)=\int_{-\infty}^{t} e^{A^*(t-s)}\,w(s)\,ds$$

and taking averages:

$$Q^g=\int_0^\infty e^{A^*t}\,Q^w\,e^{At}\,dt.$$

This equation can then be solved for Q^w giving the so called fluctuation-dissipation relation: $Q^w=-(A^*Q^g+Q^gA)$.

It is shown how the random field $g(t,x)$ can rigorously be constructed within the theory of random Schwartz distributions. I.e. only smeared averages $X(t,\varphi)=$"$\int g(t,x)\,\varphi(x)\,dx$" are considered for nice test functions φ. We construct a random process $X(t,\cdot)$ whose values are tempered distributions and show that it is a Gauss-Markov process which in an appropriate sense is a solution of the above Langevin equation, so that the heuristic theory can be justified mathematically. It is also shown that the process $X(t,\cdot)$ can be taken to have continuous trajectories with probability one. It can hence be regarded as an infinite dimensional diffusion process, and the reason that it can be constructed without too much difficulty is the fact that it is defined by a linear Langevin equation and therefore Gaussian, and the difficulties encountered e.g. in quantum field theory in constructing non Gaussian generalized random fields need not be overcome. Finally we explicitly treat some typical cases of physical interest. It is not necessary to assume that the Markov motion of the individual particles is actually a diffusion. It can be allowed to be a Markov process of quite general type.

We have only considered the case of completely non-interacting particles. It is an interesting problem to derive similar limit theorems in the more realistic situation when the particles interact. In [6] this problem is discussed for some simplified models in kinetic theory with a non-linear Boltzmann equation for the average density.

1. Construction of a Poisson System of Independent Markovian Particles

We assume that the Markov process describing the motion of each individual particle is defined by known transition probabilities $P(t,x,dy)$, $t\geqq 0$, $x,y\in R^d$.

The state space Σ is hence R^d. They define for each $x \in \Sigma$ a measure P_x on the space of paths $x(t)$, $0 \leqq t \leqq 1$, $\Omega_0 = C[0,1]$ or $D[0,1]$ depending on whether the particles move continuously or can jump. P_x is defined on the σ-algebra of Borel sets in Ω_0 and $P_x(x(0)=x)=1$. Since we want to consider a system in equilibrium we assume that there is also an invariant measure μ on Σ (finite on compact subsets of Σ and hence σ-finite).

We want to construct the Poisson process of independent particles moving according to $P(t, x, dy)$ and having at any time a Poisson distribution in Σ with density μ. It is technically convenient to do this by considering the path space $\Omega = C(-\infty, \infty)$ or $D(-\infty, \infty)$ of an individual particle equipped with the measure v generated by μ and the transition probabilities and then constructing a Poisson system in Ω with density v. In this way all particle distributions in Σ corresponding to different times are constructed simultaneously and basic properties such as the invariance of the Poisson property in time are seen very easily.

Let us first construct the measure v on Ω. If $\mu(\Sigma)$ is infinite partition Σ into $\bigcup_1^\infty \Sigma_j$ with $\mu(\Sigma_j)$ finite and $\{\Sigma_j\}$ disjoint. For each j construct v_j on

$$\bigtimes_{-\infty}^{\infty} C[n, n+1] \quad \text{or} \quad \bigtimes_{-\infty}^{\infty} D[n, n+1] = \bigtimes_{-\infty}^{\infty} \Omega_n$$

from cylinder probabilities obtained by joining the paths $x(t, \omega_n)$, $\omega_n \in \Omega_n$, $n \leqq t \leqq n+1$, giving $x(t)$ the distribution μ for t sufficiently negative, and restricting $x(0)$ to Σ_j. I.e. if C is a cylinder with base $A_m \times \cdots \times A_n$ with A_i measurable $\subset \Omega_i$ for $m \leqq i \leqq n$ and $m \leqq 0$, $n \geqq 0$ put

$$v_j(C) = \int_{(A_m \times \cdots \times A_n) \cap \{x(0) \in \Sigma_j\}} \mu(dx) P_x(d\omega_m) \ldots P_{x(n, \omega_{n-1})}(d\omega_n).$$

These cylinder probabilities are consistent and have finite total mass:

$$v_j\left(\bigtimes_{-\infty}^{\infty} \Omega_n\right) = \mu(\Sigma_j).$$

Hence, since the Ω_n are Polish spaces, the extension theorem [8] tells us that they have a unique extension to a measure on $\bigtimes_{-\infty}^{\infty} \Omega_n$ equipped with the product σ-algebra. When $\Omega_n = C[n, n+1]$ the set of paths which are continuous also when t is an integer has full measure, so v_j is indeed a measure on $\Omega \subset \bigtimes_{-\infty}^{\infty} \Omega_n$. If we finally put $v = \sum_j v_j$ we get the desired (σ-finite) measure on Ω having the property that $v(x(t) \in A) = \mu(A)$ for every t. This also shows that v is finite on compact sets $K \subset \Omega$, because K_0, the image of K under the mapping $\omega \to x(0, \omega)$ is bounded, so that $v(K) \leqq v(x(0) \in K_0) = \mu(K_0) < \infty$.

On the measure space $(\Omega, \mathscr{A})$ and with the measures v_j just constructed we can now build a Poisson process with density v using the procedure described e.g.

in [9]: For each j take a copy of the space $\bigcup_0^\infty \Omega^m$, ($\Omega^n=\Omega\times\cdots\times\Omega$, n times), equipped with the σ-algebra of sets A such that $A\cap\Omega^n\in\mathscr{A}^n$ for all n ($\mathscr{A}^n$ = product σ-algebra $\mathscr{A}\times\cdots\times\mathscr{A}$). Each point in the space can hence be specified as $(\omega_1,\ldots,\omega_N)$ with $N\geqq 0$ and $\omega_i\in\Omega$. Let P_j be the probability measure on this space describing a Poisson process with density v_j:

$$P_j=\sum_0^\infty \frac{e^{-v_j(\Omega)}}{n!}\, v_j^n.$$

I.e. the restriction of P_j to Ω^n is proportional to the product measure $v_j^n=v_j\times\cdots\times v_j$. Finally construct the Poisson process with density v by taking together the independent processes defined by the P_j, i.e. define it by P, the infinite product of the P_j on the product, Π, of the spaces just considered equipped with the product σ-algebra $\mathscr{B}$. A point of Π can hence be specified as a double array $\pi=\{\omega_{j,n},\, j=1,2,\ldots;\, n=1,\ldots,N_j\}$ describing a configuration of points in Ω, and the measure is defined by the recipie: For each $j\geqq 1$ let N_j be independent Poisson variables with averages $v_j(\Omega)$. When the N_j are fixed, for each j the $\omega_{j,1},\ldots,\omega_{j,N_j}$ are chosen independently in Ω according to the probability measure

$$\frac{v_j(d\omega)}{v_j(\Omega)},$$

and the random configuration in Ω is specified by $\pi=\{\omega_{j,n}\}$. The fact that P describes a Poisson process with density v is seen as follows:

Lemma 1. (*Compare* [9].) *For any* $\varphi\in L^1(\Omega,\mathscr{A},v)$, $\pi=\{\omega_{j,n}\}$ *put* $S(\varphi,\pi)=\sum_{j,n}\varphi(\omega_{j,n})$. *Then* $E(e^{i\theta S(\varphi)})=\exp\int(e^{i\theta\varphi(\omega)}-1)\,v(d\omega)$. *In particular, if for any* $A\in\mathscr{A}$ *with* $v(A)<\infty$ *we put* $S(A,\pi)=S(\chi_A,\pi)$ *then for any finite family of such sets* $\{A_i\}$ $\{S(A_i)\}$ *are independent Poisson variables with averages* $v(A_i)$.

Proof. Put first $S_j(\varphi,\pi)=\sum_n\varphi(\omega_{j,n})$. Then

$$E(e^{i\theta S_j(\varphi)})=\sum_0^\infty\frac{e^{-v_j(\Omega)}}{n!}\left(\int e^{i\theta\varphi(\omega)}\,v_j(d\omega)\right)^n=\exp\int(e^{i\theta\varphi(\omega)}-1)\,v_j(d\omega).$$

The $S_j(\varphi)$ are independent, and $S(\varphi)=\sum_j S_j(\varphi)$. The sum converges a.s. independently of the order of the terms if $\sum_j|E(e^{i\theta S_j(\varphi)})-1|$ converges uniformly on every interval $|\theta|\leqq T$ [1]. Using the fact that $|e^z-1|\leqq|z|$ when $Re\,z\leqq 0$ we see that the above series is majorized by $\sum_j\int|e^{i\theta\varphi(\omega)}-1|\,v_j(d\omega)\leqq|\theta|\int|\varphi(\omega)|\,v(d\omega)$, so it is in fact uniformly convergent. The characteristic function of $S(\varphi)$ is then

$$\prod_1^\infty E(e^{i\theta S_j(\varphi)})=\exp\int(e^{i\theta\varphi(\omega)}-1)\,v(d\omega).$$

The average of $S(\varphi)$ and the covariance of $S(\varphi_1)$ and $S(\varphi_2)$ are obtained directly from Lemma 1 by considering also $\varphi_0=t_1\varphi_1+t_2\varphi_2$:

Corollary 1. *For* $\varphi, \varphi_1, \varphi_2 \in L^1(\Omega, \mathscr{A}, \nu)$

$$E(S(\varphi)) = \int \varphi(\omega)\, \nu(d\omega)$$

$$\operatorname{Cov}(S(\varphi_1) S(\varphi_2)) = \int \varphi_1(\omega)\, \varphi_2(\omega)\, \nu(d\omega).$$

The Poisson process just constructed we can now take as the description of a system of independent Markovian particles moving in Σ according to $P(t, x, dy)$ and distributed in Σ according to μ, because of the following facts:

Lemma 2. *For any t the points $x(t, \omega_{j,n})$ form a Poisson system in Σ with density μ, i.e. for any $\varphi \in L^1(\Sigma, \mu)$ putting*

$$S(t, \varphi, \pi) = S(\varphi(x(t, \cdot))) = \sum_{j,n} S(\varphi(x(t, \omega_{j,n})))$$

we have

$$E(e^{i\theta S(t, \varphi)}) = \exp \int (e^{i\theta\varphi(x)} - 1)\, \mu(dx).$$

Proof. This follows immediately from Lemma 1 and the fact that

$$\int (e^{i\theta\varphi(x(t,\omega))} - 1)\, \nu(d\omega) = \int (e^{i\theta\varphi(x)} - 1)\, \mu(dx) \qquad \text{for all } t.$$

In order to express the Markov property of the evolution of these Poisson systems in Σ let $\mathscr{A}_{t_1, t_2} \subset \mathscr{A}$ be the σ-algebra generated by the variables $x(s, \omega)$ with $t_1 < s \leqq t_2$, and $\mathscr{B}_{t_1, t_2} \subset \mathscr{B}$ the one generated by the random variables $S(\varphi, \pi)$ with $\varphi \in L^1(\Omega, \mathscr{A}_{t_1, t_2}, \nu)$. We then have the following Markov property:

$$(\mathscr{A}_t = \mathscr{A}_{-\infty, t}, \mathscr{B}_t = \mathscr{B}_{-\infty, t}).$$

Lemma 3. *For $t > u$ and $\varphi \in L^1(\Omega, \mathscr{A}_{u,t}, \nu)$ we have $E(S(\varphi) | \mathscr{B}_u) = S(u, \varphi')$ with $\varphi'(x) = \int P_{u,x}(d\omega)\, \varphi(\omega)$. ($P_{u,x}$ is the measure on $(\Omega, \mathscr{A}_{u,\infty})$ generated by the transition probabilities and the initial condition $x(u) = x$.) Hence $E(S(\varphi) | \mathscr{B}_u)$ depends only on the Poisson system in Σ at time u and not on its previous values.*

Proof. Take any $\psi \in L^1(\Omega, \mathscr{A}_u, \nu)$. As in Corollary 1 we have

$$E(e^{i\theta S(\psi)} S(\varphi)) = \int e^{i\theta\psi(\omega)} \varphi(\omega)\, \nu(d\omega) \cdot \exp \int (e^{i\theta\psi(\omega)} - 1)\, \nu(d\omega),$$

and by Markov property of ν this is equal to

$$\int e^{i\theta\psi(\omega)} \varphi'(x, u, \omega))\, \nu(d\omega) \cdot \exp \int (e^{i\theta\psi(\omega)} - 1)\, \nu(d\omega) = E(e^{i\theta S(\psi)} S(u, \varphi')).$$

For any finite family $\psi_1, \ldots, \psi_n$ we can take $\psi = \sum_1^n \theta_j \psi_j$ and conclude that

$$E\left(e^{i \sum_1^n \theta_j S(\psi_j)} (S(\varphi) - S(u, \varphi'))\right) = 0$$

for all values of $\theta_1, \ldots, \theta_n$, so by the uniqueness theorem for characteristic functions

$$E(\chi_A(S(\varphi) - S(u, \varphi'))) = 0 \qquad \text{for any } A \in \mathscr{B}_u$$

depending only on $S(\psi_1), \ldots, S(\psi_u)$. Such sets generate $\mathscr{B}_u$, and

$$S(u, \varphi') \in L^1(\Pi, \mathscr{B}_u, P),$$

so it is indeed equal to $E(S(\varphi) | \mathscr{B}_u)$.

Remark. If we want to consider a non-stationary system where the particles form a Poisson system with an arbitrary density μ at time zero and then move according to $P(t,x,dy)$ we can use the same construction with $\Omega = C[0,\infty)$ or $D[0,\infty)$ and ν generated by $P(t,x,dy)$ and μ as initial measure. In this case the particles at time t form a Poisson system with density

$$\mu(t,dy) = \int \mu(dx)\,P(t,x,dy).$$

2. Limit Theorems when the Density is Large

We now look at the situation when the density becomes large and consider a family of Poisson systems defined by $P(t,x,dy)$ and $\rho\cdot\mu$, where μ is a stationary (or initial) measure and $\rho\to\infty$. The measure on Ω is hence $\rho\cdot\nu$ with ν generated by μ and the transition probabilities. ($E_\rho(\cdot)$ denotes exception with respect to this measure, and $E(\cdot)=E_1(\cdot)$.)

We first prove the law of large numbers:

Theorem 1. *For any* $\varphi\in L^1(\Omega,\mathscr{A},\nu)$ *and* $\varepsilon>0$

$$\lim_{\rho\to\infty} P_\rho(|\rho^{-1}S(\varphi)-E(S(\varphi))|>\varepsilon)=0.$$

I.e. the random configuration in Ω *defined by* $\rho^{-1}S(\varphi)=\rho^{-1}\sum_{j,n}\varphi(\omega_{j,n})$ *converges to the non-random distribution defined by* ν: $E(S(\varphi))=\int\varphi(\omega)\,\nu(d\omega)$. *In particular in the non-stationary situation* $\rho^{-1}S(t,\varphi)$ *converges to* $E(S(t,\varphi))=\int\mu(t,dx)\,\varphi(x)$ *for any* $\varphi\in L^1(\Sigma,\mu(t))$, *i.e. the random configuration in* Σ *at time* t *converges to the non-random distribution defined by* $\mu(t,dy)=\int\mu(dx)\,P(t,x,dy)$.

Proof. By Lemma 1

$$E_\rho(e^{i\theta\rho^{-1}S(\varphi)})=\exp\int(e^{i\theta\rho^{-1}\varphi(\omega)}-1)\,\rho\,\nu(d\omega)$$
$$=\exp(i\theta E(S(\varphi)))\cdot\exp\int\frac{e^{i\theta\rho^{-1}\varphi(\omega)}-1-i\theta\rho^{-1}\varphi(\omega)}{\rho^{-1}\varphi(\omega)}\,\varphi(\omega)\,\nu(d\omega),$$

and by bounded convergence the last integral goes to zero as $\rho\to\infty$, so the assertion follows from the continuity theorem for characteristic functions.

We can now continue directly and prove a central limit theorem for the scaled fluctuations $X_\rho(\varphi)=\rho^{-1/2}(S(\varphi)-E(S(\varphi)))$.

Theorem 2. *For any* $\varphi\in L^1(\Omega,\mathscr{A},\nu)\cap L^2(\Omega,\mathscr{A},\nu)$ *the distribution of* $X_\rho(\varphi)$ *converges weakly to a centered Gaussian with variance*

$$V(\varphi,\varphi)\equiv\operatorname{Var}(X_\rho(\varphi))=\int\varphi^2(\omega)\,\nu(d\omega)\qquad \text{as } \rho\to\infty,$$

and hence for any finite such family $\varphi_1,\dots,\varphi_n$ *the distribution of* $X_\rho(\varphi_1),\dots,X_\rho(\varphi_n)$ *converges weakly to a centered Gaussian with covariances*

$$V(\varphi_i,\varphi_j)=\int\varphi_i(\omega)\,\varphi_j(\omega)\,\nu(d\omega).$$

Proof. By Lemma 1

$$E_\rho(e^{i\theta X_\rho(\varphi)}) = \exp \int (e^{i\theta\rho^{-1/2}\varphi(\omega)} - 1 - i\theta\rho^{-1/2}\varphi(\omega))\,\rho\,\nu(d\omega)$$
$$= \exp\left(-\frac{\theta^2 V(\varphi,\varphi)}{2}\right)$$
$$\cdot \exp \int \frac{e^{i\theta\rho^{-1/2}\varphi(\omega)} - 1 - i\theta\rho^{-1/2}\varphi(\omega) + \theta^2(2\rho)^{-1}\varphi^2(\omega)}{\rho^{-1}\varphi^2(\omega)}\,\varphi^2(\omega)\,\nu(d\omega),$$

and as in the previous proof the last integral goes to zero, so that

$$\lim_{\rho\to\infty} E_\rho(e^{i\theta X_\rho(\varphi)}) = e^{-\frac{\theta^2 V(\varphi,\varphi)}{2}}.$$

Applying this result to $\varphi = \sum_1^n \theta_i \varphi_i$ we prove the last assertion.

The central limit theorem for the fluctuations of the distribution in Σ is obtained directly by considering $S(t,\varphi)$ in Theorem 2. For simplicity we only consider the stationary case in the following.

Corollary 2. *For any bounded* $\varphi \in L^1(\Sigma,\mu)$ *put*

$$X_\rho(t,\varphi) = \rho^{-1/2}(S(t,\varphi) - E(S(t,\varphi))),$$

then for any finite such family $\varphi_1, \ldots, \varphi_n$ *and* $t_1 < \cdots < t_n$ *the distribution of* $X_\rho(t_1,\varphi_1), \ldots, X_\rho(t_n,\varphi_n)$ *converges weakly to a centered Gaussian with covariances*

$$V(t_i,\varphi_i; t_j,\varphi_j) = \int \varphi_i(x)\,\mu(dx)\,P(t_j - t_i, x, dy)\,\varphi_j(y) \quad \text{for } i<j.$$

Also, for $\varphi_j(t,x)$ *bounded* $\in L^1(R^1\times\Sigma, dt\times d\mu)$ *the distribution of*

$$X_\rho(\varphi_j) = \int X_\rho(t,\varphi_j(t,\cdot))\,dt, \quad j=1,\ldots,n$$

converges weakly to a centered Gaussian with covariances

$$V(\varphi_i,\varphi_j) = \int_{s\le t} (\varphi_i(s,x)\,\varphi_j(t,y) + \varphi_j(s,x)\,\varphi_i(t,y))\,\mu(dx)\,P(t-s,x,dy)\,ds\,dt$$

if these are finite.

Proof. This follows directly from Theorem 2 and the Markov property of ν:

$$E(X_\rho(t_i,\varphi_i)\,X_\rho(t_j,\varphi_j)) = \int \varphi_i(x(t_i,\omega))\,\varphi_j(x(t_j,\omega))\,\nu(d\omega)$$
$$= \int \varphi_i(x)\,\mu(dx)\,P(t_j - t_i, x, dy)\,\varphi_j(y) \quad \text{for } t_i \le t_j.$$

3. Construction of the Gaussian Limit Random Field as a Generalized Stochastic Process

We now want to construct a Gaussian random field $X(t,\varphi) =$ "$\int g(t,x)\,\varphi(x)\,dx$" having the covariances $V(t,\varphi_1; t_2,\varphi_2)$ obtained for the fluctuations in Corollary 2 and show that it has the properties prescribed by the physical theory mentioned in the introduction. For $t_1=t_2$ we see from the expression for V that formally $E(g(t,x)\,g(t,y))=\mu(x)\,\delta(x-y)$ if μ has a density, so we can not hope to construct $g(t,x)$ as an ordinary stochastic process but rather as a stochastic distribution as mentioned in the introduction. A convenient space which can carry such a generalized process is the space of tempered distributions $\mathscr{S}'$ dual to the Schwartz class $\mathscr{S}$ of test functions on Σ which is widely used in analysis. Since $\mathscr{S}$ is a nuclear space this allows us to make use of the theory of probability measures on the dual of a nuclear space for which the basic theorems concerning the construction of measures, characteristic functionals and their continuity theorems are well developed and reasonably simple [3].

Let us recall the basic facts about a nuclear space and how to construct a probability measure on its dual (Minlos theorem). To conform with the notation of [3] let Φ_d be the space $\mathscr{S}$ of infinitely differentiable testfunctions φ on R^d for which $x^p D^q \varphi$ are bounded for all multiindices p, q.

$$\left(x^p=x_1^{p_1} \dots x_d^{p_d},\ D^q=\left(\frac{\partial}{\partial x_1}\right)^{q_1} \dots \left(\frac{\partial}{\partial x_d}\right)^{q_d},\ |p|=\sum_1^d p_i\right).$$

With the topology defined by the norms

$$\sup_{\substack{x \\ |p|,|q|\leqq n}} |x^p D^q \varphi|$$

or equivalently by the norms

$$\sup_{|p|,|q|\leqq n} \left[\int (x^p D^q \varphi)^2\, dx\right]^{1/2} \qquad \text{for } n=0,1,2,\dots$$

this is a nuclear space [3]. (A simple direct proof of this fact is obtained by using the sequence of norms $\|\varphi\|_n=[\int (H^n \varphi)^2\, dx]^{1/2}$, $n\geqq 0$, which are equivalent to the others. Here H is the operator defined by $H\varphi=|x|^2\varphi-\Delta\varphi$ well known in quantum mechanics, whose eigenvalues are $\lambda_p=\sum_1^d (2p_i+1)$ for $p_i=0,1,2,\dots$. K. Itô private communication). The dual space Φ_d' is the space of tempered distributions in R^d, which we consider as a measurable space $(\Phi_d', \mathscr{C}_d)$ with the σ-algebra $\mathscr{C}_d$ generated by the cylinder sets $C=\{X; X\in\Phi_d', X(\varphi_1),\dots,X(\varphi_n)\in A\}$ for arbitrary finite families $\varphi_1,\dots,\varphi_n\in\Phi_d$ and arbitrary Borel sets $A\subset R^n$. The topology of Φ_d' is generated by the norms dual to those defining the topology of Φ_d: $\|X\|_{-n}=\sup_{\|\varphi\|_n\leqq 1}|X(\varphi)|$. The norms $\|\varphi\|_n$ increase, i.e. $\|\varphi\|_n\leqq\|\varphi\|_{n+1}$ for $n\geqq 0$, and hence $\|X\|_{-n}\geqq\|X\|_{-n-1}$ for $n\geqq 0$.

Minlos theorem tells us that a probability measure on $(\Phi_d', \mathscr{C}_d)$ can be constructed if its values for all cylinder sets are prescribed in a consistent way and the following continuity condition is fulfilled: The probability measures on R^1 defined by $P_\varphi(A)=P(X(\varphi)\in A)$ are weakly continuous when φ varies in Φ_d. Alternatively the probability measure can be constructed from the characteristic functional $F(\varphi)=E(e^{iX(\varphi)})$ if it is a positive definite, continuous functional with $F(0)=1$. (These facts are proved in detail in [3].) In particular a Gaussian measure with

$E(X(\varphi))=0$, $E(X(\varphi_1)X(\varphi_2))=V(\varphi_1,\varphi_2)$ and a characteristic functional

$$F(\varphi)=e^{-\frac{V(\varphi,\varphi)}{2}}$$

can be constructed if $V(\varphi_1,\varphi_2)$ is any positive definite continuous bilinear form on $\Phi_d\times\Phi_d$.

These facts immediately allow us to construct the limit process as a generalized stochastic process in space-time.

Theorem 3. *The bilinear form of Corollary 2*

$$V(\varphi_1,\varphi_2)=\int_{s\leq t}(\varphi_1(s,x)\,\varphi_2(t,y)+\varphi_2(s,x)\,\varphi_1(t,y))\,\mu(dx)\,P(s-t,x,dy)\,ds\,dt$$

is continuous on $\Phi_{d+1}\times\Phi_{d+1}$, *and hence* $e^{-\frac{V(\varphi,\varphi)}{2}}$ *is the characteristic functional of a Gaussian measure* P *on* $(\Phi'_{d+1},\mathscr{C}_{d+1})$, *i.e.*

$$E(e^{iX(\varphi)})=e^{-\frac{V(\varphi,\varphi)}{2}}$$

if $\int\frac{\mu(dx)}{1+|x|^q}<\infty$ *for some* $q>0$. *Also, for* $\varphi\in\Phi_{d+1}$ *the distribution of* $X_\rho(\varphi)$ *defines a measure* P_ρ *on* $(\Phi'_{d+1},\mathscr{C}_{d+1})$ *with characteristic functional*

$$F_\rho(\varphi)=\exp\int(e^{i\rho^{-1/2}\int\varphi(t,x(t,\omega))\,dt}-1-i\rho^{-1/2}\int\varphi(t,x(t,\omega))\,dt)\,\rho\,\nu(d\omega),$$

and P_ρ *converges weakly to* P *as* $\rho\to\infty$.

Proof. The continuity of $V(\varphi,\varphi)$ follows from the estimate

$$\int_{s\leq t}\varphi(s,x)\,\varphi(t,y)\,\mu(dx)\,P(t-s,x,dy)\,ds\,dt$$

$$\leq\sup_{t,y}(1+t^2)\,|\varphi(t,y)|\sup_{s,x}(1+s^2)(1+|x|^q)\,|\varphi(s,x)|\int\frac{\mu(dx)}{1+|x|^q}\frac{ds}{1+s^2}\frac{dt}{1+t^2}.$$

Hence $e^{-\frac{V(\varphi,\varphi)}{2}}$ is the characteristic functional of a Gaussian measure as required. From Lemma 2 with $\varphi(\omega)=\int\varphi(t,x(t,\omega))\,dt$ follows that $F_\rho(\varphi)$ is equal to $E_\rho(e^{iX_\rho(\varphi)})$. It is continuous, because

$$|\log F_\rho(\varphi_1)-\log F_\rho(\varphi_2)|\leq 2\rho^{1/2}\int|\varphi_1(t,x(t,\omega))-\varphi_2(t,x(t,\omega))|\,\nu(d\omega)\,dt$$

$$=2\rho^{1/2}\int|\varphi_1(t,x)-\varphi_2(t,x)|\,\mu(dx)\,dt$$

$$\leq 2\rho^{1/2}\sup_{t,x}(1+t^2)(1+|x|^q)\,|\varphi_1(t,x)-\varphi_2(t,x)|\int\frac{\mu(dx)}{1+|x|^q}\frac{dt}{1+t^2},$$

and is hence the characteristic functional of a measure P_ρ on $(\Phi'_{d+1},\mathscr{C}_{d+1})$. Since $F_\rho(\varphi)\to e^{-\frac{V(\varphi,\varphi)}{2}}$ for each $\varphi\in\Phi_{d+1}$, $P_\rho\Rightarrow P$ follows from the Lévy continuity theorem, which is valid for measures on the dual of a nuclear space [7]. (Only the convergence of $F_\rho(\varphi)$ for each φ and the continuity of the limit is needed.)

We can now construct a Gaussian process $X(t,\varphi)$ with $\varphi\in\Phi_d$ having the finite dimensional distributions obtained in Corollary 2 as the limits of those of $X_\rho(t,\varphi)$.

Theorem 4. *On* $(\Phi'_{d+1}, \mathscr{C}_{d+1})$ *there is a Gaussian process* $X(t)$ *with values in* $(\Phi'_d, \mathscr{C}_d)$ *and with covariances*

$$E(X(t_1, \varphi_1)\, X(t_2, \varphi_2)) = V(t_1, \varphi_1; t_2, \varphi_2) = \int \varphi_1(x)\, \mu(dx)\, P(t_2 - t_1, x, dy)\, \varphi_2(y)$$

for $t_1 \leqq t_2$, $\varphi_1, \varphi_2 \in \Phi_d$ *if* A, *the generator of* $P(t, x, dy)$, *is defined on* Φ_d *and is continuous in the following sense:* $\sup |A\varphi(x)| \leqq C\, \|\varphi\|_p$ *for some* p.

Moreover, for some p $P(\|X(t)\|_{-p} < \infty) = 1$ *for all* t. *The distribution of* $X(t)$ *is the Gaussian measure on* $(\Phi'_d, \mathscr{C}_d)$ *with covariances* $E(X(t, \varphi_1)\, X(t, \varphi_2)) = \int \varphi_1(x)\, \varphi_2(x)\, \mu(dx)$ *for each* t. *We still assume that* $\int \frac{\mu(dx)}{1+|x|^q} < \infty$ *for some* $q \geqq 0$. *Also, for* $\varphi \in \Phi_d$ *the distribution of* $X_\rho(t, \varphi)$ *defines a measure on* $(\Phi'_d, \mathscr{C}_d)$ *with characteristic functional*

$$\exp \int (e^{i\rho^{-1/2}\varphi(x)} - 1 - i\rho^{-1/2}\varphi(x))\, \rho \cdot \mu(dx),$$

and this measure converges weakly to the Gaussian distribution of $X(t)$ *as* $\rho \to \infty$.

Proof. We construct the random variable $X(t_0, \varphi)$ on $(\Phi'_{d+1}, \mathscr{C}_{d+1})$ as the limit of $X(\varphi_n)$ for some sequence $\varphi_n(t, x) \in \Phi_{d+1}$ converging to $\delta(t - t_0)\, \varphi(x)$. Take e.g.

$$\varphi_n(t, x) = g_n(t - t_0)\, \varphi(x) \quad \text{with} \quad g_n(t) = (\sqrt{2\pi}\, \sigma_n)^{-1}\, e^{-\frac{t^2}{2\sigma_n^2}} \quad \text{and} \quad \sigma_n \downarrow 0.$$

Then

$$\begin{aligned} E(X(\varphi_n) - X(\varphi_m))^2 &= E(X(\varphi_n - \varphi_m)^2) \\ &= 2 \int_{s \leqq t} (g_n(s) - g_m(s))\, \varphi(x) (g_n(t) - g_m(t))\, \varphi(y)\, \mu(dx)\, P(t-s, x, dy)\, ds\, dt. \end{aligned}$$

Each term, e.g.

$$\int \varphi(x)\, \mu(dx) \int_{s \leqq t} g_n(s)\, g_m(t)\, ds\, dt \int P(t-s, x, dy)\, \varphi(y)$$

converges to $\frac{1}{2} \int \varphi^2(x)\, \mu(dx)$, because the innermost integral is continuous in $t - s$. Hence $X(\varphi_n)$ converges in mean square, and for some subsequence $\{n'\}$ $X(\varphi_{n'})$ converges a.s. to some limit $X(t_0, \varphi)$, and because Φ_d is separable we can assume that this holds simultaneously for all φ in a dense subset. For each n the functional $\varphi \to X_n(\varphi) = X(g_n \cdot \varphi)$ is continuous on Φ_d and hence an element of Φ'_d. Hence if we can show that for some p $\sup_n \|X_n\|_{-p} < \infty$ a.s. we know that the $X_{n'}(\varphi)$ converge for all $\varphi \in \Phi_d$ a.s., and then the limit $X(t_0)$ is an element of Φ'_d, because Φ'_d is weakly complete [3]. In this way we have constructed $X(t_0)$ as a random variable on $(\Phi'_{d+1}, \mathscr{C}_{d+1})$ with values in Φ'_d and such that $\|X(t_0)\|_{-p} < \infty$ a.s. as required. By the mean square convergence the distribution of any finite family $X(t_1, \varphi_1), \ldots, X(t_n, \varphi_n)$ is the Gaussian one with covariances $V(t_i, \varphi_i; t_j, \varphi_j)$ obtained by letting $n \to \infty$ as indicated above. The statements about $X_\rho(t, \varphi)$ follow as in Theorem 3.

To show the boundedness of $\|X_n\|_{-p}$ for some p we need the following lemma whose proof is given afterwards.

Lemma 4. *If* X *has a Gaussian distribution on* $(\Phi'_d, \mathscr{C}_d)$ *with covariances* $E(X(\varphi_1)\, X(\varphi_2)) = C(\varphi_1, \varphi_2)$, *and if* $\sum_i C(\varphi_i, \varphi_i) = D < \infty$ *for some p-orthonormal* *sequence (i.e. $(\varphi_i, \varphi_j)_p = \delta_{ij}$) then*

$$P(\|X\|_{-p} > R) \leqq 2e^{-\frac{3R^2}{8D}}$$

We show that $\sum_1^\infty \|X_{n+1} - X_n\|_{-p} < \infty$ and $\|X_1\|_{-p} < \infty$ a.s. for some p. $\operatorname{Sup}_n \|X_n\|_{-p}$ will then be bounded a.s. Put $Y_n = X_{n+1} - X_n$. Then

$$\begin{aligned} C_n(\varphi, \varphi) &\equiv E(Y_n^2(\varphi)) \\ &= 2 \int_{s \leqq t} (g_{n+1}(s) - g_n(s))(g_{n+1}(t) - g_n(t))\, V(s, \varphi; t, \varphi)\, ds\, dt \\ &= 2 \int_{s \leqq t} (g_{n+1}(s) - g_n(s))(g_{n+1}(t) - g_n(t))(V(s, \varphi; t, \varphi) - V(0, \varphi; 0, \varphi))\, ds\, dt, \end{aligned}$$

so if we can show that $\sum_i |V(s, \varphi_i; t, \varphi_i) - V(0, \varphi_i; 0, \varphi_i)| \leqq C'(t-s)$ for some p-orthonormal sequence $\{\varphi_i\}$ we get

$$\sum_i C_n(\varphi_i, \varphi_i) \leqq 2C' \int_{s \leqq t} (g_{n+1}(s) + g_n(s))(g_{n+1}(t) + g_n(t))(t-s)\, ds\, dt \leqq 4\sqrt{2}\, C' \sigma_n.$$

Then from Lemma 4 we can conclude that

$$P(\|Y_n\|_{-p} > R_n) \leqq 2e^{-(\text{const}) \frac{R_n^2}{\sigma_n}}$$

and if we put $R_n = n^{-2}$, $\sigma_n = n^{-5}$ e.g. Borel-Cantelli's lemma tells us that

$$\sum_n \|Y_n\|_{-p} < \infty \quad \text{a.s.}$$

The estimate for V can be derived as follows. For any transition semigroup T_t on R^d with generator A we have

$$(T_t - I)\,\varphi = \int_0^t T_s A \varphi\, ds$$

if φ is in the domain of A. Hence

$$\begin{aligned} |V(s, \varphi; t, \varphi) - V(0, \varphi; 0, \varphi)| &= \left| \int \varphi(x)\, \mu(dx) \int_0^{t-s} P(u, x, dy)\, A\varphi(y)\, du \right| \\ &\leqq C(t-s)\, \|\varphi\|_{p'} \int |\varphi(x)|\, \mu(dx) \leqq C'(t-s)\, \|\varphi\|_{p'}^2 \end{aligned}$$

if A is continuous in the sense postulated and p' is big enough. By the nuclearity of Φ_d for some $p \geqq p'$ there is a p-orthonormal sequence $\{\varphi_i\}$ such that $\sum_i \|\varphi_i\|_{p'}^2 < \infty$, and the estimate is hence valid.

The a.s. boundedness of $\|X_1\|_{-p}$ follows from Lemma 4 if p can be chosen so that $D < \infty$. As in the proof of Theorem 3 it follows that $E(X_1(\varphi)^2) \leqq \text{const}\, \|\varphi\|_{p'}^2$ for some p', and hence again by the nuclearity $D < \infty$ for some $p \geqq p'$. Since $\|X\|_{-p}$ decreases as p increases the largest of the two p-values can be used.

Proof of Lemma 4. Let S be the sphere $\{X; \|X\|_{-p} \leqq R\}$ and S_N the sphere of radius R in R^N. Let B be a cylinder set of the form $B = \{X; X(f_1), \ldots, X(f_N) \in A\}$ with $f_i \in \Phi_d$, $i = 1, \ldots, N$ and A a Borel set in R^N. We can assume that $(f_i, f_j)_p = \delta_{ij}$.

Under the mapping $X \to (X(f_1), \ldots, X(f_N)) \in R^N$ S is mapped onto S_N. Indeed, $X \in S$ implies

$$\left(\sum_1^N x_i X(f_i)\right)^2 = X\left(\sum_1^N x_i f_i\right)^2 \leqq R^2 \left\| \sum_1^N x_i f_i \right\|_p^2 = R^2 \sum_1^N x_i^2,$$

so putting $x_i = X(f_i)$ we see that $\sum_1^N x_i^2 \leqq R^2$. Conversely, let X_i, $i=1, \ldots, N$ be such that $X_i(f_j) = \delta_{ij}$ and $X_i(f) = 0$ if f is not in the linear hull of $f_1, \ldots, f_N$. For any $(x_1, \ldots, x_N) \in S_N$ put $X = \sum_1^N x_i X_i$. Then X is mapped onto $(x_1, \ldots, x_N)$, and for any f with $\|f\|_p \leqq 1$ we have $f = \sum_1^N c_i f_i + \tilde{f}$ with $c_i = (f, f_i)_p$, $\sum_1^N c_i^2 \leqq 1$ and $X(\tilde{f}) = 0$, so $|X(f)| \leqq \left|\sum_1^N c_i x_i\right| \leqq R$ and $\|X\|_{-p} \leqq R$. Hence we can conclude that if B is disjoint from S then A is disjoint from S_N, and

$$P(B) = P(A) \leqq P\left(\sum_1^N X^2(f_i) > R^2\right).$$

The variables $X_i = X(f_i)$ are Gaussian with $E(X_i) = 0$, $E(X_i^2) = C(f_i, f_i) \equiv d_i$, and $d \equiv \sum_1^N d_i \leqq D$. For any such family

$$P\left(\sum_1^N X_i^2 > R^2\right) \leqq 2 e^{-\frac{3R^2}{8d}}.$$

This can be seen as follows: By a rotation in R^N which does not change $\sum_1^N X_i^2$ or d we can make the variables independent, so we need only consider that case. Then for each i:

$$E(e^{sX_i^2}) = \int \frac{e^{-\frac{x^2}{2}\left(\frac{1}{d_i} - 2s\right)}}{\sqrt{2\pi d_i}} dx = (1 - 2s d_i)^{-1/2} \quad \text{if } 2s d_i < 1,$$

and

$$E\left(e^{s \sum_1^N X_i^2}\right) = \prod_1^N (1 - 2s d_i)^{-1/2} \leqq (1 - 2s d)^{-1/2} \quad \text{if } 2s d < 1$$

by the elementary inequality $\prod_1^N (1 - a_i) \geqq 1 - \sum_1^N a_i$. We can then conclude that

$$P\left(\sum_1^N X_i^2 > R^2\right) \leqq (1 - 2s d)^{-1/2} e^{-sR^2},$$

so if we put $2s d = \frac{3}{4}$ we get the estimate above. Thus we see that

$$P(B) \leqq 2 e^{-\frac{3R^2}{8D}}$$

for any cylinder $B \subseteq S^c$, and it follows that the same bound holds for $P(S^c)$ too.

We next want to show that the process $X(t)$ is also Markovian and in an appropriate sense a solution of a stochastic differential equation corresponding to the Langevin equation for the formal density $g(t,x)$: $\frac{dg}{dt}=A^* g+w$ with

$$E(w(t,x)\,w(s,y))=-\delta(t-s)(A^*\,\delta(x-y)\,\mu(x)+\delta(x-y)\,\mu(x)\,A).$$

If we multiply both sides by an arbitrary testfunction $\varphi(t,x)\in\Phi_{d+1}$ and integrate by parts formally, we see that the equation satisfied by $X(\varphi)=$"$\int g(t,x)\varphi(t,x)\,dt\,dx$" and $W(\varphi)=$"$\int w(t,x)\,\varphi(t,x)\,dt\,dx$" is:

$$X\left(\frac{-\partial\varphi}{\partial t}\right)=X(A\,\varphi)+W(\varphi)$$

and that the covariances of W are defined by

$$E(W(\varphi_1)\,W(\varphi_2))=-\int(\varphi_1(t,x)\,A\,\varphi_2(t,x)+\varphi_2(t,x)\,A\,\varphi_1(t,x))\,dt\,\mu(dx).$$

(That this is actually a positive definite form can be seen from the expression

$$-2\int\varphi(x)\,A\,\varphi(x)\,\mu(dx)=\lim_{t\downarrow 0}t^{-1}\int E_x(\varphi(x(t))-\varphi(x))^2\,\mu(dx).)$$

It is now straighforward to verify that this equation holds for $X(\varphi)$.

Theorem 5. *If A is defined in Φ_d and is continuous, then the random field $X(\varphi)$ in space-time satisfies the Langevin equation in the weak sense:*

$$X\left(\frac{-\partial\varphi}{\partial t}\right)=X(A\,\varphi)+W(\varphi),$$

where $W(\varphi)$ is a Gaussian random field in space-time with covariances

$$E(W(\varphi_1)\,W(\varphi_2))=-\int(\varphi_1(t,x)\,A\,\varphi_2(t,x)+\varphi_2(t,x)\,A\,\varphi_1(t,x))\,dt\,\mu(dx).$$

Proof. Because A is assumed to be continuous the equation

$$W(\varphi)=X\left(\frac{-\partial\varphi}{\partial t}\right)-X(A\,\varphi)$$

actually defines a Gaussian random field, and we only have to check that it has the above covariances. It is enough to check the case $\varphi_1=\varphi_2$:

$$E(W(\varphi)^2)=E\left(X\left(\frac{\partial\varphi}{\partial t}+A\,\varphi\right)^2\right)$$

$$=2\int\left(\frac{\partial\varphi(s,x)}{\partial s}+A\,\varphi(s,x)\right)\mu(dx)\,ds\int_{t\geq s}P(t-s,x,dy)\left(\frac{\partial\varphi(t,y)}{\partial\varphi}+A\,\varphi(t,y)\right)dt.$$

As already mentioned, for any transition semigroup T_t on R^d with generator A we have $(T_t-I)\,\varphi=\int_0^t T_s\,A\,\varphi\,ds$ if φ is in the domain of A. (We here use operator notation and indicate the time dependence by a subscript.) Hence for $\varphi,\psi\in\Phi_{d+1}$

we have:

$$\int_{s\leqq t} d\mu\,\psi_s\,T_{t-s}\,A\,\varphi_t\,ds\,dt = -\int_{s\leqq t\leqq u} d\mu\,\psi_s\,T_{t-s}\,A\,\frac{\partial\varphi_u}{\partial u}\,ds\,dt\,du$$

$$= -\int_{s\leqq u} d\mu\,\psi_s\left(\int_s^u T_{t-s}\,A\,\frac{\partial\varphi_u}{\partial u}\,dt\right) ds\,du = -\int_{s\leqq u} d\mu\,\psi_s (T_{u-s}-I)\,\frac{\partial\varphi_u}{\partial u}\,ds\,du,$$

or

$$\int_{s\leqq t} d\mu\,\psi_s\,T_{t-s}\left(A\,\varphi_t+\frac{\partial\varphi_t}{\partial t}\right) ds\,dt = -\int d\mu\,\psi_t\,\varphi_t\,dt.$$

Putting $\psi_t=\dfrac{\partial\varphi_t}{\partial t}+A\,\varphi_t$ we hence see that

$$E(W(\varphi)^2) = -2\int d\mu\left(\frac{\partial\varphi_t}{\partial t}+A\,\varphi_t\right)\varphi_t\,dt = -2\int d\mu\,\varphi_t\,A\,\varphi_t\,dt$$

as claimed above.

The Markov property of the process $X(t)$ is heuristically "obvious" from the Langevin equation by the same argument as for an ordinary finite dimensional such equation: The solution can be written

$$g(t,y)\,dy = \int_{-\infty}^{t} ds \int w(s,x)\,dx\,P(t-s,x,dy).$$

Hence the conditional average of $g(t,y)$ for $t>0$ given $w(s,x)$ for $s\leqq 0$ should be given by

$$\hat{g}(t,y)\,dy = \int_{-\infty}^{0} ds \int w(s,x)\,dx\,P(t-s,x,dy) = \int g(0,z)\,dz\,P(t,z,dy)$$

because $w(s,x)$ and $w(s',x')$ are independent when $s<0<s'$. This suggests that the conditional average of $X(t,\varphi)$ given all $X(s,\psi)$ for $s\leqq 0$ is given by

$$\hat{X}(t,\varphi) = \int g(0,z)\,dz\,P(t,z,dy)\,\varphi(y) = X(0,\varphi_t)$$

with $\varphi_t(z)=\int P(t,z,dy)\,\varphi(y)$. This is indeed easy to verify directly.

Theorem 6. *The Markov property of* $X(t)$: For $t>t_0$ *the conditional average of* $X(t,\varphi)$ *given* $X(s,\psi)$ *for all* $s\leqq t_0$, $\psi\in\Phi_d$ *is given by* $\hat{X}(t,\varphi)=X(t_0,\varphi_{t-t_0})$ *with* φ_t *defined above.*

Proof. Because all variables are Gaussian it is enough to check that

$$E((X(t,\varphi)-\hat{X}(t,\varphi))\,X(s,\psi))=0 \qquad \text{for } s\leqq t_0,\ \psi\in\Phi_d:$$

$$\begin{aligned} E(X(s,\psi)\hat{X}(t,\varphi)) &= E(X(s,\psi)\,X(t_0,\varphi_{t-t_0}))\\ &= \int \psi(x)\,\mu(dx)\,P(t_0-s,x,dz)\,P(t-t_0,z,dy)\,\varphi(y)\\ &= \int \psi(x)\,\mu(dx)\,P(t-s,x,dy)\,\varphi(y) = E(X(s,\psi)\,X(t,\varphi)) \end{aligned}$$

by the semigroup equation for $P(t,x,dy)$.

Finally we prove that there is a version of the process $X(t)$ which is a.s. continuous in the topology of Φ_d'.

Theorem 7. *Under the same assumptions as in Theorem 4 a modified process $X'(t)$ can be constructed such that for some p:*

$$P(\|X'(t)-X(t)\|_{-p}=0)=1 \qquad \text{for any } t,$$

and

$$P(\lim_{h\downarrow 0} \sup_{|t|\leqq T} \|X'(t+h)-X'(t)\|_{-p}=0)=1$$

$$P(\sup_{|t|\leqq T} \|X'(t)\|_{-p}<\infty)=1$$

for any $T>0$. In particular, $X'(t)$ is a.s. continuous in the topology of Φ_d'.

Proof. Since $E(X(t+h,\varphi)-X(t,\varphi))^2=2(V(0,\varphi;0,\varphi)-V(0,\varphi;h,\varphi))$ it follows from the estimate $\sum_i V(0,\varphi_i;0,\varphi_i)-V(0,\varphi_i;h,\varphi_i)\leqq c'h$ and Lemma 4 that

$$P(\|X(t+h)-X(t)\|_{-p}>R)\leqq 2e^{-\frac{3R^2}{16c'h}} \qquad \text{for some } p.$$

We use the well known method of constructing $X'(t)$ by interpolating $X(t)$ linearly between the points $t_{n,k}=2^{-n}\cdot k$ and then considering the limit $n\to\infty$. Let $X_n(t)$ be the interpolating process:

$$X_n(t)=(1-\lambda)X(t_{n,k})+\lambda X(t_{n,k+1})$$

for

$$t=(1-\lambda)t_{n,k}+\lambda t_{n,k+1}, \qquad 0\leqq\lambda\leqq 1.$$

It is easy to check that for three consecutive points

$$t_1=t_{n,k}=t_{n+1,2k}, \qquad t_2=t_{n+1,2k+1}, \qquad t_3=t_{n,k+1}=t_{n+1,2k+2}$$

we have

$$\sup_{t_1\leqq t\leqq t_3} \|X_{n+1}(t)-X_n(t)\|_{-p}\leqq\tfrac{1}{2}(\|X(t_2)-X(t_1)\|_{-p}+\|X(t_3)-X(t_2)\|_{-p})$$

if $\|X(t)\|_{-p}<\infty$ a.s. for all t.

Hence

$$P(\sup_{t_1\leqq t\leqq t_3} \|X_{n+1}(t)-X_n(t)\|_{-p}>R)\leqq 4e^{-(\text{const})\,2^n R^2}$$

and

$$P(\sup_{|t|\leqq T} \|X_{n+1}(t)-X_n(t)\|_{-p}>R_n)\leqq 4\cdot T\cdot 2^n\, e^{-(\text{const})\,2^n R_n^2},$$

so with $R_n=n^{-2}$ e.g. Borel-Cantelli's lemma tells us that

$$\sum_n \sup_{|t|\leqq T} \|X_{n+1}(t)-X_n(t)\|_{-p}<\infty \qquad \text{a.s.}$$

It follows that with probability one $\lim_{n\to\infty} X_n(t,\varphi)=X'(t,\varphi)$ exists for all φ and is hence an element of Φ'_d, and that

$$\lim_{n\to\infty}\sup_{|t|\leqq T}\|X_n(t)-X'(t)\|_{-p}=0, \qquad \sup_{|t|\leqq T}\|X'(t)\|_{-p}<\infty .$$

Since

$$\begin{aligned}\sup_{|t|\leqq T}&\|X'(t+h)-X'(t)\|_{-p}\\ &\leqq \sup_{|t|\leqq T}\|X'(t+h)-X_n(t+h)\|_{-p}+\sup_{|t|\leqq T}\|X_n(t+h)-X_n(t)\|_{-p}\\ &\quad+\sup_{|t|\leqq T}\|X_n(t)-X'(t)\|_{-p}\end{aligned}$$

and $X_n(t)$ is continuous in $\|\cdot\|_{-p}$ uniformly for $|t|\leqq T$ a.s. it follows that $X'(t)$ has the same property as stated.

4. Some Examples

4.1. Brownian Motion

In this case $P(t,x,dy)=(2\pi t)^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{2t}}\,dy$ and the invariant measure is ordinary Lebesque measure, $\mu(dx)=dx$. The generator A is the Laplacian $\frac{\Delta}{2}$, and the continuity condition of Theorem 4 is easily verified. The Langevin equation is hence the ordinary heat equation with white noise added:

$$\frac{dg}{dt}=\frac{1}{2}\Delta g+w,$$

and the covariances of w are determined by

$$\begin{aligned}E\left(\int w(t,x)\,\varphi(t,x)\,dt\,dx\right)^2&=-\int\varphi(t,x)\,\Delta\,\varphi(t,x)\,dt\,dx\\ &=\int|\mathrm{grad}_x\,\varphi(t,x)|^2\,dt\,dx.\end{aligned}$$

4.2. A General Diffusion

We can more generally consider a diffusion with a generator defined by

$$A\,\varphi(x)=\sum_{ij}a_{ij}(x)\frac{\partial^2\varphi(x)}{\partial x_i\,\partial x_j}+\sum_i b_i(x)\frac{\partial\varphi(x)}{\partial x_i}$$

with e.g. infinitely differentiable coefficients of at most polynomial growth at infinity and an equilibrium density $\mu(x)$ satisfying the stationary forward equation

$$\sum_{ij}\frac{\partial^2(a_{ij}\mu)}{\partial x_i\,\partial x_j}-\sum_i\frac{\partial(b_i\mu)}{\partial x_i}=0.$$

The Langevin equation is the forward (Fokker-Planck) equation driven by white noise whose covariances are determined by

$$E(\int w(t,x)\,\varphi(t,x)\,dt\,dx)^2$$
$$= -2\int \varphi(t,x)\left(\sum_{ij} a_{ij}(x)\frac{\partial^2\varphi(t,x)}{\partial x_i\,\partial x_j}+\sum_i b_i(x)\frac{\partial\varphi(t,x)}{\partial x_i}\right)\mu(x)\,dt\,dx$$
$$=2\int\left(\sum_{ij} a_{ij}(x)\frac{\partial\varphi(t,x)}{\partial x_i}\frac{\partial\varphi(t,x)}{\partial x_j}\right)\mu(x)\,dt\,dx.$$

The last expression which is manifestly non-negative is obtained by partial integration using the equation for $\mu(x)$.

4.3. Random Flight with Independent Velocities

The state of each particle is determined by its position x and velocity $u\in R^{d/2}$. The velocity is only changed at collisions with some medium. These form a Poisson process in time with intensity λ, and after each collision the velocity is independent of everything in the past with a given distribution $F(du)$. The position is determined by

$$x(t)=x(0)+\int_0^t u(s)\,ds.$$

The velocity process is Markovian with transition probabilities

$$P(t,u,dv)=e^{-\lambda t}\,\delta(u,dv)+(1-e^{-\lambda t})\,F(dv).$$

($\delta(u,dv)$ is the distribution function of a unit mass at u.) The stationary measure is F, and the generator is defined by

$$\tilde{A}\varphi(u)=\lambda\int(F(dv)-\delta(u,dv))\,\varphi(v).$$

The generator of the process $(x(t),u(t))$ is hence defined by

$$A\varphi(x,u)=u\cdot\mathrm{grad}_x\,\varphi(x,u)+\lambda\int(F(dv)-\delta(u,dv))\,\varphi(x,v)$$

and the invariant measure by $\mu(dx,du)=dx\cdot F(du)$. The forward equation for a distribution $f(t,x,du)\,dx$ which has a density in x is hence given by

$$\frac{df(t,x,du)}{dt}=-u\cdot\mathrm{grad}_x f(t,x,du)+\lambda\int f(t,x,dv)(F(du)-\delta(v,du)).$$

The Langevin equation for the formal random distribution $g(t,x,du)\,dx$ having a density in x is hence obtained by adding white noise $w(t,x,du)\,dx$ whose covariances are determined by

$$E(\int w(t,x,du)\,\varphi(t,x,u)\,dx)^2$$
$$=-2\int\varphi(t,x,u)\,(u\cdot\mathrm{grad}_x\,\varphi(t,x,u)$$
$$+\lambda\int(F(dv)-\delta(u,dv))\,\varphi(x,v))\,dt\,dx\,F(du)$$
$$=2\lambda\int\varphi(t,x,u)\,(F(du)\,\delta(u,dv)-F(du)\,F(dv))\,\varphi(t,x,v)\,dt\,dx$$
$$=2\lambda\int\left[\int\varphi^2(t,x,u)\,F(du)-\left[\int\varphi(t,x,u)\,F(du)\right]^2\right]dt\,dx.$$

A process of this type is the so called wind-tree model in the "Boltzmann limit", which has been studied in the physics literature [4]. In the three dimensional wind-tree model for example F is the uniform distribution on a sphere.

4.4. An Arbitrary Markov Process with a Finite Number of States

If Σ is a finite set the generator A is a matrix with elements $A(x,y)$ such that $A(x,y)\geqq 0$, $x\neq y$, $\sum_y A(x,y)=0$, and the invariant measure has a density such that $\sum_x \mu(x)A(x,y)=0$. The process $g(t,x)$ in this case is an ordinary finite dimensional Gauss-Markov process with covariances $E(g(s,x)\,g(t,y))=\mu(x)P(t-s,x,y)$ for $t\geqq s$, so it need not be constructed as a generalized stochastic process. The Langevin equation is

$$\frac{dg(t,x)}{dt}=\sum_y g(t,y)A(y,x)+w(t,x)$$

with

$$E(w(s,x)\,w(t,y))=-\delta(t-s)(\mu(x)A(x,y)+\mu(y)A(y,x)).$$

(That these covariances actually form a positive definite matrix follows from the fact that $-2\sum_{x,y}\varphi(x)\mu(x)A(x,y)\varphi(y)=\sum_{x,y}(\varphi(x)-\varphi(y))^2\mu(x)A(x,y)$.) The fact that g is defined by the above Langevin equation can be seen directly by the following heuristic argument. Consider the gas of particles and let $S(t,x)$ be the number of particles at x. As $\rho\to\infty$ we have approximatively

$$S(t,x)=\rho\cdot\mu(x)+\rho^{1/2}g(t,x).$$

The number of particles $dS(t,x,y)$ jumping from x to y in a small interval dt is the sum of $S(t,x)$ independent Poisson variables with mean and variance $A(x,y)\,dt$, so when ρ is large, $dS(t,x,y)$ is approximatively Gaussian with mean and variance $S(t,x)A(x,y)\,dt$, and with sufficient accuracy we can write

$$dS(t,x,y)=(\rho\,\mu(x)+\rho^{1/2}g(t,x))A(x,y)\,dt+(\rho\,\mu(x)A(x,y))^{1/2}\,dW(t,x,y),$$

where $W(t,x,y)$ are independent Wiener processes with $E(dW^2)=dt$. In the flow equation

$$dS(t,x)=\sum_{y\neq x} dS(t,y,x)-dS(t,x,y)$$

we can then equate the terms proportional to $\rho^{1/2}$ and get

$$\begin{aligned}dg(t,x)=\sum_y g(t,y)A(y,x)\,dt+\sum_{y\neq x}&(\mu(y)A(y,x))^{1/2}\,dW(t,y,x)\\&-(\mu(x)A(x,y))^{1/2}\,dW(t,x,y).\end{aligned}$$

Call the last term $dW(t,x)$. $W(t,x)$ are then Wiener processes, and it is easy to check that

$$E(dW(t,x)\,dW(t,y)) = -(\mu(x)\,A(x,y)+\mu(y)\,A(y,x))\,dt$$

so we have the above Langevin equation with $w(t,x)=\dfrac{dW(t,x)}{dt}$.

Acknowledgements. This work was done when the author was staying at Cornell University in 1973 with support from the N.S.F. The author benefitted very much from clarifying discussions with K. Itô and F. Spitzer.

References

1. Doob, J.L.: Stochastic Processes. New York: Wiley 1953
2. Fox, R.F., Uhlenbeck, G.E.: Contributions to the Theory of Non-Equilibrium Thermodynamics I and II. Phys. Fluids **13**, 1893–1902 and 2881–2890 (1970)
3. Gelfand, I.M., Vilenkin, N.J.: Generalized Functions, Vol. 4. New York-London: Academic Press 1966
4. Hauge, E.H.: What Can One Learn from Lorentz Models? In: Transport Phenomena. G. Kirczenow, J. Marro eds. Lecture Notes in Physics **31**, Berlin-Heidelberg-New York: Springer 1974
5. Hauge, E.H., Martin-Löf, A.: Fluctuating Hydrodynamics and Brownian Motion. J. Statist. Phys. **7** 259–281 (1973)
6. Kac, M.: Some Probabilistic Aspects of the Boltzmann Equation, Acta Phys. Austriaca, Suppl. X, 379–400 (1973)
7. Meyer, P.-A.: Le Théorème de Continuité de P. Lévy sur les Espaces Nucléaires. Sém. Bourbaki, 18e année, 1965/66, no. 311
8. Neveu, J.: Mathematical Foundations of the Calculus of Probability. San Francisco: Holden Day 1965
9. Neveu, J.: Processus Aléatoires Gaussiens. Les Presses de l'Université de Montréal 1968

Received September 30, 1975