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On the Last Time and the Number of Boundary Crossings Related to the Strong Law of Large Numbers and the Law of the Iterated Logarithm

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1. Introduction and Summary

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with $EX_1=0$, $S_n=X_1+\cdots+X_n$, and let $(b(n))_{n\geqq 1}$ be a sequence of real numbers. Define

$$N(b(n))=\sum_{n=1}^{\infty} I_{[S_n\geqq b(n)]}, \tag{1.1a}$$

$$\tilde N(b(n))=\sum_{n=1}^{\infty} I_{[|S_n|\geqq b(n)]}. \tag{1.1b}$$

Kolmogorov's strong law of large numbers can be stated in terms of the random variable $\tilde N(\varepsilon n)$ as follows:

$$P[\tilde N(\varepsilon n)<\infty]=1 \qquad \text{for every } \varepsilon>0. \tag{1.2}$$

To obtain a stronger result than Kolmogorov's strong law, Slivka and Severo [12] have discussed the moments of $\tilde N(\varepsilon n)$ and they proved that for $r\geqq 1$,

$$E|X_1|^{r+1}<\infty \Rightarrow E\tilde N^r(\varepsilon n)<\infty \qquad \text{for all } \varepsilon>0. \tag{1.3}$$

Motivated by the Marcinkiewicz-Zygmund extension of Kolmogorov's strong law, Stratton [14] proved that for $r\geqq 1$ and $\alpha>\frac{1}{2}$,

$$E|X_1|^{(r+1)/\alpha}<\infty \Leftrightarrow E\tilde N^r(\varepsilon n^{\alpha})<\infty \qquad \text{for all } \varepsilon>0. \tag{1.4}$$

There is in fact a misprint in [14] where the requirement $\alpha>0$ should be replaced by $\alpha>\frac{1}{2}$.

An earlier result to strengthen almost sure convergence for normalized sample sums was due to Strassen [13] who, instead of $N(b(n))$ and $\tilde N(b(n))$, considered

$$L(b(n))=\sup\{n\geqq 1: S_n\geqq b(n)\}, \tag{1.5a}$$

$$\tilde L(b(n))=\sup\{n\geqq 1: |S_n|\geqq b(n)\} \qquad (\sup\emptyset=0). \tag{1.5b}$$

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In other words, Strassen considers the last exit time $\tilde{L}(b(n))$ while Slivka and Severo consider the number of exits $\tilde{N}(b(n))$. Again Kolmogorov's strong law can be restated in terms of the last exit time $\tilde{L}(\varepsilon n)$ as $P[\tilde{L}(\varepsilon n)<\infty]=1$ for all $\varepsilon>0$, and it follows easily from the results of Baum and Katz (see [1, 9, 13] or Section 4 below) that for $\alpha>\frac{1}{2}$ and $r>0$,

$$E|X_1|^{(r+1)/\alpha}<\infty \Leftrightarrow E\tilde{L}^r(\varepsilon n^\alpha)<\infty \quad \text{for all } \varepsilon>0. \tag{1.6}$$

An obvious connection between $\tilde{L}(b(n))$ and $\tilde{N}(b(n))$ is the following:

$$\tilde{N}(b(n))\leqq \tilde{L}(b(n)). \tag{1.7}$$

Likewise we have the inequality:

$$N(b(n))\leqq L(b(n)). \tag{1.8}$$

Making use of (1.6) and (1.7), together with Lemma 3 in Section 2, we can prove that in (1.4), the condition $r\geqq 1$ as imposed by Stratton can be dropped. In fact we shall prove in Section 3 the following one-sided theorem involving $N(\varepsilon n^\alpha)$ which then immediately implies the corresponding two-sided result involving $\tilde{N}(\varepsilon n^\alpha)$.

Theorem 1. *Let $X_1, X_2, \ldots$ be i.i.d. random variables with $EX_1=0$, $E|X_1|^q<\infty$ for some $1\leqq q\leqq 2$. Let $\alpha>1/q$ and $r>0$. Set $S_n=X_1+\cdots+X_n$ and define $N(\varepsilon n^\alpha)$ as in* (1.1a). *Then*

$$\begin{aligned} E(X_1^+)^{(r+1)/\alpha}<\infty &\Leftrightarrow EN^r(\varepsilon n^\alpha) \quad \textit{for all} \quad \varepsilon>0 \\ &\Leftrightarrow EN^r(\varepsilon n^\alpha) \quad \textit{for some} \ \ \varepsilon>0. \end{aligned}$$

We remark that for the one-sided result in Theorem 1, we require the two-sided moment condition $E|X_1|^q<\infty$ for some $1\leqq q\leqq 2$. A counter-example to show that this condition cannot be dropped can be found in Section 2 of [2].

The law of the iterated logarithm can be formulated in terms of $\tilde{N}(\varepsilon(2n\log\log n)^{1/2})$ as follows: If $EX_1=0$ and $EX_1^2=\sigma^2$, then

$$\begin{aligned} P[\tilde{N}(\varepsilon(2n\log\log n)^{1/2})<\infty &=1 \quad \text{if } \varepsilon>\sigma, \\ &=0 \quad \text{if } \varepsilon<\sigma. \end{aligned} \tag{1.9}$$

However, Slivka [11] showed that $E\tilde{N}^r(\varepsilon(2n\log\log n)^{1/2})=\infty$ for all $r>0$ and $\varepsilon>0$. In [14], Stratton sharpened Slivka's result and found that if X_1 is symmetric with $EX_1=0$ and $EX_1^{2(m+1)}<\infty$ for some positive integer m, then

$$E\tilde{N}^r(\sigma(2(1+\delta)n\log n)^{1/2})<\infty \quad \text{if } 1\leqq r<\min(m, 1+\delta); \tag{1.10a}$$

$$E\tilde{N}^r(\sigma(2(1+\delta)n\log n)^{1/2})=\infty \quad \text{if } r>1+\delta. \tag{1.10b}$$

There is in fact a misprint in Stratton's paper [14, p. 1012] where the moment condition $EX_1^{2m}<\infty$ should be changed to $EX_1^{2(m+1)}<\infty$ and the factor σ should be added to $(2(1+\delta)n\log n)^{1/2}$. Stratton's proof makes use of the Berry-Esseen bound, the form he quotes being the following result of Katz [7]:

$$E|X_1|^p<\infty \Rightarrow \sup_x |\Phi(x)-P[S_n\leqq \sigma n^{1/2}x]|\leqq C(p, \sigma^{-p}E|X_1|^p)n^{-(p-2)/2}, \tag{1.11}$$

where $C(p,\eta)$ is a universal constant depending only on p and η and Φ is the distribution function of the standard normal distribution. Now (1.11) is valid for $2<p\leqq 3$ and, as is well known (cf. [4, p. 53]), it cannot be extended to the case

$p>3$. However, Stratton's proof deals with the case $p=2(m+1)\geqq 4$ and therefore (1.11) cannot be applied as he stated. What should be applied instead is the following result due to Esseen [4, p. 73]: If $E|X_1|^k<\infty$ for some integer $k\geqq 3$, then for $|x|\geqq\{(1+\delta)(k-2)\log n\}^{1/2}$,

$$|\Phi(x)-P[S_n\leqq\sigma n^{1/2}x]|\leqq c(\delta,\beta_1,\ldots,\beta_k)\,n^{-(k-2)/2}/(1+|x|^k), \tag{1.12}$$

where δ is any fixed number with $0<\delta<1$ and $c(\delta,\beta_1,\ldots,\beta_k)$ is a finite constant depending only on δ and the moments $\beta_i=E|X_1|^i$, $i=1,\ldots,k$.

If we apply (1.12) in place of (1.11), we can indeed prove Stratton's result. However, by applying a result of [2] instead, we can drop the assumption that X_1 is symmetric and weaken the moment condition $EX_1^{2(m+1)}<\infty$; also we shall no longer require m to be an integer and we can prove the one-sided theorem involving $N(\sigma(2(1+\delta)n\log n)^{1/2})$ which then immediately implies the corresponding result for the number of exits $\tilde N$ of the two-sided region studied by Stratton. Our result, which will be proved in Section 3, is the following theorem:

Theorem 2. *Let $X_1, X_2, \ldots$ be i.i.d. random variables such that $EX_1=0$, $\infty>EX_1^2=\sigma^2>0$. Set $S_n=X_1+\cdots+X_n$ and define $N(\varepsilon(n\log n)^{1/2})$ as in* (1.1a) *and $L(\varepsilon(n\log n)^{1/2})$ as in* (1.5a). *Then for any $r>0$, the following statements are equivalent:*

$$\int_{[X_1>e]} X_1^{2(r+1)}(\log X_1)^{-(r+1)}\,dP<\infty. \tag{1.13}$$

$$EL^r(\varepsilon(n\log n)^{1/2})<\infty \quad \textit{for all } \varepsilon>(2r)^{1/2}\sigma. \tag{1.14}$$

$$EN^r(\varepsilon(n\log n)^{1/2})<\infty \quad \textit{for all } \varepsilon>(2r)^{1/2}\sigma. \tag{1.15}$$

$$EN^r(\varepsilon(n\log n)^{1/2})<\infty \quad \textit{for some } \varepsilon>0. \tag{1.16}$$

$$EL^r(\varepsilon(n\log n)^{1/2})<\infty \quad \textit{for some } \varepsilon>0. \tag{1.17}$$

As an easy corollary of Theorem 2, we obtain the following analogue for $\tilde L$ and $\tilde N$:

Theorem 3. *Let $X_1, X_2, \ldots$ be i.i.d., $S_n=X_1+\cdots+X_n$ and define $\tilde N(\varepsilon(n\log n)^{1/2})$, $\tilde L(\varepsilon(n\log n)^{1/2})$ as in* (1.1b) *and* (1.5b) *respectively. Then for any $r>0$, the following statements are equivalent:*

$$EX_1=0,\quad EX_1^2=\sigma^2 \quad \textit{and} \quad E|X_1|^{2(r+1)}(1+\log^+|X_1|)^{-(r+1)}<\infty. \tag{1.18}$$

$$E\tilde L^r(\varepsilon(n\log n)^{1/2})<\infty \quad \textit{for all } \varepsilon>(2r)^{1/2}\sigma. \tag{1.19}$$

$$E\tilde N^r(\varepsilon(n\log n)^{1/2})<\infty \quad \textit{for all } \varepsilon>(2r)^{1/2}\sigma. \tag{1.20}$$

$$E\tilde N^r(\varepsilon(n\log n)^{1/2})<\infty \quad \textit{for some } \varepsilon>0. \tag{1.21}$$

$$E\tilde L^r(\varepsilon(n\log n)^{1/2})<\infty \quad \textit{for some } \varepsilon>0. \tag{1.22}$$

The equivalence between (1.18) and (1.22) was first discovered in [9] where it was proved by applying Theorem 3 of [8]. This equivalence, as pointed out in [9], sharpens an earlier result of Strassen [13] who, by embedding the sample sum process in Brownian motion, has shown that if $EX_1=0$, $EX_1^2=\sigma^2$ and

$E|X_1|^p<\infty$ for some $p>2(r+1)$, then (1.19) holds. In [9], it is also proved that under the condition (1.18),

$$E\tilde{L}^r(\varepsilon(n\log n)^{1/2})=\infty \quad \text{if } \varepsilon<(2r)^{1/2}\sigma, \tag{1.23}$$

a result first obtained by Strassen under the stronger moment condition $E|X_1|^p<\infty$ for some $p>2(r+1)$. In view of (1.7), Stratton's result (1.10b) implies (1.23), but Stratton has to assume that X_1 is symmetric with $EX_1^{2(m+1)}<\infty$ for some integer $m>r$. In Section 3, we shall prove the following theorem which completely generalizes the results of Slivka and Stratton.

Theorem 4. *Let $X_1, X_2, \ldots$ be i.i.d. random variables, $S_n=X_1+\cdots+X_n$ and define $N(\varepsilon(n\log n)^{1/2})$ as in* (1.1a). *Let $r>0$.*

(i) *If X_1 is symmetric, then*

$$EN^r(\varepsilon(n\log n)^{1/2})=\infty \quad \textit{for all } 0<\varepsilon<(2r\,EX_1^2)^{1/2}. \tag{1.24}$$

(Since we do not assume any moment condition on X_1, EX_1^2 in (1.24) *can be infinite.)*

(ii) *If X_1 satisfies the moment condition* (1.18), *then* (1.24) *still holds.*

From Theorems 3 and 4, we see that for sample sums, Strassen's strengthening of the law of the iterated logarithm in terms of the finiteness of the r-th moment of the last exit time $\tilde{L}(\varepsilon(n\log n)^{1/2})$, a concept which he calls in [13] the r-quick lim sup, turns out to be equivalent to the finiteness of the r-th moment of the number of exits $\tilde{N}(\varepsilon(n\log n)^{1/2})$ considered by Stratton. Furthermore, in view of (1.6) and Theorem 1, the refinement of the strong law of large numbers by Severo, Slivka and Stratton, who consider the finiteness of r-th moment of $\tilde{N}(\varepsilon n^\alpha)$, again turns out to be equivalent to the notion of r-quick convergence studied by Strassen [13] and Lai [9], who consider the r-th moment of $\tilde{L}(\varepsilon n^\alpha)$ instead. Some statistical applications showing the usefulness of the concept of r-quick convergence can be found in [9] and the references thereof.

So far we have discussed boundaries of the form εn^α $(\alpha>1/2)$ and $\varepsilon(n\log n)^{1/2}$ to obtain the r-quick analogues of the Marcinkiewicz-Zygmund strong law and the law of the itered logarithm. In Section 4, we shall extend our results to general boundaries and in this connection, we obtain a general form of the Hsu-Robbins-Erdös-Baum-Katz theorem on convergence rates for the tail probabilities of sample sums.

2. Some Preliminary Lemmas

Lemma 1. *Let $X_1, X_2, \ldots$ be i.i.d. random variables, $S_n=X_1+\cdots+X_n$ and let $(b(n))_{n\geq 1}$ be a nondecreasing sequence of nonnegative numbers. Define $N=N(b(n))$ as in* (1.1a) *and let $N_m=\sum_{n=1}^{m} I_{[S_n\geq b(n)]}$. Assume that $EN^r<\infty$ for some $r>0$.*

(i) *If $r\geq 1$, then $\sum_{n=1}^{\infty} EN_n^{r-1} I_{[S_n\geq b(n)]}<\infty$.*

(ii) *If* $r>0$ *and* r *is not an integer, then* $\sum_{n=1}^{\infty} n^{r-[r]-1} EN_n^{[r]} I_{[S_n \geqq b(n)]} < \infty$.

Proof. (i) is an immediate consequence of the following relation:

$$N^r = N^{r-1} \sum_{n=1}^{\infty} I_{[S_n \geqq b(n)]} \geqq \sum_{n=1}^{\infty} N_n^{r-1} I_{[S_n \geqq b(n)]}. \tag{2.1}$$

If $r>1$, then (ii) follows directly from (i). Assume that $0<r<1$. It is easy to see that $(x+1)^r - x^r$ is decreasing in $x>0$. Let $N_0=0$. Then

$$EN_m^r = E \sum_{n=1}^{m} (N_n^r - N_{n-1}^r) = E \sum_{n=1}^{m} (N_n^r - N_{n-1}^r) I_{[S_n \geqq b(n)]}$$
$$\geqq E \sum_{n=1}^{m} (n^r - (n-1)^r) I_{[S_n \geqq b(n)]} \geqq r \sum_{n=1}^{m} n^{r-1} EI_{[S_n \geqq b(n)]}.$$

Letting $m \to \infty$ above gives the desired conclusion.

Lemma 2. *With the same notations as in Lemma* 1, *assume that* $EX_1=0$, $E|X_1|^q < \infty$ *for some* $1 \leqq q \leqq 2$ *and* $b^q(n) \geqq n$ *for all* n. *Then given any positive integer* r *and* $0<\delta<1$, *there exist positive constants* α, C *and integer* m_0 *depending only on* r, δ, q *and* $E|X_1|^q$ *such that* $1-\delta<\alpha<1$ *and for all* $m \geqq m_0$ *and* $\varepsilon>0$,

$$EN_m^r I_{[S_m \geqq \varepsilon b(m)]} \geqq C m^r P[S_{[\alpha m]} \geqq ((r+1)\delta + \varepsilon) b(m)]. \tag{2.2}$$

Proof. Let $1-\delta<A<1$ be a constant such that

$$4(1-A) E|X_1|^q \leqq \delta^q. \tag{2.3}$$

Take any positive integer k and positive integers $1 \leqq i_1 \leqq \cdots \leqq i_k$ satisfying $i_k \geqq A\, i_{k+1}$. Then by the Markov inequality and an inequality due to von Bahr and Esseen [5],

$$P[S_{i_k} - S_{i_{k-1}} \geqq -\delta b(i_k)] \geqq 1 - P[|S_{i_k} - S_{i_{k-1}}| > \delta b(i_k)]$$
$$\geqq 1 - 2(\delta b(i_k))^{-q} (i_k - i_{k-1}) E|X_1|^q \geqq \tfrac{1}{2}. \tag{2.4}$$

The last inequality above follows from (2.3) and the assumption that $b^q(n) \geqq n$. Therefore

$$P\left(\bigcap_{j=1}^{k} [S_{i_j} \geqq \varepsilon b(i_j)]\right) \geqq P\left(\bigcap_{j=1}^{k} [S_{i_j} \geqq \varepsilon b(i_k)]\right)$$
$$\geqq P[S_{i_k} - S_{i_{k-1}} \geqq -\delta b(i_k),\ S_{i_{k-1}} \geqq (\delta+\varepsilon) b(i_k),\ S_{i_{k-2}} \geqq \varepsilon b(i_k), \ldots, S_{i_1} \geqq \varepsilon b(i_k)]$$
$$\geqq \tfrac{1}{2} P\left(\bigcap_{j=1}^{k-1} [S_{i_j} \geqq (\delta+\varepsilon) b(i_k)]\right)$$
$$\geqq \cdots \geqq (\tfrac{1}{2})^{k-1} P[S_{i_1} \geqq ((k-1)\delta + \varepsilon) b(i_k)]. \tag{2.5}$$

Let $\alpha = (1+A)/2$ and $m_0 \geqq 2/(1-A)$. Then for $m \geqq m_0$, $[\alpha m] \geqq Am$ and

$$EN_m^r I_{[S_m \geqq \varepsilon b(m)]} \geqq E\left\{ I_{[S_{[\alpha m]} \geqq \varepsilon b([\alpha m])]} \left(\sum_{i=[\alpha m]}^{m} I_{[S_i \geqq \varepsilon b(i)]}\right)^r I_{[S_m \geqq \varepsilon b(m)]}\right\}$$
$$\geqq ((1-\alpha)m)^r (\tfrac{1}{2})^{r+1} P[S_{[\alpha m]} \geqq ((r+1)\delta + \varepsilon) b(m)]. \tag{2.6}$$

The last relation follows from (2.5) and the multinomial expansion of $\left(\sum_{i=[\alpha m]}^{m} I_{[S_i \geqq \varepsilon b(i)]}\right)^r$ since there are at most r distinct factors in each of the $(m-[\alpha m])^r$ terms (not necessarily distinct) of the multinomial expansion.

Lemma 3. *With the same notations as in Lemma* 1, *if* $EN^\gamma < \infty$ *for some* $0<\gamma<1$, *then* $\sum n^{\gamma-1} P[S_n \geqq b(n)] < \infty$. *Now assume that*

$$EN^r < \infty \quad \text{for some } r \geqq 1. \tag{2.7}$$

(i) *If* $EX_1 = 0$, $E|X_1|^q < \infty$ *for some* $1 \leqq q \leqq 2$ *and* $b^q(n) \geqq n$ *for all large* n, *then given any* $\delta > 0$, *there exists* α *such that* $1-\delta<\alpha<1$ *and*

$$\sum n^{r-1} P[S_{[\alpha n]} \geqq ((r+1)\delta+1)\, b(n)] < \infty. \tag{2.8}$$

(ii) *If* X_1 *is symmetric, then given any* $0<\alpha<1$,

$$\sum n^{r-1} P[S_{[\alpha n]} \geqq b(n)] < \infty. \tag{2.9}$$

Proof. By Lemma 1 (ii), $EN^\gamma < \infty$ implies that $\sum n^{\gamma-1} P[S_n \geqq b(n)] < \infty$ if $0<\gamma<1$. Now assume (2.7). To prove (i), since changing $b(n)$ for finitely many n's does not change (2.7), we can without loss of generality assume that $b^q(n) \geqq n$ for all n. If r is an integer, then (2.8) follows from Lemma 1 (i) and Lemma 2. If r is not an integer, then by Lemma 1 (ii), $\sum n^{r-[r]-1} EN_n^{[r]} I_{[S_n \geqq b(n)]} < \infty$, so an application of Lemma 2 gives (2.8). To prove (ii), since X_1 and $-X_1$ have the same distribution, (2.7) implies that $E\tilde{N}^r < \infty$. An application of Lemmas 2 and 3 of [14] then proves (2.9) in a similar manner.

Lemma 4. *Suppose* $X_1, X_2, \ldots$ *are independent symmetric random variables and* $a_1, a_2, \ldots$ *are positive constants. Let* $X_n' = X_n I_{[|X_n| \leqq a_n]}$, $S_n' = X_1' + \cdots + X_n'$ *and* $S_n = X_1 + \cdots + X_n$. *Then for any* $\varepsilon > 0$,

$$P[S_n \geqq \varepsilon] \geqq \tfrac{1}{2} P[S_n' \geqq \varepsilon]. \tag{2.10}$$

Proof. Let $X_n'' = X_n - X_n'$, $S_n'' = S_n - S_n'$. By symmetry, (X_n', X_n'') and $(X_n', -X_n'')$ have the same distribution. Hence by independence, (S_n', S_n'') and $(S_n', -S_n'')$ have the same distribution. Therefore

$$\begin{aligned} P[S_n' \geqq \varepsilon] &\leqq P[S_n' \geqq \varepsilon, S_n'' \geqq 0] + P[S_n' \geqq \varepsilon, S_n'' \leqq 0] \\ &= 2P[S_n' \geqq \varepsilon, S_n'' \geqq 0] \leqq 2P[S_n \geqq \varepsilon]. \end{aligned}$$

3. Proof of Theorems

Proof of Theorem 1. Under the assumptions $EX_1 = 0$, $E|X_1|^q$ for some $1 \leqq q \leqq 2$ and $\alpha > 1/q$, it follows from the corollary to Theorem 1 of [2] (see also Theorem 5 of that paper) that the following statements are equivalent:

$$E(X_1^+)^{(r+1)/\alpha} < \infty; \tag{3.1}$$

$$EL^r(\varepsilon n^\alpha) < \infty \quad \text{for all } \varepsilon > 0; \tag{3.2}$$

$$\sum n^{r-1} P[S_n \geqq \varepsilon n^\alpha] < \infty \quad \text{for some } \varepsilon > 0. \tag{3.3}$$

We shall now show that these statements are equivalent to each of the following statements:

$$EN^r(\varepsilon n^\alpha)<\infty \quad \text{for all } \varepsilon>0; \tag{3.4}$$

$$EN^r(\varepsilon n^\alpha)<\infty \quad \text{for some } \varepsilon>0. \tag{3.5}$$

Since $N(\varepsilon n^\alpha)\leqq L(\varepsilon n^\alpha)$, (3.2) implies (3.4). Clearly (3.4) implies (3.5). Since $q\alpha>1$, it follows from Lemma 3 that (3.5) implies (3.3). The proof of Theorem 1 is complete.

Proof of Theorem 2. By Theorem 4 of [2], (1.13) is equivalent to each of the following statements:

$$\sum n^{r-1} P[\sup_{k\geqq n}(k\log k)^{-1/2} S_k\geqq\varepsilon]<\infty \quad \text{for all } \varepsilon>(2r)^{1/2}\sigma; \tag{3.6}$$

$$\sum n^{r-1} P[S_n\geqq\varepsilon(n\log n)^{1/2}]<\infty \quad \text{for some } \varepsilon>0. \tag{3.7}$$

Clearly (3.6) $\Leftrightarrow$ (1.14) $\Rightarrow$ (1.17) $\Rightarrow$ (3.7). It is obvious from (1.8) that (1.14) $\Rightarrow$ (1.15). Since (1.15) $\Rightarrow$ (1.16), it remains to prove (1.16) $\Rightarrow$ (3.7). This implication follows from Lemma 3 since $\varepsilon^2 n\log n>n$ for all large n.

Proof of Theorem 4. To prove (i), since $0<\varepsilon<(2rEX_1^2)^{1/2}$, we can choose $\alpha>1$ and c such that

$$\alpha\varepsilon<(2r\operatorname{Var} X_1 I_{[|X_1|\leqq c]})^{1/2}. \tag{3.8}$$

Define

$$X_n'=X_n I_{[|X_n|\leqq c]},\ S_n'=X_1'+\cdots+X_n',\ \tau^2=E(X_1')^2. \tag{3.9}$$

Since X_1 is symmetric, $EX_1'=0$ and we obtain by Lemma 4 that

$$P[S_n\geqq\alpha\varepsilon(n\log n)^{1/2}]\geqq\tfrac{1}{2}P[S_n'\geqq\alpha\varepsilon(n\log n)^{1/2}]. \tag{3.10}$$

Choose $k\geqq 6$ such that $\alpha\varepsilon/\tau<(k-2)^{1/2}$. Then by an inequality due to Esseen [4, p. 75–76],

$$\begin{aligned} P[S_n'/(\tau n^{1/2})&\geqq(\alpha\varepsilon/\tau)(\log n)^{1/2}]\\ &\geqq 1-\Phi((\alpha\varepsilon/\tau)(\log n)^{1/2})-c_1 n^{-1/2}\{1+(\alpha\varepsilon/\tau)^3(\log n)^{3/2}\}\\ &\quad\cdot\exp(-\alpha^2\varepsilon^2\log n/(2\tau^2))-c_2 n^{-(k-2)/2}\\ &=(c+o(1))(\log n)^{-1/2}\exp(-\alpha^2\varepsilon^2\log n/(2\tau^2)), \end{aligned} \tag{3.11}$$

where c, c_1, c_2 are positive constants and Φ is the distribution function of the standard normal distribution. From (3.8) and (3.11), we obtain that

$$\sum n^{r-1} P[S_n'\geqq\alpha\varepsilon(n\log n)^{1/2}]=\infty,$$

so by (3.10),

$$\sum n^{r-1} P[S_n\geqq\alpha\varepsilon(n\log n)^{1/2}]=\infty. \tag{3.12}$$

If $0<r<1$, then by Lemma 3, (3.12) implies that $EN^r(\varepsilon(n\log n)^{1/2})=\infty$. Now assume $r \geqq 1$ and take $1 < \beta < \alpha$. It is easy to see that (3.12) implies that

$$\sum n^{r-1} P[S_{[\beta^{-2}n]} \geqq \varepsilon(n \log n)^{1/2}] = \infty. \tag{3.13}$$

Therefore by Lemma 3 (ii), $EN^r(\varepsilon(n \log n)^{1/2}) = \infty$. Hence we have proved part (i) of Theorem 4.

To prove part (ii) of the theorem, since X_1 need not be symmetric, instead of defining X'_n by (3.9), we define

$$\begin{aligned} &X'_n = X_n I_{[|X_n| \leqq c]} - EX_1 I_{[|X_1| \leqq c]}, \qquad \tau^2 = E(X'_1)^2, \\ &X''_n = X_n - X'_n, \qquad S'_n = X'_1 + \cdots + X'_n, \qquad S''_n = X''_1 + \cdots + X''_n, \end{aligned} \tag{3.14}$$

where we choose $\alpha > 1$ and c so large that (3.8) is satisfied and

$$2r E(X''_1)^2 < (\theta \varepsilon)^2 \qquad \text{with } 1 < 1 + \theta < \alpha. \tag{3.15}$$

Take $\beta > 1$ such that $\beta + \theta < \alpha$. Then

$$\begin{aligned} \sum n^{r-1} P[S_n \geqq \beta \varepsilon(n \log n)^{1/2}] + \sum n^{r-1} P[|S''_n| \geqq \theta \varepsilon(n \log n)^{1/2}] \\ \geqq \sum n^{r-1} P[S'_n \geqq \alpha \varepsilon (n \log n)^{1/2}]. \end{aligned} \tag{3.16}$$

As before, making use of Esseen's inequality as in (3.11), we obtain that

$$\sum n^{r-1} P[S'_n \geqq \alpha \varepsilon (n \log n)^{1/2}] = \infty. \tag{3.17}$$

In view of (3.15), it follows from Theorem 3 of [8] that

$$\sum n^{r-1} P[|S''_n| \geqq \theta \varepsilon (n \log n)^{1/2}] < \infty. \tag{3.18}$$

From (3.16), (3.17) and (3.18), it follows that

$$\sum n^{r-1} P[S_n \geqq \beta \varepsilon (n \log n)^{1/2}] = \infty. \tag{3.19}$$

Take $0 < \delta < 1$ such that $(1-\delta)^{-1/2}\{(r+1)\delta + 1\} < \beta$. It is easy to see from (3.19) that for all α with $1 - \delta < \alpha < 1$,

$$\sum n^{r-1} P[S_{[\alpha n]} \geqq ((r+1)\delta + 1)\, \varepsilon (n \log n)^{1/2}] = \infty. \tag{3.20}$$

Since $\varepsilon^2 (n \log n) > n$ for all large n, it follows from Lemma 3(i) that

$$EN^r(\varepsilon(n \log n)^{1/2}) = \infty \tag{3.21}$$

when $r \geqq 1$. When $0 < r < 1$, (3.19) implies (3.21) by Lemma 3.

4. General Upper-Class Boundaries and the Convergence Rate of Tail Probabilities for Sample Sums

In [13, pp. 339–340], Strassen has proved the following theorem: If $S_n = X_1 + \cdots + X_n$, where $X_1, X_2, \ldots$ are i.i.d. with $EX_1 = 0$, $EX_1^2 = \sigma^2$ and and $E|X_1|^p < \infty$ for some $p > 2(r+1)$, and if $t^{-1/2} b(t) \uparrow \infty$ while $t^{-q} b(t) \downarrow 0$ as $t \uparrow \infty$

for some $q>1/2$, then the following two statements are equivalent:

$$\int_1^\infty t^{r-3/2} b(t) \exp(-b^2(t)/2\sigma^2 t)\, dt < \infty; \tag{4.1}$$

$$E\tilde{L}^r(b(n)) < \infty. \tag{4.2}$$

In particular, if for all large t,

$$b(t) = \{2\sigma^2 t(r \log t + \tfrac{3}{2}\log_2 t + \log_3 t + \cdots + \log_{k-1} t + \delta \log_k t)\}^{1/2},$$

where $\log_2 t = \log\log t$, etc., then

$$E\tilde{L}^r(b(n)) = \infty \quad \text{or} \quad < \infty \quad \text{according as} \quad \delta \leqq 1 \quad \text{or} \quad \delta > 1. \tag{4.3}$$

In what follows, we shall assume $EX_1 = 0$ and consider boundaries $b(t)$ which satisfy (4.1). We have seen in (1.6) that if $b(t) = t^\alpha$ ($\alpha > \frac{1}{2}$), then (4.2) holds if and only if $E|X_1|^{(r+1)/\alpha} < \infty$, i.e., $E(\Psi(|X_1|))^{r+1} < \infty$, where $\Psi(t) = t^{1/\alpha}$ is the inverse function of $b(t)$. Likewise Theorem 3 asserts that if $b(t) = \varepsilon(t \log t)^{1/2}$, $t \geqq 1$, where $\varepsilon > (2r)^{1/2}\sigma$, then (4.2) holds if and only if $E(\Psi(|X_1|))^{r+1} < \infty$, where

$$\Psi(t) \sim t^2/(2\varepsilon^2 \log t)$$

is the inverse function of $b(t)$. This observation suggests the general analogue in Theorems 5 and 6 below for the last exit time of the region bounded by general boundaries $b(t)$ and $-b(t)$.

In the proof of Theorems 1, 2 and 4 in Section 3, we have seen that the relation (4.2) is closely related to the convergence of a certain type of series. In [1], Baum and Katz studied some series of this type and proved that for $\alpha > \frac{1}{2}$ and $r > 0$, the following three statements are equivalent:

$$E|X_1|^{(r+1)/\alpha} < \infty \qquad \text{and in the case } \alpha \leqq 1,\ EX_1 = 0; \tag{4.4}$$

$$\sum n^{r-1} P[\sup_{k \geqq n} k^{-\alpha}|S_k| \geqq \varepsilon] < \infty \qquad \text{for all } \varepsilon > 0; \tag{4.5}$$

$$\sum n^{r-1} P[|S_n| \geqq \varepsilon n^\alpha] < \infty \qquad \text{for some } \varepsilon > 0. \tag{4.6}$$

This result generalizes an earlier theorem of Hsu and Robbins [6] and Erdös [3] who consider the special case $\alpha = r = 1$, and the proof given by Baum and Katz follows closely that of Erdös. In [2], by using a different approach, Chow and Lai have obtained inequalities, called Paley-type inequalities, which relate the series in (4.5) or (4.6) with $E|X_1|^{(r+1)/\alpha}$ and thereby give another proof of the result of Baum and Katz. By an extension of the method of [2], we can prove the following generalization of the Hsu-Robbins-Erdös-Baum-Katz theorem.

Theorem 5. *Let $X_1, X_2, \ldots$ be i.i.d. random variables such that $EX_1 = 0$ and $E|X_1|^q < \infty$ for some $1 \leqq q \leqq 2$, and let $S_n = X_1 + \cdots + X_n$. Let b be a real-valued function on $[1, \infty)$ satisfying the following condition:*

$b(t)$ is ultimately nondecreasing, $\liminf_{t\to\infty} b(\delta t)/b(t) > 1$ for all large δ, and there exists $\beta > 1/q$ such that $\lim_{t\to\infty} t^{-\beta} b(t) = \infty$. (4.7)

Define $\Psi(x)=\inf\{t: b(t)>x\}$ *for* $x\geqq 0$. *Then for any* $r>0$, *the following statements are equivalent:*

$$E(\Psi(X_1^+))^{r+1}<\infty; \tag{4.8}$$

$$\sum n^{r-1} P[\max_{j\leqq n} S_j\geqq b(\varepsilon n)]<\infty \quad \textit{for all } \varepsilon>0; \tag{4.9}$$

$$\sum n^{r-1} P[S_n\geqq \varepsilon b(\varepsilon n)]<\infty \quad \textit{for some } \varepsilon>0; \tag{4.10}$$

$$EL^r(b(n))<\infty; \tag{4.11}$$

$$EN^r(b(n))<\infty. \tag{4.12}$$

Proof. We shall prove (4.8) ⇒ (4.9) ⇒ (4.11) ⇒ (4.12) ⇒ (4.10) ⇒ (4.8). To prove (4.8) ⇒ (4.9), without loss of generality, we can assume that $E|X_1|\neq 0$. Let k be a positive integer such that $k(q\beta-1)>r$. We note that

$$P[\max_{j\leqq n} S_j\geqq b(\varepsilon n)]\leqq P[\max_{j\leqq n} X_j\geqq b(\varepsilon n)/2k] + P[\max_{j\leqq n} S_j\geqq b(\varepsilon n), \max_{j\leqq n} X_j\leqq b(\varepsilon n)/2k]. \tag{4.13}$$

For large t, $x\geqq b(t)$ implies that $\Psi(x)\geqq t$. Therefore for all large n,

$$P[\max_{j\leqq n} X_j\geqq b(\varepsilon n)/2k]\leqq nP[X_1\geqq b(\varepsilon n)/2k]\leqq nP[\Psi(2kX_1^+)\geqq \varepsilon n]. \tag{4.14}$$

Define $\tau_1^{(n)}=\inf\{j\geqq 1: S_j\geqq b(\varepsilon n)/2k\}$, $\tau_2^{(n)}=\inf\{j\geqq 1: S_{\tau_1^{(n)}+j}-S_{\tau_1^{(n)}}\geqq b(\varepsilon n)/2k\}$, etc. Then

$$\begin{aligned} P[\max_{j\leqq n} S_j\geqq b(\varepsilon n), \max_{j\leqq n} X_j\leqq b(\varepsilon n)/2k] &\leqq P[\tau_i^{(n)}\leqq n \text{ for } i=1,\ldots,k] \\ &= P^k[\tau_1^{(n)}\leqq n]=P^k[\max_{j\leqq n} S_j\geqq b(\varepsilon n)/2k] \\ &\leqq \{(2k/b(\varepsilon n))^q E|S_n|^q\}^k, \quad \text{by the submartingale inequality} \\ &\leqq \{(4k/b(\varepsilon n))^q nE|X_1|^q\}^k, \quad \text{by the Esseen-von Bahr inequality [5]} \\ &= o(n^{-k(q\beta-1)}). \end{aligned} \tag{4.15}$$

The last relation above follows from the assumption that $\lim_{n\to\infty} n^{-\beta} b(n)=\infty$. It is easy to see from (4.7) that

$$\limsup_{x\to\infty} \Psi(\eta x)/\Psi(x)<\infty \quad \text{for any } \eta>1. \tag{4.16}$$

Hence (4.8) implies that $E(\Psi(2kX_1^+))^{r+1}<\infty$, and therefore

$$\sum n^r P[\Psi(2kX_1^+)\geqq \varepsilon n]<\infty. \tag{4.17}$$

Since $k(q\beta-1)>r$, it follows from (4.15) that

$$\sum n^{r-1} P[\max_{j\leqq n} S_j\geqq b(\varepsilon n), \max_{j\leqq n} X_j\leqq b(\varepsilon n)/2k]<\infty. \tag{4.18}$$

From (4.13), (4.14), (4.17) and (4.18), we obtain (4.9).

We shall now prove that (4.9) ⇒ (4.11). Without loss of generality, we can assume that $b(t)$ is nondecreasing for $t \geqq 1$. We need only note that

$$\sum_{i=1}^{\infty} 2^{ri} P[S_n \geqq b(n) \text{ for some } n \geqq 2^i] \leqq \sum_{i=1}^{\infty} 2^{ri} \sum_{j=i}^{\infty} P[\max_{n \leqq 2^{j+1}} S_n \geqq b(2^j)]$$

$$\leqq c \sum_{j=1}^{\infty} 2^{rj} P[\max_{n \leqq 2^{j+1}} S_n \geqq b(2^j)]$$

$$\leqq c_1 \sum_{j=0}^{\infty} \sum_{2^{j+1} \leqq m < 2^{j+2}} m^{r-1} P[\max_{n \leqq m} S_n \geqq b(m/4)]$$

$$= c_1 \sum_{m=2}^{\infty} m^{r-1} P[\max_{n \leqq m} S_n \geqq b(m/4)].$$

It is obvious from (1.8) that (4.11) $\Rightarrow$ (4.12). To prove (4.12) $\Rightarrow$ (4.10), we apply Lemma 3. If $0<r<1$, then the desired result is immediate from Lemma 3. Now assume that $r \geqq 1$. By (4.7), $b^q(n) > n$ for all large n. Therefore if (4.12) holds, then by Lemma 3, there exists $\frac{1}{2} < \alpha < 1$ such that

$$\sum n^{r-1} P[S_{[\alpha n]} \geqq (\tfrac{1}{2}(r+1)+1) b(n)] < \infty. \tag{4.19}$$

It is obvious that (4.19) $\Rightarrow$ (4.10).

To prove (4.10) $\Rightarrow$ (4.8), we follow the argument due to Erdös [3]. For $k=1, \ldots, n$, let $A_k^{(n)} = [X_k \geqq 2\varepsilon b(\varepsilon n)]$, $B_k^{(n)} = [|\sum_{1 \leqq j \leqq n,\, j \neq k} X_j| \leqq \varepsilon b(\varepsilon n)]$. Since $\beta > 1/q$ and $E|X_1|^q < \infty$, $n^{-\beta} S_{n-1} \to 0$ a.s. by the Marcinkiewicz-Zygmund strong law of large numbers. Therefore by (4.7), $\lim_{n\to\infty} P(B_1^{(n)}) = 1$. Also for all large n, $P(A_1^{(n)}) \leqq P[X_1 \geqq n^{1/q}] = o(1/n)$ since $E|X_1|^q < \infty$. Therefore we can choose n_0 such that for $n \geqq n_0$, $P(B_1^{(n)} - n P(A^{(n)}) \leqq 1/2$. Hence for $n \geqq n_0$,

$$P[S_n \geqq \varepsilon b(\varepsilon n)] \geqq \sum_{k=1}^{n} \{P(A_k^{(n)} \cap B_k^{(n)}) - P(A_k^{(n)} \cap (A_1^{(n)} \cup \cdots \cup A_{k-1}^{(n)}))\}$$

$$\geqq \sum_{k=1}^{n} P(A_k^{(n)})(P(B_1^{(n)}) - n P(A_1^{(n)})) \geqq \tfrac{1}{2} n P(A_1^{(n)}).$$

Hence (4.10) implies that

$$\sum n^r P[X_1 \geqq 2\varepsilon b(\varepsilon n)] < \infty. \tag{4.20}$$

Since $\Psi(x) > t$ implies $x \geqq b(t)$ for all large t, it is easy to see using (4.16) that (4.20) implies (4.8).

By a modification of the proof of Theorem 5, we obtain the following theorem which considers boundaries $b(t)$ satisfying weaker growth conditions than (4.7) but under stronger two-sided moment conditions on X_1.

Theorem 6. *Let $X_1, X_2, \ldots$ be i.i.d. random variables such that $EX_1 = 0$ and $E|X_1|^{2+\theta} < \infty$ for some $\theta > 0$. Let $S_n = X_1 + \cdots + X_n$, and let b be a real-valued function on $[1, \infty)$ satisfying the following condition:*

$b(t)$ is ultimately nondecreasing, $\liminf_{t\to\infty} b(\delta t)/b(t) > 1$

for all large δ and $\lim_{t\to\infty} (t \log t)^{-1/2} b(t) = \infty$. (4.21)

Define $\Psi(x)=\inf\{t\colon b(t)>x\}$ *for* $x\geqq 0$. *Then for any* $r>0$, *the statements* (4.8), (4.9), (4.10), (4.11) *and* (4.12) *are equivalent.*

Proof. We first prove (4.8) $\Rightarrow$ (4.9). Without loss of generality, we can assume that $0<\theta<1$. Choose a positive integer k such that $k\theta/2>r$. Letting $\sigma^2=EX_1^2$, we can without loss of generality assume that $\sigma\neq 0$. As in the proof of Theorem 5, we obtain that

$$P[\max_{j\leqq n} S_j\geqq b(\varepsilon n), \max_{j\leqq n} X_j\leqq b(\varepsilon n)/2k]\leqq P^k[\max_{j\leqq n} S_j\geqq b(\varepsilon n)/2k]. \tag{4.22}$$

By the Lévy inequality (cf. [10, p. 248]),

$$\begin{aligned} P[\max_{j\leqq n} S_j\geqq b(\varepsilon n)/2k] &\leqq 2P[S_n\geqq b(\varepsilon n)/2k-(2n\sigma^2)^{1/2}] \\ &\leqq 2P[S_n\geqq b(\varepsilon n)/3k]. \end{aligned} \tag{4.23}$$

Using (1.11), we obtain that for all large n

$$\begin{aligned} P^k[S_n\geqq b(\varepsilon n)/3k] &\leqq \{1-\Phi(b(\varepsilon n)/(3k\sigma n^{1/2}))+c n^{-\theta/2}\}^k \\ &\leqq c_1 \exp(-b^2(\varepsilon n)/(18k\sigma^2 n))+c_2 n^{-k\theta/2} \end{aligned} \tag{4.24}$$

where c, c_1 and c_2 are positive constants. Since $\lim_{n\to\infty} b^2(\varepsilon n)/(n\log n)=\infty$ and $k\theta/2>r$, it follows from (4.23) and (4.24) that

$$\sum n^{r-1} P^k[\max_{j\leqq n} S_j\geqq b(\varepsilon n)/2k]<\infty. \tag{4.25}$$

Using (4.22) and (4.25), we can prove the implication (4.8) $\Rightarrow$ (4.9) as in Theorem 5.

The proof of (4.9) $\Rightarrow$ (4.11) in Theorem 5 carries over and since $\lim_{n\to\infty} b^2(n)/n=\infty$, so does the proof of (4.12) $\Rightarrow$ (4.10). Since $EX_1^2<\infty$, $(n\log n)^{-1/2} S_n\to 0$ a.s. and $P[X_1\geqq n^{1/2}]=o(1/n)$. Therefore we can again apply the argument due to Erdös as in Theorem 5 to prove (4.10) $\Rightarrow$ (4.8).

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References

1. Baum, L., Katz, M.: Convergence rates in the law of large numbers. Trans. Amer. Math. Soc. **120**, 108–123 (1965)
2. Chow, Y. S., Lai, T. L.: Some one-sided theorems on the tail distribution of sample sums with applications to the last time and largest excess of boundary crossings. Trans. Amer. Math. Soc. **208**, 51–72 (1975)
3. Erdös, P.: On a theorem of Hsu and Robbins. Ann. Math. Statist. **20**, 286–291 (1949)
4. Esseen, C. G.: Fourier analysis of distributions of functions. A mathematical study of the Laplace-Gaussian law. Acta Math. **11**, 1–125 (1945)
5. Esseen, C. G., von Bahr, B.: Inequalities for the rth absolute moment of a sum of random variables, $1<r<2$. Ann. Math. Statist. **36**, 299–303 (1965)
6. Hsu, P. L., Robbins, H.: Complete convergence and the law of large numbers. Proc. Nat. Acad. Sci. U.S.A. **33**, 25–31 (1947)
7. Katz, M.: The probability in the tail of a distribution. Ann. Math. Statist. **34**, 312–318 (1963)
8. Lai, T. L.: Limit theorems for delayed sums. Ann. Probability **2**, 432–440 (1974)

9. Lai, T. L.: On r-quick convergence and a conjecture of Strassen. To appear in Ann. Probability **4** (1976)
10. Loève, M.: Probability Theory. Princeton: Van Nostrand (1963)
11. Slivka, J.: On the law of the iterated logarithm. Proc. Nat. Acad. Sci. U.S.A. **63**, 289–291 (1969)
12. Slivka, J., Severo, N. C.: On the strong law of large numbers. Proc. Amer. Math. Soc. **24**, 729–734 (1970)
13. Strassen, V.: Almost sure behavior of sums of independent random variables and martingales. Proc. 6th Berkeley Sympos. Math. Statist. Probab. Univ. Calif., **2**, 315–343 (1967)
14. Stratton, H.: Moments of oscillations and ruled sums. Ann. Math. Statist. **43**, 1012–1016 (1972)

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