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Occupation-Times for Functions with Countable Level Sets and the Regeneration of Stationary Processes *

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The first part of this paper gives the Lebesgue decomposition relative to a Radon measure π on $\mathbb{R}$ of the occupation-time measure $B \to m(x^{-1}(B))$ (m Lebesgue measure, B a Borel set) of a real function $x(t)$ which satisfies a certain condition on the size of its level sets. When $x(t)$ has an approximate derivative $\dot{x}_{\mathrm{ap}}(t)$ for m-a.e. t the decomposition can be written explicitly in terms of $\dot{x}_{\mathrm{ap}}(t)$ and the multiplicity kernel $\nu(y, A)$ = cardinality of $\{t \in A: x(t) = y\}$. In the second part, we first give probabilistic conditions which allow the real-variable results to be applied to almost every trajectory of a stationary random process $X_t(\omega)$. We then exhibit various random times at which the process regenerates itself probabilistically: letting $\tau(\omega)$ be the first $t > 0$ at which $X_t(\omega) = X_0(\omega)$ and the (approximate) derivative $\dot{X}_t(\omega)$ exists, we characterize the class of measures which are dominated by the law P of $X_t(\omega)$ and invariant under the random time shift which moves the origin to $\tau(\omega)$; we also give a necessary and sufficient condition for a "random iterate" of this transformation to preserve P itself and describe the invariant σ-fields for these transformations.

§ 0. Introduction

This paper extends our earlier work [8] on occupation-times for stationary random processes and investigates certain random shifts of the time origin which preserve the law of the process. Part I (§§ 1–3) is concerned entirely with non-random functions of a real variable; the results are applied to processes in Part II (§§ 4–9).

Let π be a Radon measure (i.e. finite on compacts) on the Borel σ-field $\mathscr{B}$ of the real line $\mathbb{R}$ and $x(t)$ a real-valued Borel function on $\mathbb{R}$. We consider (§ 2) the Lebesgue decomposition relative to π of the *occupation-time measure* $\mu_t(\Gamma) = m(x^{-1}(\Gamma) \cap (0, t])$, $\Gamma \in \mathscr{B}$, m Lebesgue measure, when $x(t)$ satisfies the condition $(T_1\, G, \pi)$ of § 1 having to do with the size of the level sets. When $\pi = m$, $(T_1\, G, m)$

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lies between the classical conditions (T_1) and (T_2) of Banach. We also prove that Lusin's condition (N) implies $(T_1\, G, m)$, and proceed to identify the components of μ_t more explicitly for functions which have an approximate derivative a.e. Despite its purely analytical character, the occupation-time measure does not seem to be in the literature on functions of a real variable; of course, it is well-known in probability theory (see [10] and the references therein). Connections with related results in real variables (especially Sarkhel's [22]) are drawn in § 3, and several open questions are indicated.

In Part II the foregoing results are applied to the trajectories of random processes. We begin (§ 4) by relating the present work to our earlier results [10] on local times and giving concrete illustrations of the abstract results in [10, 12]. Starting with § 5 we concentrate on *homogeneous* stationary processes, i.e. those of the form $X_t = X \circ \theta_t$, X being a random variable over a dynamical system $(\Omega, \mathscr{F}^0, P, \theta_t)$. We restrict attention to trajectories which are (approximately) differentiable a.e.; Section 5 gives probabilistic conditions which imply this for almost every trajectory, and establishes the good measurability properties of the derivative process. There is also a short discussion which indicates that approximate differentiability is a natural concept in the subject of random processes rather than an idle generalization.

The main applications to processes are in §§ 6, 7. Let $\tau(\omega)$ be the first $t>0$ at which $X_t(\omega) = X_0(\omega)$ and the (approximate) derivative $\dot{X}_t(\omega)$ exists. We describe the class of measures on Ω which are dominated by P and invariant under the random time shift $\phi(\omega) = \theta_{\tau(\omega)}(\omega)$, and give a necessary and sufficient condition for a "random iterate" of ϕ to preserve P itself. In effect we are exhibiting random times at which the process "starts over" probabilistically – one such is the time of first return of the vector process $(X_t, |\dot{X}_t|)$ to its initial position. As by-products we obtain extensions of the formulae of Rice (mean number of level crossings) and Kac (mean return time) and of an observation of Neveu [19] which may be of interest on its own.

Two examples are given in § 8, the second of which, involving the torus, shows how awkward it may be to deal directly with even a very simple process. Similar examples, having discontinuous but still relatively "smooth" trajectories motivated the problem of extending the results in [8]. Finally, in § 9, we briefly discuss a continuous-parameter flow which arises in place of ϕ when the level sets of X_t are no longer countable.

I. Occupation-Times for Approximately Differentiable Functions

§ 1. Preliminaries

Let $x(t)$ be a real-valued Borel function on $\mathbb{R}$. We write m for Lebesgue measure on $\mathscr{B}$, the Borel σ-field on $\mathbb{R}$. For each real t (construed as "time") we define the *occupation-time measure*

$$\mu_t(\Gamma) = \int_0^t I_\Gamma(x(s))\, ds = m(x^{-1}(\Gamma) \cap (0, t]), \qquad \Gamma \in \mathscr{B}.$$

Our primary interest is the Lebesgue decomposition

$$\mu_t(dy) = \alpha_t(y)\,\pi(dy) + \bar{\mu}_t(dy)$$

of μ_t relative to a Radon measure π on $\mathscr{B}$. When the μ-singular component $\bar{\mu}_t$ vanishes, the density $\alpha_t(y)$ of the π-absolutely continuous component is called *local time* (at y) and we say that $x(t)$ satisfies the condition (LT) on $[0, t]$. It is easy to see that $\alpha_t(y)$ may always be "regularized" to be a non-decreasing, right-continuous function of t for every y, hence jointly (t, y)-measurable. We always assume this to be done and freely regard $\alpha.(y)$ as a measure on $\mathscr{B}$.

The condition (LT) is a kind of obverse to the classical condition (N) of Lusin, viz. $m(E)=0$ implies $m(x(E))=0$ for $E \in \mathscr{B}$. Intuitively, it means that a traveller whose position at time t is $x(t)$ spends (Lebesgue measure) zero time in any π-negligible set.

The *multiplicity kernel* of $x(t)$ is $\nu(y, B) = \operatorname{card}\{t \in B\colon x(t) = y\}$, $y \in \mathbb{R}$, $B \in \mathscr{B}$. To justify the integrals below involving ν we will need:

(1) **Proposition.** *For each y fixed, $B \to \nu(y, B)$ is a measure on $\mathscr{B}$; for each $B \in \mathscr{B}$ fixed, $y \to \nu(y, B)$ is universally measurable.*

Note. "Universal measurability" means measurability relative to the universal completion $\mathscr{B}^*$ of $\mathscr{B}$, defined as the intersection of all the σ-fields $\mathscr{B}_\mu$ where μ is a finite Borel measure and $\mathscr{B}_\mu$ the μ-completion of $\mathscr{B}$.

Proof. The first assertion is obvious. For the second, if $x(t)$ is continuous and B is an interval, then a classical argument [11, p. 280] shows that $y \to \nu(y, B)$ is actually $\mathscr{B}$-measurable. In general, if $B \in \mathscr{B}$, recall that $x(B)$ is an analytic set since $x(t)$ is Borel [17, p. 35] and every analytic set is in $\mathscr{B}^*$ [17, p. 44]. From here the argument in [6, p. 176] concludes the proof.

We will say that $x(t)$ satisfies the condition (T_1, π) (or simply "is (T_1, π)") on $E \in \mathscr{B}$ if $\pi\{y\colon \nu(y, E) = \infty\} = 0$; replacing "$\nu(y, E) = \infty$" by "$x^{-1}(y)$ is uncountable" we arrive at the definition of (T_2, π). Similarly (N, π) means that $\pi(x(E)) = 0$ whenever $m(E) = 0$, $E \in \mathscr{B}$, noting that $x(E) \in \mathscr{B}^*$. Finally we say $x(t)$ is $(T_1 G, \pi)$ on E (G for "generalized") if there exists a countable partition of E into Borel sets on each of which $x(t)$ is (T_1, π). Clearly $(T_1 G, \pi)$ implies (T_2, π), but we do not know whether the converse is true. When $\pi = m$, each of the above conditions (except $T_1 G$) reduces to a classical one, (T_1) and (T_2) being Banach's and (N) being Lusin's condition. Suppose F is any distribution function of π, i.e. $\pi(a, b] = F(b) - F(a)$ for every interval $(a, b]$. If π has no point masses, F is continuous and [21, p. 100] $m(F(E)) = \pi(E)$, for $E \in \mathscr{B}$. It follows, then, that $x(t)$ is (T_1, π) iff the function $\xi(t) = F \circ x(t)$ is (T_1), and similarly for the others.

Consider the set function $\zeta\colon \mathscr{B} \to [0, \infty]$ given by $\zeta(B) = \pi(x(B))$ and the outer measure ψ obtained by Carathéodory's construction from ζ; as in 2.10.10 of [6] we have

(2) $\psi(B) = \int \nu(y, B)\,\pi(dy), \qquad B \in \mathscr{B}.$

Notice that $\pi(x(B)) = 0$ implies $\psi(B) = 0$ – this will be very useful later.

(3) **Proposition.** *The measure ψ on $\mathscr{B}$ is σ-finite* iff *$x(t)$ is $(T_1 G, \pi)$ on* $\mathbb{R}$.

Proof. It is clear that $x(t)$ is $(T_1\,G, \pi)$ on any set of finite ψ-measure. If, on the other hand, $x(t)$ is $(T_1\,G, \pi)$, we have a Borel partition $\{E_n\}$ of $\mathbb{R}$ with $\nu(y, E_n)<\infty$, π-a.e. for each n, thus the sets

$$A_{n,m,k}=\{y\colon |y|\leqq m, \nu(y, E_n)=k-1\}, \qquad n, m, k\geqq 1,$$

and

$$A=\{y\colon \nu(y, E_n)=\infty \text{ for some } n\}$$

form a partition of $\mathbb{R}$, and are in $\mathscr{B}^*$. Let $B, B_{n,m,k}$ be Borel sets which enclose the corresponding A's and have the same π-measure. Then, by (2), $\psi(x^{-1}(B))=0$ since $\pi(A)=0$, $\psi(E_n\cap x^{-1}(B_{n,m,k}))=(k-1)\pi[-m,m]<\infty$, and the σ-finiteness of ψ follows.

The importance of (3) lies in the fact that we will want to use the Lebesgue decomposition of ψ when $x(t)$ is a trajectory of a random process having locally finite level sets.

It is known [7] that (N) implies (T_2), whence (N, π) implies (T_2, π) if π is continuous; in fact,

(4) **Theorem.** *If π is continuous, (N, π) implies $(T_1\,G, \pi)$.*

Because of our earlier remarks, this may or may not be a better result than what was already known; in any case it shows that ψ is σ-finite when $x(t)$ is (N, π). We will indicate below how $(T_1\,G, \pi)$ is related to some of the other classical conditions such as *VBG*.

Proof. It suffices to deal with $\pi=m$ and to restrict attention to $x(t)$ for $t\in U=[0,1]$. Thus suppose $x(t)$ is (N) and choose a Lebesgue measurable set $A_1\subset U$ such that $x(A_1)=x(U)$ and $x(t)$ is univalent on A_1 – this is possible by [7]. Next, for each countable ordinal number $\alpha>1$, let $B_\alpha=\bigcup_{\beta<\alpha} A_\alpha$. If $m(x(U\setminus B_\alpha))>0$, choose $A_\alpha\subset U\setminus B_\alpha$ such that A_α is measurable, $x(A_\alpha)=x(U\setminus B_\alpha)$, and $x(t)$ is univalent on A_α; by (N) we have $m(A_\alpha)>0$ in this case. On the other hand, if $m(x(U\setminus B_\alpha))=0$, take $A_\alpha=U\setminus B_\alpha$. Since the A_α are disjoint, the second case must occur for some countable ordinal γ, and then $m(A_\gamma)=m(x(A_\gamma))=0$; $x(t)$ is (T_1) on each A_α, $\alpha<\gamma$, and the result follows.

§ 2. Main Results

Write $\pi=\pi_c+\pi_d$, where π_d is purely discrete and π_c is continuous, i.e. $\pi_c(\{x\})=0$ for all $x\in\mathbb{R}$. We can quickly dispose of π_d. Let $A=\{a_1, a_2, \ldots\}$ be the set of atoms of π, with a_n carrying mass $\pi_n>0$. Since $\mu_t(\Gamma)=\mu_t(\Gamma\cap A)+\mu_t(\Gamma\cap A^c)$, it is clear that we can treat π_d and π_c separately. One easily checks that

$$\mu_t(\Gamma\cap A)=\int_\Gamma \sum_n \left(\frac{m(x^{-1}(a_n)\cap[0,t])}{\pi_n}\right) I_{\{a_n\}}(y)\,\pi(dy).$$

So now *we assume π is continuous and ψ is σ-finite.* Let

(5) $\psi(dt)=h(t)\,m(dt)+\bar{\psi}(dt)$

be the Lebesgue decomposition of ψ relative to m. The function $h(t)$ is a version of the Radon-Nikodym derivative $d\psi/dm$ (computed as in § 8.2 of [20] if ψ is Radon) and $\bar{\psi}$ is m-singular. The function h is an a.e. finite, non-negative Borel function and we may take $h=\infty$, $\bar{\psi}$-a.e. Any version of $d\psi/dm$ having this last property will agree with h a.e. *m and* ψ and may be used in the results below.

(6) **Theorem.** *The Lebesgue decomposition* $\mu_t(dy)=\alpha_t(y)\pi(dy)+\bar{\mu}_t(dy)$ *is given by*

$$(7)\quad \alpha_t(y)=\int_0^t I_{(0,\infty)}(h(s))(h(s))^{-1}\,\nu(y,ds)$$

$$\bar{\mu}_t(\Gamma)=m(x^{-1}(\Gamma)\cap H_0\cap(0,t]),\qquad \Gamma\in\mathscr{B},$$

where $H_0=\{t\colon h(t)=0\}$, *and the interval of integration is* $(0,t]$. *The condition* (LT) *holds iff* $m(H_0)=0$.

Proof. We begin by showing that $\bar{\mu}_t$ is indeed a π-singular measure. Clearly $\bar{\mu}_t$ lives on $x(H_0)$; but $\psi(H_0)=0$ and $\psi(H_0)\geqq\pi(x(H_0))$ by (2), so $\bar{\mu}_t$ is singular.

Next, by (2) and (5), we have

$$\iint f(s)\,\nu(y,ds)\,\pi(dy)=\int h(s)f(s)\,ds+\int f(s)\,\bar{\psi}(ds)$$

for any Borel function $f\geqq 0$. Choosing $f(s)=I_{(0,t]}(s)\,I_{(0,\infty)}(h(s))(h(s))^{-1}I_\Gamma(x(s))$, we obtain

$$\int_0^t I_{H_0^c}(s)\,I_\Gamma(x(s))\,ds=\int_\Gamma\int_0^t I_{(0,\infty)}(h(s))(h(s))^{-1}\,\nu(y,ds)\,\pi(dy),$$

and adding $\bar{\mu}_t(\Gamma)$ to both sides gives the result.

Notice that ψ has no point masses, and further, is *m-absolutely continuous iff* $x(t)$ *is* (N,π). This may be construed as a kind of dual to the statement that μ_t is π-absolutely continuous iff (LT) holds.

Theorem (6) will be useful in applications to the extent that we can more explicitly identify $h(t)$ in terms of $x(t)$ and π. For example, when $\pi=m$ and $x(t)$ is continuous and of locally bounded variation, then ψ is just the total variation measure of $x(t)$ and the de la Vallée Poisson decomposition [21, p. 127] shows that $h(t)=|\dot{x}(t)|$, $\dot{x}(t)$ being the ordinary derivative of $x(t)$. We now proceed to identify $h(t)$ for a much wider class of functions.

The derivative of $x(t)$ at t, denoted $\dot{x}(t)$, is defined as the common value – when such exists – of the four Dini derivates of $x(t)$. Using approximate Dini derivates instead, one defines the approximate derivative $\dot{x}_{\mathrm{ap}}(t)$ [21]. All of the (approximate) Dini derivates inherit the Borel measurability of $x(t)$ [1], consequently each of the following is a Borel set:

$C\ =\{t\colon x \text{ is continuous at } t\}$,

$D\ =\{t\colon \dot{x}(t) \text{ exists, finite or not}\}$,

$D^0\ =\{t\in D\colon \dot{x}(t)=0\}$,

$D^\infty=\{t\in D\colon |\dot{x}(t)|=\infty\}$,

$D^*\ =\{t\in D\colon 0<|\dot{x}(t)|<\infty\}$,

as well as D_{ap}, D^0_{ap}, etc. defined using $\dot{x}_{\mathrm{ap}}$ above.

Let F be any distribution function for π and $\xi(t)=F\circ x(t)$. The following results are indifferent to the particular choice of F.

(8) **Theorem.** If $\xi(t)$ has an approximate derivative a.e., then (7) holds with $h(t)=|\dot{\xi}_{\mathrm{ap}}(t)|$.

Before proving (8) we remark that, taking $n=m=1$ in the Hausdorff area theorem ((3.2.3) and p. 241 of [6]), we find $\psi(B)=\int_B |\dot{x}_{\mathrm{ap}}(t)|\,dt$ for any Borel set $B\subset\{t: |\dot{x}_{\mathrm{ap}}(t)|<\infty\}$, where ψ is given by (2) with $\pi=m$. Because of the exceptional set of measure 0 where $\dot{\xi}_{\mathrm{ap}}(t)$ may not exist, we must be somewhat circumspect in applying this theorem. In fact, we will give a slightly different proof which will render the result a little more accessible and allow us to keep track of the singular part $\bar{\psi}$. We also note that, being approximately differentiable a.e., $\xi(t)$ agrees a.e. with a function which is VBG, and any VBG function is easily seen to be $(T_1 G, m)$ – see [21, p. 279].

Proof of (8). We will show that the function h in (5) may be identified with $|\dot{\xi}_{\mathrm{ap}}|$, except that it need not be true that $|\dot{\xi}_{\mathrm{ap}}|=\infty$, $\bar{\psi}$-a.e. As will be clear from the proof this will not affect formula (7). The proof is broken into three steps.

1°. Assume $\pi=m$ and $x(t)$ is of bounded variation. The de la Vallée Poussin decomposition of the absolute variation measure $V(dt)$ [21, p. 127] yields

(9) $$V(B\cap C)=\int_B |\dot{x}(s)|\,ds+V(B\cap C\cap D^{\infty}),\qquad B\in\mathscr{B},$$

while 2.10.4 (p. 177) of [6] gives

(10) $$V([a,b])=\int_{\mathbb{R}} v(y,[a,b])\,dy+\sum_{t\in[a,b)}|x(t+)-x(t)|+\sum_{t\in(a,b]}|x(t-)-x(t)|.$$

Since $x(C^c)$ is countable, (9) and (10) imply

(11) $$\int_{\mathbb{R}}\int_{\mathbb{R}} f(s)\,v(y,ds)\,dy=\int_{\mathbb{R}} f(s)|\dot{x}(s)|\,ds+\int_{C\cap D^{\infty}} f(s)\,V(ds)$$

for any non-negative Borel function f. Here ψ is just the continuous part of the measure V, and $h=|\dot{x}|$ satisfies the conditions at (5). Notice that whenever $\dot{x}$ exists, $\dot{x}_{\mathrm{ap}}$ also exists and coincides with $\dot{x}$.

2°. Now assume that $\pi=m$ and $\dot{x}_{\mathrm{ap}}$ exists a.e. The notation VBG, etc. are from [21], from which we know that $x(t)$ is VBG on D^*_{ap}.

There is a partition of D^*_{ap}, say $D^*_{\mathrm{ap}}=\bigcup_1^{\infty} K_n$, $K_n\in\mathscr{B}$, such that $x(t)$ is VB and $\dot{x}_{\mathrm{ap}}$ is bounded and never zero on each K_n. Using [21, p. 221] we find a function $x_n(t)$ of bounded variation such that

(12) $$x_n(t)=x(t)\qquad \text{on } K_n.$$

For each n, define v_n, D_n, etc. for $x_n(t)$. Let L_n be the set of density points of K_n *in* K_n, so $L_n\in\mathscr{B}$, $m(K_n\setminus L_n)=0$. The true derivative $\dot{x}_n$ exists a.e., hence [21, p. 220]

(13) $$\dot{x}_n(t)=\dot{x}_{\mathrm{ap}}(t)\qquad \text{for all } t\in L_n\cap D_n.$$

Now apply (11) to x_n with $f(s)=g(s)I_{L_n\cap D_n^*}(s)$, g a non-negative Borel function. Noting that $D_n^*\cap D_n^\infty=\phi$, $m(L_n\setminus D_n^*)=0$, and using (12), (13), we obtain

$$\int_{L_n} g(s)|\dot{x}_{\mathrm{ap}}(s)|\,ds=\int_{\mathbb{R}}\int_{L_n\cap D_n^*} g(s)\nu(y,ds)\,dy.$$

Since $m(K_n\setminus(L_n\cap D_n^*))=0$, we have $m(x(K_n\setminus(L_n\cap D_n^*)))=0$ [21, p. 292], hence both L_n and $L_n\cap D_n^*$ may be changed to K_n. Summing the resulting equation on n gives

$$(14)\qquad \int_{D_{\mathrm{ap}}^*} g(s)|\dot{x}_{\mathrm{ap}}(s)|\,ds=\int_{\mathbb{R}}\int_{D_{\mathrm{ap}}^*} g(s)\nu(y,ds)\,dy.$$

We may further change D_{ap}^* to $D_{\mathrm{ap}}^*\cup D_{\mathrm{ap}}^0$ in (14) since $m(x(D_{\mathrm{ap}}^0))=0$. It follows that

$$\int_{\mathbb{R}} g\,d\psi=\int_{D_{\mathrm{ap}}^*\cup D_{\mathrm{ap}}^0} g(s)|\dot{x}_{\mathrm{ap}}(s)|\,ds+\int_{D_{\mathrm{ap}}^\infty\cup D_{\mathrm{ap}}^c} g\,d\psi$$

or, since $m(\mathbb{R}\setminus(D_{\mathrm{ap}}^*\cup D_{\mathrm{ap}}^0))=0$ by hypothesis,

$$(15)\qquad \int_{\mathbb{R}} g\,d\psi=\int_{\mathbb{R}} g(s)|\dot{x}_{\mathrm{ap}}(s)|\,ds+\int_{D_{\mathrm{ap}}^\infty\cup D_{\mathrm{ap}}^c} g\,d\psi.$$

This shows that $|\dot{x}_{\mathrm{ap}}|$ will serve for h in (5), though it may not be infinite $\bar{\psi}$-a.e. Consider instead $h=|\bar{x}_{\mathrm{ap}}|$, $\bar{x}_{\mathrm{ap}}$ being the approximate upper bilaterial Dini derivative (the corresponding lower derivative would do just as well). Putting this in for $|\dot{x}_{\mathrm{ap}}|$ does not affect the first right-hand term in (15). It remains only to show that $h=\infty$, $\bar{\psi}$-a.s. From (15) we find $\bar{\psi}$ to be the restriction of ψ to the (Lebesgue null) set $D_{\mathrm{ap}}^\infty\cup D_{\mathrm{ap}}^c$. Certainly $|\bar{x}_{\mathrm{ap}}|=\infty$ on D_{ap}^∞ since $\bar{x}_{\mathrm{ap}}=\dot{x}_{\mathrm{ap}}$ when the latter exists. Now the analogue of the Denjoy-Saks-Young Theorem for approximate derivates [21, p. 295] tells us that, for a.e. t, $\dot{x}_{\mathrm{ap}}(t)$ exists finite, or $\bar{x}_{\mathrm{ap}}(t)=\infty=-\underline{x}_{\mathrm{ap}}(t)$. Let B be the exceptional null set, so $B=D_{\mathrm{ap}}^c\cap\{t\colon \bar{x}_{\mathrm{ap}}(t)<\infty$ or $\underline{x}_{\mathrm{ap}}(t)>-\infty\}$. Then [21, p. 292] $x(t)$ is (N) on B, $m(B)=0$, so $m(x(B))=0$, and we conclude $\bar{\psi}$ is supported by $\{t\colon \bar{x}_{\mathrm{ap}}(t)=\infty=-\underline{x}_{\mathrm{ap}}(t)\}\cup D_{\mathrm{ap}}^\infty$.

3°. Now let π be arbitrary and assume $\dot{\xi}_{\mathrm{ap}}$ exists a.e. We write (2) in terms of the distribution function F,

$$(16)\qquad \psi(B)=\int_{\mathbb{R}} \nu(y,B)\,dF(y),\qquad B\in\mathscr{B},$$

and make the change of variable $y=\hat{F}(z)$, where $\hat{F}$ is the right-continuous "inverse" of F ($\hat{F}(z)=\inf\{y\colon F(y)>z\}$), which transforms (16) into

$$\psi(B)=\int_{\mathbb{R}} \nu(\hat{F}(z),B)\,dz.$$

Let $n(z,ds)$ be the multiplicity kernel of the function $\xi(t)$. One may easily verify that, if z is not one of the countably many flat levels on the graph of F, $\nu(\hat{F}(z),B)=n(z,B)$, whence $\psi(B)=\int n(z,B)\,dz$. Applying step 2° we have

$$\psi(B)=\int_B |\dot{\xi}_{\mathrm{ap}}(s)|\,ds+\bar{\psi}(B)$$

with $\bar{\psi}$ being the restriction of ψ to the set $D_{\mathrm{ap}}^\infty\cup D_{\mathrm{ap}}^c$ (now referring to $\xi(t)$). Replacing $\dot{\xi}_{\mathrm{ap}}$ (as in 2°) we again get $|\bar{\xi}_{\mathrm{ap}}|=\infty$, $\bar{\psi}$-a.e., but we note that in the local time formula (7) we may use $h=|\dot{\xi}_{\mathrm{ap}}|$ rather than $|\bar{\xi}_{\mathrm{ap}}|$. Q.E.D.

Remark. Here is another approach to (8) when $\pi=m$. However, as above, care must be taken to keep track of various null sets whose images may not be null. Suppose $B\subset\{t: |\dot{x}(t)| \text{ exists finite}\}$. Then [21, p. 227] $m(x(B))\leqq\int_B|\dot{x}(s)|\,ds$, and it is easy to conclude $\psi(B)\leqq\int_B|\dot{x}(s)|\,ds$, and further $h(s)\leqq|\dot{x}(s)|$ a.e. where $|\dot{x}|<\infty$. Using a theorem of Hinčin (cf. § 3) we may extend the conclusion to $h(s)\leqq|\dot{x}_{\mathrm{ap}}(s)|$ a.e. where $|\dot{x}_{\mathrm{ap}}|$ exists and is finite. Next, let $x(t)$ be continuous. It is then VBG on the set $D_{\mathrm{ap}}\smallsetminus D_{\mathrm{ap}}^{\infty}$, and by a theorem of Cesari [2] one obtains $\psi(B)\geqq\int_B|\dot{x}_{\mathrm{ap}}(s)|\,ds$ for $B\subset D_{\mathrm{ap}}\smallsetminus D_{\mathrm{ap}}^{\infty}$, whence $h(s)\geqq|\dot{x}_{\mathrm{ap}}(s)|$ a.e. on $D_{\mathrm{ap}}\smallsetminus D_{\mathrm{ap}}^{\infty}$. Using Lusin's theorem we could get rid of the continuity assumption, and so obtain $h=|\dot{x}_{\mathrm{ap}}|$ a.e. where the latter exists.

We conclude this section with a proposition relating the differentiability properties of ξ, F, and x.

(17) **Proposition.** *If the true derivative $\dot{x}$ (respectively $\dot{\xi}$) exists $\neq 0$* a.e. *then $\dot{\xi}$ (respectively $\dot{x}$) exists a.e. and the "chain rule" $\dot{\xi}(s)=F'(x(s))\dot{x}(s)$ holds* a.e. *(F' is the ordinary derivative of F).*

Proof. First, if $\dot{x}$ exists $\neq 0$ a.e. then $x(t)$ has Lebesgue local time and so spends zero time outside $\{y: F'(y) \text{ exists finite}\}$. It follows that $\dot{\xi}$ exists $\neq 0$ a.e. and equals $F'(x(s))\dot{x}(s)$ a.e. (see also [23]).

Now suppose $\dot{\xi}$ exists $\neq 0$ a.e. and let $Z=\{y: F'(y) \text{ does not exist, finite or infinite}\}$, $Z^0=\{y: F'(y)=0\}$. Since π is continuous, $\pi(Z^0)=m(F(Z^0))=0$ [21, p. 100, 226], and $\pi(Z)=0$ by [21, p. 125]. By (8), x has a local time relative to π; consequently, $m(x^{-1}(Z\cup Z^0))=0$. Let $W=\{t: x(t)\notin Z\cup Z^0, |\dot{\xi}(t)|\neq 0, \infty\}$ so $m(W^c)=0$. Fix $t\in W$ and suppose for some sequence $s_n\to t$ we had $x(s_n)=x(t)$. Then $\dot{\xi}(t)=0$, contradicting $t\in W$. Consider the quotient $(\xi(s)-\xi(t))/(x(s)-x(t))$ for $s\to t$. Clearly

$$\overline{\lim_{s\to t}}\frac{\xi(s)-\xi(t)}{x(s)-x(t)}\leqq\bar{F}(x(t)),\qquad \underline{\lim_{s\to t}}\frac{\xi(s)-\xi(t)}{x(s)-x(t)}\geqq\underline{F}(x(t))$$

where $\bar{F}$, $\underline{F}$ are the bilateral derivatives of F. But $t\in W$ implies $F'(x(t))$ exists, $0<F'(x(t))\leqq\infty$. We conclude

$$\lim_{s\to t}\frac{\xi(s)-\xi(t)}{x(s)-x(t)}=F'(x(t))$$

and the rest is easy.

§ 3. Remarks

(a) The following theorem is given in Serrin and Varberg [23]: if $\dot{x}(t)$ exists a.e., and $m(\Gamma)=0$, then $\dot{x}(t)=0$ a.e. on $x^{-1}(\Gamma)$. For absolutely continuous (AC) functions this is given in [16, p. 213].

The theorem is still true if $\dot{x}$ is replaced by $\dot{x}_{\mathrm{ap}}$. Indeed, it follows easily (as do several other theorems in the literature) from the AC case and Hinčin's theorem

[14]: $\dot{x}_{\mathrm{ap}}$ exists a.e. iff for each $\varepsilon>0$ there is a compact $K\subset[a,b]$ and an AC function $y(t)$ on $[a,b]$ such that $m([a,b]\setminus K)<\varepsilon$ and $x(t)=y(t)$ on K.

We now can state a generalization of the Serrin-Varberg Theorem: $m(\Gamma)=0$ implies $h(t)=0$ a.e. on $x^{-1}(\Gamma)$ where h appears in (5).

(b) The proof of the Serrin-Varberg Theorem actually shows that $m(\Gamma)=0$ implies $\liminf_{s\to t}|x(s)-x(t)|/|s-t|=0$ a.e. on $x^{-1}(\Gamma)$. In conjunction with the Denjoy-Saks-Young Theorem [21, p. 271] we then have: if $m(\Gamma)=0$, then at a.e. $t\in x^{-1}(\Gamma)$ one of the four conditions below must hold:

(i) $\dot{x}(t)$ exists $=0$,

(ii) $\bar{x}(t)=+\infty=-\underline{x}(t)$,

(iii) $\bar{x}^{+}(t)=\underline{x}_{-}(t)=0,\quad \underline{x}_{+}(t)=-\infty,\quad \bar{x}^{-}(t)=+\infty,$

(iv) $\underline{x}_{+}(t)=\bar{x}^{-}(t)=0,\quad \bar{x}^{+}(t)=+\infty,\quad \underline{x}_{-}(t)=-\infty.$

Thus, if at a.e. t at least one Dini derivate is finite, (LT) holds (with $\pi=m$) iff *that* derivate $\neq 0$ a.e. (there is no ambiguity as to *which* derivate is finite, again by Denjoy-Saks-Young). We note that almost every Brownian motion trajectory satisfies (ii) a.e. *and* satisfies (LT); on the other hand, Cesari [2] gives an example of a continuous function $x(t)$ satisfying (ii) (in fact $\bar{x}_{\mathrm{ap}}(t)=+\infty=-\underline{x}_{\mathrm{ap}}(t)$) a.e. on a set E of positive measure, and such that $m(x(E))=0$, hence (LT) fails.

(c) An idea related to occupation-times appears in a paper [22] by Sarkhel – to our knowledge the only one on real variables which deals with such matters. Sarkhel defines the "upper right metric density" relative to a set B of an (arbitrary) function $x(t)$ at t for $y\in\mathbb{R}$ as

$$\rho^{+}(t,y,B)=\lim_{h\to 0^{+}}\varlimsup_{k\to 0^{+}}k^{-1}\bar{m}\{s\in B\colon y\leqq x(s)\leqq y+k,\, t-h<s<t+h\},$$

where $\bar{m}$ is Lebesgue outer measure. Three other densities $\rho^{-},\rho_{-},\rho_{+}$ are analogously defined, whose common value – if it exists – is called the *metric density* at (t,y), denoted $\rho(t,y,B)$. For simplicity we assume $B=\mathbb{R}$ and suppress it in the notation. The main results of [22] of interest here are: (i) $\rho(t,y)$ exists (finite) for a.e. y (t fixed); (ii) if $\dot{x}(t)$ exists $\neq 0$, then $\rho(t,x(t))$ exists and equals $|\dot{x}(t)|^{-1}$ ($=0$ if $|\dot{x}(t)|=\infty$); (iii) if $\dot{x}(t)=0$, $\rho^{+}(t,x(t))=\rho_{+}(t,x(t))=\infty$ or $\rho_{-}(t,x(t))=\rho^{-}(t,x(t))=\infty$. (These results extend to approximate derivatives as well.)

In our notation, $\rho^{+}(t,y)=\lim_{h\to 0^{+}}\varlimsup_{k\to 0^{+}}k^{-1}[\mu_{t+h}(y,y+k)-\mu_{t-h}(y,y+k)]$ and (i) follows from general theorems on differentiation of measures. More interesting is that $\rho(t,y)$ (when it exists) is the jump of $\alpha_{\cdot}(y)$ at t. Consequently, the identification of $\alpha_t(y)$ in (8) for $\pi=m$ can be derived from (i)-(iii), and conversely (though not trivially in either direction). One may therefore regard Sarkhel's work as a "local" study of the occupation-time measure when $\pi=m$.

(d) The local time decomposition (7) will be valid when $\pi(x(D^{c}_{\mathrm{ap}}))=0$, where D^{c}_{ap} refers to $\xi(t)$, if we change the singular component to

$$\bar{\mu}_t(\Gamma)=m(x^{-1}(\Gamma)\cap H_0\cap(0,t])+m(x^{-1}(\Gamma)\cap D^{c}_{\mathrm{ap}}\cap(0,t]).$$

(e) The comment in (d) applies to a continuous function $x(t)$ which is (T_1) on every interval and $\pi=m$; indeed, for $x(t)$ continuous, we have (T_1) on every interval iff $m(x(D^c))=0$. Now a similar remark applies to continuous functions which are (T_1, π) on finite intervals.

It would be interesting to identify $h(t)$ explicitly, or to discover the occupation-time decomposition for Borel functions $x(t)$, under any of the following conditions (assume $\pi=m$):

(A) (T_1) on finite intervals, but not necessarily continuous,
(B) (N) functions,
(C) (T_2) functions.

Each of (A) and (B) imply (C), which is essentially the weakest hypothesis under which results of the type considered here, namely those involving $\nu(y, ds)$ and having only jumps in $\alpha.(y)$, can be valid.

II. Occupation-Times and Regeneration of Random Processes

§ 4. Local Times

Let $(\Omega, \mathscr{F}^0, P)$ be a probability space, the superscript ^{0} indicating that the σ-field $\mathscr{F}^0$ is not assumed complete; the completion is denoted $\mathscr{F}$. Further, let (X_t), $t\in\mathbb{R}$, be a real-valued measurable random process on $(\Omega, \mathscr{F}^0, P)$, meaning that the mapping $(t, \omega)\to X_t(\omega)$ is $\mathscr{B}\otimes\mathscr{F}^0/\mathscr{B}$-measurable. Every trajectory $t\to X_t(\omega)$ is a Borel function and the occupation-time measure $\mu_t(\Gamma)$ is now also a function of $\omega\in\Omega$ (often suppressed from the notation).

We are going to apply the results of Part I to individual trajectories but first wish to place the present work in the context of the general theory of local times for random processes (which we gave in [10]), leaving aside some technical details. Consider the Lebesgue decomposition $\mu_t(dy, \omega)=\alpha_t(y, \omega)\pi(dy)+\bar{\mu}_t(dy, \omega)$ of μ_t relative to the Radon measure π on $\mathscr{B}$. The condition (LT) is now interpreted to mean: for almost every $\omega\in\Omega$ the trajectory $X.(\omega)$ has a local time relative to π.

For simplicity we assume $\pi=m$, that each X_s has a density $p_s(x)$, and that almost every trajectory $X_t(\omega)$ has a derivative $\dot{X}_t(\omega)$ a.e. It will also be convenient to (temporarily) take $\mathbb{R}_+=[0, \infty)$ with its Borel sets $\mathscr{B}_+$ as our time set. We have, for $t\in\mathbb{R}_+$, $\Gamma\in\mathscr{B}$,

$$(18)\quad \int_t^\infty e^{-s}P(X_s\in\Gamma, A)\,ds=\int_\Gamma\left[\int_t^\infty e^{-s}p_s(x)P(A\,|\,X_s=x)\,ds\right]dx.$$

Define a measure Q_x on $\mathscr{B}_+\otimes\mathscr{F}^0$ as follows: for sets of the form $B=(t, \infty)\times A$ $(t\in\mathbb{R}_+, A\in\mathscr{F}^0)$, $Q_x(B)$ is given by the expression in brackets in (18). Using the general theory of processes [3] and [10] one can show that (LT) holds iff for a.e. x, the measure Q_x charges no "evanescent" set B (i.e. $I_B(s, \omega)\equiv 0$ for almost every $\omega\in\Omega$). Now, by the decomposition (7) we may rewrite (18) as

$$(19)\quad \int_t^\infty e^{-s}P(X_s\in\Gamma, A)\,ds$$
$$=\int_\Gamma\left\{E\left[\int_t^\infty e^{-s}\,d\alpha_s(x); A\right]+\int_t^\infty e^{-s}p_s(x)P(\dot{X}_s=0, A\,|\,X_s=x)\,ds\right\}dx;$$

in short, for a.e. x, $Q_x(B)$ is given by the expression in curly brackets in (19) for $B=(t,\infty)\times A$. Again from the general theory of processes it is known that the measure Q'_x on $\mathscr{B}_+\times\mathscr{F}^0$ determined by $E\left[\int_t^\infty e^{-s}\,d\alpha_s(x); A\right]$ charges no evanescent set. On the other hand, the measure Q''_x determined by

$$\int_t^\infty e^{-s} p_s(x) P(\dot{X}_s=0, A \mid X_s=x)\,ds$$

is evidently supported by the set $N_x=\{(s,\omega)\colon X_s(\omega)=x, \dot{X}_s(\omega)=0\}$ which is evanescent for a.e. x: for each ω, the measure of the image of $\{s\colon \dot{X}_s(\omega)=0\}$ under $X.(\omega)$ is 0. Consequently, by Fubini, for almost every x, the set $\{s\colon X_s(\omega)=x, \dot{X}_s(\omega)=0\}$ is empty a.s., i.e. N_x is evanescent. Thus condition (LT) is equivalent to the disappearance of the measure Q''_x for a.e. x.

If, in addition, (X_t) is stationary, the entire analysis can be carried out on the space Ω rather than $\mathbb{R}_+\times\Omega$. The condition for (LT) becomes very simple: $P\{\dot{X}_0=0\}=0$. Instead of the decomposition $Q_x=Q'_x+Q''_x$ above we obtain $P_x(A)=P_x^*(A)+P_x(A,\dot{X}_0=0)$, $A\in\mathscr{F}^0$, where $P_x(A)$ is the (regular) conditional probability given $X_0=x$, P_x^* is a certain "Palm measure" (cf. § 6) and the measure $A\to P_x(A,\dot{X}_0=0)$ is carried by the "polar" set $\{X_0=x, \dot{X}_0=0\}$. (Anticipating the notation of § 5, a set $B\subset\Omega$ is *polar* relative to a flow θ_t if, a.s., $I_B(\theta_t\omega)\equiv 0$.) The proof that $\{X_0=x, \dot{X}_0=0\}$ is polar is similar to the proof that N_x above is evanescent. The decomposition of P_x provides an explicit example of the type of results given in [12].

In what follows we will apply the results of Part I to the case in which $\pi=m$ and the approximate derivative $\dot{X}_{\mathrm{ap}}(t,\omega)$ exists at a.e. t, for almost every trajectory. The case of a general π is handled similarly using the process $\xi_t=F(X_t)$ in analogy with Part I. Finally we recall that in (8) one may use the approximate upper bilateral derivative $\bar{x}_{\mathrm{ap}}(t)$ instead of $\dot{x}_{\mathrm{ap}}(t)$; when working with processes it will sometimes be convenient to make a similar replacement of $\dot{X}_{\mathrm{ap}}(t,\omega)$ by $\bar{X}_{\mathrm{ap}}(t,\omega)$ without writing it explicitly, the advantage being that the latter is defined for *every* t.

§ 5. Existence and Measurability of Derivatives

We now turn to stationary processes for the remainder of the paper. Assume that θ_t, $t\in\mathbb{R}$, is a *flow* on the probability space $(\Omega,\mathscr{F}^0,P)$, i.e. a one-parameter group (under composition) of measure-preserving bijections of Ω such that $\theta_0=$ identity and the mapping $(t,\omega)\to\theta_t(\omega)$ is $\mathscr{B}\otimes\mathscr{F}^0/\mathscr{F}^0$-measurable. Other terminology pertaining to flows and Palm measures is explained in [9, 12, 15]. For a real-valued random variable X on $(\Omega,\mathscr{F}^0)$ we define a strictly stationary, measurable process $X_t(\omega)=X\circ\theta_t(\omega)$, $t\in\mathbb{R}$, $\omega\in\Omega$. The results of this section will apply as well to the process $\xi_t=F(X_t)$, *F as in* § 4.

We begin by giving "local" probabilistic conditions on behavior of X_t near $t=0$ which imply the "global" property of (approximate) differentiability a.e. for almost every trajectory. Next we show that the *approximate* derivative process always has good measurability; and then conclude with some remarks on separability and approximate differentiability for random processes.

Let the measurable process (Y_t) be a "strictly separable" modification of (X_t), meaning that, for each t, $X_t = Y_t$ a.s., and, for some fixed countable set $S \subset \mathbb{R}$,

(20) $$Y_t(\omega) \in \bigcap_I \overline{Y_{I \cap S}(\omega)} \quad \text{for every } t \in \mathbb{R},\ \omega \in \Omega,$$

the intersection being over all intervals I containing t, and $\overline{Y_{I \cap S}(\omega)}$ being the closure in the extended reals of the image of $I \cap S$ under $Y.(\omega)$. The existence of (Y_t) is proven as in [17, pp. 57–8] taking care to retain $\mathscr{B} \otimes \mathscr{F}^0$-measurability. The process (Y_t) is also stationary, but we can only affirm $Y_t = Y_0 \circ \theta_t$ a.s. for each t.

Next, for every $\omega \in \Omega$, let

$$\bar{X}(\omega) = \limsup_{t \to 0} \frac{X_t(\omega) - X_0(\omega)}{t}, \qquad \bar{Y}(\omega) = \limsup_{t \to 0} \frac{Y_t(\omega) - Y_0(\omega)}{t}.$$

There is no harm in assuming $0 \in S$ and $X_t \equiv Y_t$ for $t \in S$; by separability we then have

$$\bar{Y}(\omega) = \limsup_{\substack{t \to 0 \\ t \in S}} \frac{X_t(\omega) - X_0(\omega)}{t},$$

whence $\bar{Y} \leqq \bar{X}$. Notice that $\bar{Y}$ is $\mathscr{F}^0$-measurable, but $\bar{X}$ need not even be $\mathscr{F}$-measurable. We will write $\bar{X}_t(\omega)$, $\bar{Y}_t(\omega)$ for the upper bilateral derivates at t for the trajectories $X.(\omega)$, $Y.(\omega)$, so that $\bar{X} = \bar{X}_0$, $\bar{Y} = \bar{Y}_0$; we have also $\bar{X}_t = \bar{X} \circ \theta_t$, but the analogous statement for Y may not be valid. Similar remarks and results (see below) apply to the lower derivates.

The following easily proven fact will be useful momentarily.

(21) **Lemma.** *If Z is an $\mathscr{F}$-measurable function, the mapping $(t, \omega) \to Z \circ \theta_t(\omega)$ is $\overline{\mathscr{B} \otimes \mathscr{F}^0}$-measurable, the bar signifying completion under $m \times P$.*

(22) **Theorem.** *Suppose $K = \{\bar{X} < \infty\}$ (respectively $L = \{\bar{Y} < \infty\}$) is full; then almost every trajectory $X.(\omega)$ is (approximately) differentiable a.e.*

Recall that a set is *full* if its complement has measure zero. Notice that $L \in \mathscr{F}^0$ automatically, but K need not be measurable. By Fubini and the Denjoy-Saks-Young theorem one can show $P\{\bar{X} = -\infty\} = 0$; another consequence is that if $K \in \mathscr{F}$ and $X.(\omega)$ is non-differentiable a.e. for almost all ω, then $\bar{X} = \infty$ a.s.

Proof. If K is full (hence in $\mathscr{F}$), then by (21) and Fubini $A = \{(t, \omega)\colon \bar{X}_t(\omega) < \infty\}$ is full under $m \times P$. Hence, for almost every $\omega \in \Omega$, $A_\omega = \{t\colon \bar{X}_t(\omega) < \infty\}$ is full in $\mathbb{R}$, and $X.(\omega)$ is differentiable a.e. on A_ω by Denjoy-Saks-Young.

Next, in view of [21, p. 220] and Fubini, it suffices to prove that if L is full, $Y.(\omega)$ has a true derivative a.e. for almost every ω. Because (Y_t) is strictly separable and measurable, $B = \{(t, \omega)\colon \bar{Y}_t(\omega) < \infty\}$ is in $\mathscr{B} \otimes \mathscr{F}^0$. Let $B^t = \{\omega\colon \bar{Y}_t(\omega) < \infty\}$, noting $B^0 = L$. For each t,

$$s^{-1}(Y_{s+t} - Y_t) = s^{-1}(X_{s+t} - X_t) \qquad \text{for all } s \in S \setminus \{0\} \text{ a.s.},$$

and the right member $= s^{-1}(X_s \circ \theta_t - X_0 \circ \theta_t) = s^{-1}(Y_s \circ \theta_t - Y_0 \circ \theta_t)$ on Ω; in short, $B^t = \theta_t^{-1} L$ a.s. Thus $P(B^t) = 1$ for every t, B is full, and the argument above yields the result.

(23) **Theorem.** *The process $\bar{X}_{\mathrm{ap}}(t,\omega)$ is measurable.*

This implies that the approximate derivative process $\dot{X}_{\mathrm{ap}}(t,\omega)$ is measurable.

Consider the measurable process (Z_t) with $Z_t = t^{-1}(X_t - X_0)$ for $t \neq 0$ and Z_0 any $\mathscr{F}^0$-measurable function. According to [5] or [24] (using their topology T_d) the process $Z_t^* = \mathrm{ap}-\limsup_{s\to t} Z_s$ is measurable, whence the $\mathscr{F}^0$-measurability of $\bar{X}_{\mathrm{ap}}(0,\omega) = Z_0^*$. Since $\bar{X}_{\mathrm{ap}}(t,\omega) = \bar{X}_{\mathrm{ap}}(0,\theta_t\omega)$, (23) follows.

Suppose $X_t = X\circ\theta_t$ as above and that Y_t is a separable, measurable modification with the added property that, for almost every $\omega\in\Omega$, there exists for each t a Borel set $B_t(\omega)$ such that $\lim_{\delta\downarrow 0}\delta^{-1} m((t,t+\delta)\cap B_t(\omega)) = 1$ and $Y_t(\omega) = \lim_{\substack{s\to t\\ s\in B_t(\omega)}} Y_s(\omega)$. Then there also exists a separable, measurable, *homogeneous* modification under very mild conditions, e.g. $X\in L^1$. To see this, define $Z = \limsup_{\delta\downarrow 0}\delta^{-1}\int_0^\delta X\circ\theta_s\,ds$; Z is $\mathscr{F}^0$-measurable, and $Z = X$ a.s. (local ergodic theorem). Then, for almost every ω we will have $Z\circ\theta_t(\omega) = \limsup_{\delta\downarrow 0}\delta^{-1}\int_0^\delta Y_{t+s}(\omega)\,ds = Y_t(\omega)$ for all t, i.e. the process $Z\circ\theta_t$ and Y_t are *indistinguishable*, and $Z\circ\theta_t$ is the desired modification. It would be nice to know general conditions when such a (Y_t) process exists.

We may apply this remark as follows: let (X_t) be a second order strictly stationary process having a finite second spectral moment. Doob [4, p. 536] shows that any separable, measurable modification must in fact have absolutely continuous trajectories a.s. The above result now implies that there is a *homogeneous* modification with absolutely continuous trajectories.

The reader may have noticed that the use of the separable modification Y_t requires some care. For instance, in passing from X_t to Y_t, the homogeneity property $X_t \equiv X_0\circ\theta_t$ is lost—this can be troublesome, e.g., in the next section. Other essential features of the original process may also be destroyed. Consider a continuous process Z_t subjected to random "impulses" represented by a "point process" Δ_t which is zero except on a t-set having no finite accumulation point. The sum $X_t = Z_t + \Delta_t$ is not separable. If, as in the stationary case, $P(\Delta_t \neq 0) = 0$, Z_t is the separable version obtained by the usual construction, but this erases the perturbations Δ_t which may be the object of primary interest. Thus there may be theoretical or practical reasons for not replacing X_t by a separable modification. As noted in the proof of (22), if the separable version Y_t is differentiable a.e., then X_t is approximately differentiable a.e. (in fact, X_t is a.e. equal to an a.e. differentiable function), so approximate differentiability arises as a natural analytical property when we are constrained to the original process. The results of Part I and §§ 6, 7 then show that approximate differentiability serves almost as well as true differentiability for many purposes.

In the following sections, we will assume the existence of $\dot{X}_{\mathrm{ap}}(t,\omega)$ at a.e. t, for almost every trajectory, but will usually drop the subscript "ap".

§ 6. Regeneration

Throughout this section we take $\pi = m$ and define $D^* = D^*(\omega)$ as in Part I relative to the trajectory $X_\cdot(\omega)$. The results remain valid for any π provided $\dot{X}$ is replaced by $\dot{\xi}$, D^* by $D^*(\xi)$, etc. For each $\omega \in \Omega$, let $M(\omega) = \{t \in D^*(\omega): X_t(\omega) = X_0(\omega)\}$. We will consider the point transformation $\theta_\tau(\omega) = \theta_{\tau(\omega)}(\omega)$ on Ω, where $\tau(\omega)$ is the first positive $t \in M(\omega)$ (a more precise definition is given below). This transformation has the effect of shifting the time origin to $\tau(\omega)$, and we are interested in the extent to which the process "starts over" probabilistically, i.e. we will study the class of measures dominated by P and preserved by θ_τ or its (random) iterates. The basic results are (29), (31), (37), (41).

Define $\nu^*(x, \omega, B) = \nu(x, \omega, B \cap D^*(\omega))$, $B \in \mathscr{B}$, and let $(\mathscr{B} \otimes \mathscr{F}^0)^*$ denote, as in § 1, the universal completion of $\mathscr{B} \otimes \mathscr{F}^0$.

(24) **Lemma.** *For each $B \in \mathscr{B}$, the mappings $(x, \omega) \to \nu(x, \omega, B)$ and $(x, \omega) \to \nu^*(x, \omega, B)$ are $(\mathscr{B} \otimes \mathscr{F}^0)^*$-measurable.*

Proof. Consider the $\mathscr{B} \otimes (\mathscr{B} \otimes \mathscr{F}^0)$-measurable process $Z_t(x, \omega) = X_t(\omega) - x$ (the first $\mathscr{B}$ refers to "time"). Let $\tau_n(x, \omega)$ be the "n-debut" [3] of the subset $\{(t, x, \omega): Z_t(x, \omega) = 0, t \in B\}$ of $\mathbb{R} \times (\mathbb{R} \times \Omega)$. For any probability measure Q on $\mathscr{B} \otimes \mathscr{F}^0$, the set $\{\tau_n < \infty\}$ is in the Q-completion of $\mathscr{B} \otimes \mathscr{F}^0$ [3, p. 51], and one checks $\{\tau_n < \infty\} = \{\nu(\cdot, \cdot, B) \geqq n\}$. The same proof works for ν^* if one uses $\{(t, x, \omega): Z_t(x, \omega) = 0, t \in B \cap D^*(\omega)\}$.

For each x, $\nu(x, \omega, ds)$ and $\nu^*(x, \omega, ds)$ are *homogeneous random measures*, meaning, e.g. $\nu(x, \theta_t \omega, B) = \nu(x, \omega, B + t)$ for every $\omega \in \Omega$, $t \in \mathbb{R}$, and $B \in \mathscr{B}$. (This would no longer be true without the homogeneity discussed in § 5.) We denote the Palm measure of $\nu^*(x, \omega, ds)$ by P_x^* and integration relative to P_x^* by E_x^*.

Let $u(s, \omega)$ be a non-negative, measurable process. Since the map $s \to u(s, \omega)$ is Borel ($\omega \in \Omega$ fixed), (8) and especially (14) imply, for $\Gamma \in \mathscr{B}$,

$$(25) \quad \int_0^t I_\Gamma(X_s(\omega)) \, |\dot{X}_s(\omega)| \, u(s, \omega) \, ds = \int_\Gamma \int_0^t u(s, \omega) \, \nu^*(x, \omega, ds) \, dx.$$

Taking $t = 1$, $u(s, \omega) = Z \circ \theta_s(\omega)$ (Z an $\mathscr{F}^0$-measurable function) and integrating with P, we obtain (writing $\dot{X}$ for $\dot{X}_0$)

$$(26) \quad E(Z \, |\dot{X}|; X \in \Gamma) = \int_\Gamma E_x^*(Z) \, dx.$$

Putting $Z = I_{\{|\dot{X}| \leqq n\}}$, we reach the conclusion that P_x^* is σ-finite for a.e. x.

Before considering the regeneration properties of our process, we pause to remark several consequences of (26).

(a) A similar equation holds with π and $\dot{\xi}$ replacing respectively m, $\dot{X}$.

(b) Let $\pi(dx) = P(X \in dx)$. For this choice of the measure π we have

$$E(Z \, |\dot{\xi}| \mid X = x) = E_x^*(Z)$$

for π-a.e. x, and with $Z = 1$ this becomes

$$(27) \quad E \nu^*(x, (0, 1]) = E(|\dot{\xi}| \mid X = x) \qquad \pi\text{-a.e.}$$

This is a general version of Rice's formula (as we observed in [8] under additional assumptions). In the standard version, one has a continuous second order process X_t, ν^* is replaced by ν, and $\dot{\xi}$ by the quadratic mean derivative at $t = 0$; in the Gaussian case *or* under still further assumptions the formula is valid for every x.

We now impose a requirement which will be referred to as (T_1^*): $\nu^*(x, \omega, B) < \infty$ for every bounded set $B (m \times P)$-a.e. This seems to be the weakest condition which allows the use of the mass-preservation properties of certain random time shifts. In general we have $\nu^* \leqq \nu$; if, for every $B \in \mathscr{B}$ and $m \times P$-a.e. pair (x, ω),

(28) $\quad \nu^*(x, \omega, B) = \nu(x, \omega, B)$,

then we may take $D^*(\omega) = \mathbb{R}$ in what follows. For $\omega \in \Omega$ fixed, (28) will hold iff x is not in the image of $(D^*(\omega))^c \cap B$ under $X.(\omega)$. Recall that

$$(D^*(\omega))^c = D^0(\omega) \cup D^\infty(\omega) \cup D^c(\omega);$$

of these, D^∞ and D^c are Lebesgue null, the first by Denjoy-Saks-Young, and the second by assumption, while the *image* of $D^0(\omega)$ is always Lebesgue null. A sufficient condition for (28) is, therefore, that $X.(\omega)$ be an (N)-function a.s. On the other hand, if $X.(\omega)$ is VBG_* [21] or is continuous, locally (T_1) and has a true derivative a.e., then the image of $D^c(\omega)$ will be null, but this may not be true of $D^\infty(\omega)$.

Let V be the set of $\omega \in \Omega$ for which $M(\omega)$, defined above, is unbounded in both directions and has no finite accumulation point, i.e.

$$\begin{aligned} V = \{\omega\colon & \nu^*(X(\omega), \omega, [0, \infty)) \\ & = \nu^*(X(\omega), \omega, (-\infty, 0)) = \infty, \ \nu^*(X(\omega), \omega, [-n, n]) < \infty \ \text{ for all } n \geqq 1\}. \end{aligned}$$

By (24), $V \in (\mathscr{F}^0)^*$. Next, define $\tau(\omega) = I_V(\omega) \cdot \inf(M(\omega) \cap (0, \infty))$ and $\phi\colon \Omega \to \Omega$ by $\phi(\omega) = \theta_{\tau(\omega)}(\omega)$. Evidently τ is $(\mathscr{F}^0)^*$-measurable, ϕ is invertible, and ϕ is $(\mathscr{F}^0)^*/\mathscr{F}^0$-measurable.

Theorem. *The set V is full for the measure $d\mu = |\dot{X}|\, dP$ and ϕ preserves μ.*

Proof. We recall first that the Palm measure P_x^* lives on the set

$$\Omega_x = \{\omega\colon \nu^*(x, \omega, \{0\}) = 1\}.$$

Define, for each $x \in \mathbb{R}$,

$$A_x = \{\omega\colon \nu^*(x, \omega, [0, \infty)) = \nu^*(x, \omega, (-\infty, 0)) = \infty\},$$

$$B_x = \{\omega\colon \nu^*(x, \omega, \mathbb{R}) > 0\},$$

$$C_x = \{\omega\colon \nu^*(x, \omega, B) < \infty \ \text{ for every bounded } B\}.$$

Each of these sets is invariant under θ_t and is $(\mathscr{F}^0)^*$-measurable. (For the latter assertion, use (21) and consider the completion of $\mathscr{B} \otimes \mathscr{F}^0$ by measure of the form $\delta_x \times Q$, where δ_x is unit mass on x and Q any probability law on $\mathscr{F}^0$.) It is standard that $P(B_x \setminus A_x) = 0$ and by assumption $P(C_x) = 1$ a.e. Under the measure P_x^* we have $V^c = A_x^c \cup C_x^c$ P_x^*-a.e. But P_x^* charges no invariant null set so $P_x^*(C_x^c) = 0$ (for a.e. x) and $P_x^*(B_x \setminus A_x) = 0$, whence $P_x^*(A_x^c) = P_x^*(\Omega_x \setminus A_x) \leqq P_x^*(B_x \setminus A_x) = 0$, i.e. $P_x^*(V^c) = 0$ a.e. By (26) we then get $\mu(V^c) = 0$.

Next, let $\tau_x(\omega)=\inf\{t>0: \nu^*(x,\omega,(0,t])>0\}$ ($=0$ if the indicated set is empty). It is again standard that θ_{τ_x} preserves P_x^* for every x such that $P(C_x)=1$. The result now follows quickly from (26).

We can now state a few more consequences of (26):

(c) The formula $\int_{\{X\in\Gamma\}} \tau\, d\mu = \int_\Gamma P(B_x)\, dx$, $\Gamma\in\mathscr{B}$, may be construed as a "continuous" version of Kac's formula for the mean return time to a set of positive measure.

(d) Consider a stationary, ergodic, Gaussian process having continuous trajectories, in which case $P(B_x)\equiv 1$. The formula in (c) gives us $E(\tau\,|\dot{X}|)=\infty$ which implies

(30) $\quad E(\tau^{1+\delta})=\infty \qquad$ for every $\delta>0$.

(Use Hölder's inequality and the existence of all moments of $\dot{X}$.)

In general, ϕ does not preserve P itself—(29) suggests that $|\dot{X}\circ\phi|=|\dot{X}|$ would be necessary (also see §7). Let $\sigma\colon \Omega\to[0,\infty)$ be such that $\sigma(\omega)\in M(\omega)$ for each $\omega\in V$ and $\sigma(\omega)=0$ for $\omega\notin V$; also let σ be $(\mathscr{F}^0)^*/\mathscr{B}$-measurable.

(31) **Theorem.** *The mapping $\theta_\sigma\colon \Omega\to\Omega$ preserves $d\dot{P}=I_{\{|\dot{X}|>0\}}\,dP$ iff θ_σ is almost invertible and leaves $|\dot{X}|$ invariant on the set $\{|\dot{X}|>0\}$.*

Before proving (31) we need an auxiliary result. Suppose $(W,\mathscr{W},Q)$ is a σ-finite measure space equipped with a measure-preserving bijection $s\colon W\to W$ and a random variable $K\colon W\to\mathbb{Z}$ ($\mathbb{Z}$ denoting the integers). For any $0<\zeta\in L^1(Q)$:

(32) **Theorem.** *The function $s^K\colon W\to W$ given by $s^K(\omega)=s^{K(\omega)}(\omega)$ preserves the measure $\zeta\, dQ$ iff s^K is almost invertible and $\zeta\circ s^K=\zeta$* a.s.

The meaning of "almost invertible" is that there exists a random variable $L=K$ a.s. such that s^L is bijective. This is equivalent to

(33) $\quad \sum_{k\in\mathbb{Z}} I_{\{K\circ s^{-k}=k\}}=1$ a.s.,

which says that the sets $A_k=\{K\circ s^{-k}=k\}$ partition W up to a null set (cf. Prop. 1 of [19]). Equation (33) is also equivalent to

(34) $\quad s^K$ preserves Q.

This makes clear that s^K preserves $\zeta\, dQ$ when it is almost invertible and (almost) preserves ζ. Now we assume that s^K preserves $\zeta\, dQ$; we will first show (33) holds.

Let $Z\geqq 0$ be a random variable on W. Then

$$\begin{aligned}\int Z\circ s^K\,\zeta\, dQ &= \sum_{k\in\mathbb{Z}} \int_{\{K=k\}} Z\circ s^k\,\zeta\, dQ\\ &= \sum \int_{A_k} Z\cdot\zeta\circ s^{-k}\, dQ\\ &= \int Z\cdot\eta\, dQ,\end{aligned}$$

where $\eta=\sum\zeta\circ s^{-k} I_{A_k}$. Since s^K preserves $\zeta\, dQ$ we conclude $\zeta=\eta$, and then $\sum I_{A_k}\geqq 1$ because $\zeta>0$. Write G for the set on which $\sum I_{A_k}>1$. If $Q(G)=0$, then

s^K preserves both Q (take $\zeta=1$ above) and $\zeta\,dQ$; consequently, for Z as above:

$$\int Z\zeta\,dQ=\int Z\circ s^K\,\zeta\,dQ=\int Z\circ s^L\,\zeta\,dQ=\int Z\zeta\circ(s^L)^{-1}\,dQ.$$

Hence $\zeta=\zeta\circ s^L=\zeta\circ s^K$ a.s. Now suppose $Q(G)>0$. Since $\zeta>0$, we have $Q((s^K)^{-1}G)>0$ as well. This also follows from

$$\begin{aligned} Q((s^K)^{-1}B)&=\sum_k\int I_B\circ s^K\,I_{\{K=k\}}\,dQ\\ &=\int_B\sum_k I_{A_k}\,dQ\geqq Q(B), \end{aligned}$$

which holds for any $B\in\mathscr{W}$. If Q happens to be a *finite* measure, the same inequality applied to B^c shows that in fact $Q((s^K)^{-1}B)=Q(B)$, so that (33) holds. In the general case, choose $w\in(s^K)^{-1}G$ and let $s^K w=v\in G$. Now $v\in A_{K(w)}$, hence there is some $k\neq K(w)$ such that $v\in A_k$ as well, since $v\in G$. We then have

(35) $\zeta(v)=\eta(v)\geqq\zeta\circ s^{-K(w)}(v)+\zeta\circ s^{-k}(v)>\zeta\circ s^{-K(w)}(v)$, i.e.

$\zeta(s^K w)>\zeta(w), \quad w\in(s^K)^{-1}G.$

For $w\in(S^K)^{-1}G^c$ we have a unique k such that $s^k w\in A_k$, namely $k=K(w)$, and so $\zeta(s^K w)=\eta(s^K w)=v(s^{-K(w)}s^{K(w)}w)$, i.e.

(36) $\zeta(s^K w)=\zeta(w), \quad w\in(s^K)^{-1}G^c.$

But ζ and $\zeta\circ s^K$ have the same distribution under $\zeta\,dQ$, and this is incompatible with (35) and (36) unless $Q((s^K)^{-1}G)=0$.

Note. The step involving the equidistribution of ζ and $\zeta\circ s^K$ fails without the assumption that $\zeta\in L^1(Q)$: take, e.g. $Q=m$ on $\mathbb{R}$ and $f(x)=x$, $g(x)=x+1$ to obtain two equi-distributed functions with $g>f$ everywhere.

Proof of (31). There exists a random variable $K:\Omega\to\mathbb{Z}$ such that $\theta_\sigma=\phi^K$. We can then apply (32) to the system $(\Omega,\mathscr{F}^0,\mu,\phi)$ with $\zeta=|\dot{X}|^{-1}I_{\{|\dot{X}|>0\}}$. Q.E.D.

Let $H\subseteq\Omega$ be the event $|\dot{X}\circ\phi^n|=|\dot{X}|$ infinitely often for $n\geqq 1$ and for $n\leqq -1$, and define $N(\omega)=I_H(\omega)$ (inf $\{n\geqq 1: |\dot{X}\circ\phi^n(\omega)|=|\dot{X}(\omega)|\}$). Clearly ϕ^N is invertible and has suitable measurability for

(37) **Corollary.** *ϕ^N preserves $\dot{P}$.*

Some examples which illustrate this behavior are given in § 8. We found it surprising that, even for processes with higher derivatives, mass-preservation depends only on the first derivative.

§ 7. Ergodic Properties

We now describe the invariant σ-field of ϕ and the class of P-equivalent invariant measures. The invariant σ-field for the original flow (θ_t) is denoted $\mathscr{A}$ and consists of those $A\in\mathscr{F}^0$ for which $\theta_t A=A$ for every $t\in\mathbb{R}$. Similarly, the ϕ-invariant σ-field is denoted $\mathscr{A}_\phi$ and $A\in\mathscr{A}_\phi$ iff $A\in\mathscr{F}^0$ and $\phi^{-1}A=A$. In what follows $\mathscr{A}\vee X$ denotes the σ-field generated by X and $\mathscr{A}$; $\overline{\mathscr{A}\vee X^\mu}$ means the σ-field generated by $X \vee \mathscr{A}$ together with all μ-null sets (similarly for $\bar{\mathscr{A}}_\phi^\mu$ etc.). We will assume $\mathscr{F}^0$ is separable (i.e. generated by a countable subfamily) and contains a "compact family". These conditions guarantee the existence of regular versions of conditional probabilities relative to any sub-σ-field of $\mathscr{F}^0$ (see [18]).

(38) **Theorem.** *Suppose $\mu_X(\Gamma)=\mu(X\in\Gamma)$ is a σ-finite measure; then*

$$\bar{\mathscr{A}}_\phi^\mu=\overline{\mathscr{A}\vee X}^\mu.$$

Proof. First observe that $\mathscr{A}\vee X\subset\mathscr{A}_\phi$ without any completion, so we need only prove $\bar{\mathscr{A}}_\phi^\mu\subset\overline{\mathscr{A}\vee X}^\mu$. Next, since X is $\mathscr{A}_\phi$-measurable and μ_X is σ-finite, by restricting to a set of the form $\{X\in\Gamma\}$ we may and do assume that μ_X is in fact *finite*, and, for convenience, even a probability measure.

Let $\mathscr{A}'$ be a separable sub-σ-field of $\mathscr{A}$ such that $\bar{\mathscr{A}}'=\bar{\mathscr{A}}$, the bar now signifying completion by all P-null sets. It is standard that there exists a Markov kernel $Q(\omega, A)$, $\omega\in\Omega$, $A\in\mathscr{F}^0$, such that

(39) (a) $Q(\omega, A)$ is a (regular) conditional probability of A given $\mathscr{A}'$,
(b) for almost every $\omega\in\Omega$, $Q(\omega,\cdot)$ is preserved by the flow (θ_t),
(c) for almost every $\omega\in\Omega$, $Q(\omega,\cdot)$ is ergodic, i.e., $Q(\omega, A)=0$ or 1 if $A\in\mathscr{A}$.

This is the *ergodic decomposition* of the measure P.

We write $\mu_\omega(A)=\int_A |\dot{X}(\omega')|\, Q(\omega, d\omega')$. Then, with the obvious notation,

1° $\bar{X}^{\mu_\omega}=\bar{\mathscr{A}}_\phi^{\mu_\omega}$ for almost every $\omega\in\Omega$.

To prove this, consider the dynamical system $(\Omega, \mathscr{F}^0, Q(\omega,\cdot), \theta_t)$ for a fixed $\omega\in\Omega$. This system has all the properties of our original system $(\Omega, \mathscr{F}^0, P, \theta_t)$, at least for ω satisfying (39). In particular, (26) becomes

$$(40)\qquad \int_{(X\in\Gamma)} Z\, d\mu_\omega=\int_\Gamma E^*_{x,\omega}(Z)\, dx$$

in which the subscripts ω indicate that the basic measure is $Q(\omega,\cdot)$. Now with $Q(\omega,\cdot)$ ergodic, it follows (see e.g. [9]) that $P^*_{x,\omega}$ is ergodic relative to $\phi_x=\theta_{\tau_x}$ for a.e. x (i.e. those x's for which $P^*_x(V^c_\omega)=0$).

Let $\omega\in\Omega$ be fixed with (39) in force, and put $Z=I_A$, $A\in\mathscr{A}_\phi$. Since $Z\circ\phi\equiv Z$, we have $Z\circ\phi_x=Z$, $P^*_{x,\omega}$-a.e. and so $Z=z_x$ (a constant $=0$ or 1) $P^*_{x,\omega}$-a.e. for a.e. x. Let $g_\omega(x)$ be the indicator of $\{x: z_x=1\}$. We will show that $Z=g_\omega(X)\,\mu_\omega$-a.s. A monotone class argument applied to (40) shows

$$E_\omega(f(X,\cdot)\,|\dot{X}|)=\int_{\mathbb{R}} E^*_{x,\omega} f(x,\cdot)\, dx$$

for any $\mathscr{B}\times\mathscr{F}^0$-measurable function $f\geqq 0$. Taking $f(x,\omega')=I_{\{Z\neq g_\omega\}}(x,\omega')$ immediately yields the result, and we conclude $Z=E_{\mu_\omega}(Z\mid X)$.

2° Let $Z=I_A$ as in 1°. Then

$$E(ZY\mid X\vee\mathscr{A})=ZE(Y\mid X\vee\mathscr{A})\qquad \mu\text{-a.s.},$$

where $Y=|\dot{X}|$.

We begin by observing

$$E_\omega(ZY\mid X)=E_\omega(Y\mid X)\, E_{\mu_\omega}(Z\mid X)\qquad Q(\omega,\cdot)\text{-a.e.}$$

(hence μ_ω-a.e.), which follows from the relation $\mu_\omega(d\omega')=Y(\omega')\,Q(\omega,d\omega')$. Next, arguing as in [13], we find

$$E_\omega(Y\mid X)=E(Y\mid X\vee\mathscr{A}),\qquad E_\omega(ZY\mid X)=E(ZY\mid X\vee\mathscr{A})$$

$Q(\omega,\cdot)$-a.e. (μ_ω-a.e.) for almost every $\omega\in\Omega$. From 1° we find

$$E(ZY\mid X\vee\mathscr{A})=ZE(Y\mid X\vee\mathscr{A})\qquad \mu_\omega\text{-a.e.}$$

for P-a.e. ω, and so μ-a.s.

3° Let Z be as in 1°, and $U=E(Y\mid X\vee\mathscr{A})$. Then

$$\mu(U=0)=E(Y;\,U=0)=E(U;\,U=0)=0.$$

Thus $Z=E(ZY\mid X\vee\mathscr{A})/E(Y\mid X\vee\mathscr{A})$ μ-a.s. and the theorem is proven. Q.E.D.

Theorem (38) is similar to Theorem (11) of [13], which deals with the discrete-parameter case but requires a supplementary hypothesis which is avoided in the present case because the theory of Palm measures allows the conclusion in 1° that $P^*_{x,\omega}$ is ergodic.

An immediate consequence of (38) is obtained under the additional assumption that $\dot{X}\neq 0$ a.s.

(41) **Theorem.** *A σ-finite measure Q on $\mathscr{F}^0$ is absolutely continuous relative to P and invariant under ϕ iff it is of the form $dQ=\zeta\,|\dot{X}|\,dP$ for some $\mathscr{A}\vee X$-measurable function $\zeta\geqq 0$.*

Remark. Suppose $\dot{X}$ is the true derivative, still $\neq 0$ a.s. For any π, (17) applies to almost every $X_\cdot(\omega)$ so that $\dot{\xi}_s(\omega)$ exists and $F'(X_s(\omega))\,\dot{X}_s(\omega)$ a.e. $(m\times P)$. By stationarity, $\dot{\xi}(\omega)$ exists and $F'(X(\omega))\,\dot{X}(\omega)$ a.s. Suppose $\dot{\xi}$ and $\dot{X}$ induce the *same* transformation ϕ, which then must preserve *both* $|\dot{X}|\,dP$ and $|\dot{\xi}|\,dP$. (This will be true, e.g., when (X_t) is very smooth.) In this case, (4) is satisfied for $dQ=|\dot{\xi}|\,dP$ by choosing $\zeta=F'\circ X$.

§ 8. Examples

Example 1. This is simply to illustrate the general results, particularly (37) and (38). Consider the process $X_t=A\,\cos t+B\sin t$, where A,B are independent, standard normal random variables. By using a suitable function space representation, we may assume our probability space is endowed with a flow (θ_t) such that $X_t=X_0\circ\theta_t$, for all t,ω. The trajectories $t\to X_t(\omega)$ are sinusoidal, with independent phase and amplitude, the former uniform on $[0,2\pi]$ and the latter with density $xe^{-x^2/2}$ on $[0,\infty)$. Taking $\pi=m$, $\dot{\xi}=\dot{X}_0=B$ and a picture shows that $\dot{X}_\tau=-\dot{X}_0$. According to (37), P is preserved by θ_τ; in particular, $(X_{t+\tau})$ has the same law as (X_t). By (38), we have $\mathscr{A}_\phi=\overline{\mathscr{A}\vee A}$ (the bar may refer to μ or P indifferently). Since $|B|\in\mathscr{A}_\phi$, there must be a function $f(x,\omega)$ which is $\mathscr{B}\times\mathscr{A}$-measurable and such that $|B(\omega)|=f(A(\omega),\omega)$ a.s. Indeed, the function $f(x,\omega)=(A^2(\omega)+B^2(\omega)-x^2)^{1/2}$ has these properties since $(A^2+B^2)^{1/2}$ is the amplitude of the sinusoid, and hence invariant under shifts of the time origin. More interesting examples are easily generated by superimposing independent copies of (X_t) of varying (non-random) frequencies, although we shall not pursue these.

Suppose now that (X_t) is any *ergodic*, separable stationary Gaussian process with a finite second spectral moment. (Almost every trajectory is then absolutely continuous.) It is then *impossible* for $\phi = \theta_\tau$ to preserve P itself: if it did, then $|\dot{X}_\tau| = |\dot{X}_0|$ a.s. by (31) so that $|\dot{X}_0|$ would be $\bar{\mathscr{A}}_\phi$-measurable. But then (38) would imply $|\dot{X}_0|$ is measurable over the σ-field generated by X_0 and the null sets – which is impossible in view of the independence of X_0 and $\dot{X}_0$. This does not preclude the repetition of $(\dot{X}_t)$ later in the time set M; it would be interesting to know if this could indeed occur.

Example 2. Let Ω be the torus, conceived as the unit square with properly pasted boundaries, $\mathscr{F}^0$ its Borel σ-field, and P Lebesgue measure. For $(x, y) \in \Omega$, let $\theta_t(x, y) = (x+t, y+\gamma t)$, the addition being mod 1, γ a fixed number. This is well-known to be a flow which is ergodic iff γ is irrational. The flow lines are straight lines moving "diagonally" across the square with slope γ. A random variable $X = X(x, y)$ on Ω induces a family of "level curves" on the square: $\tau(x, y)$ is then the first time that the flow moves back into the level curve from which it started. It was an attempt to work with such examples which led to the extension of the results in [8] to the more general situation treated in this paper.

We will show directly that ϕ preserves $|\dot{X}|\,dP$ in the special case where $\gamma = 1$ and X is the restriction to the square of a smooth (continuous first partials) function on $\mathbb{R}^2$. Obviously $\dot{X}(x, y) = \dfrac{\partial X}{\partial x} + \gamma \dfrac{\partial X}{\partial y}$, except possibly at the boundary. Define $G\colon \Omega \to \mathbb{R}^2$ by $G(x, y) = (X(x, y), y - x)$. One verifies easily that $|\dot{X}|$ is the absolute value of the Jacobian determinant of G. Since G is smooth along with X, we have from [6]

$$(42)\quad \int_A |\dot{X}|\,dP = \int_{\mathbb{R}} \nu(u, v, A)\,du\,dv$$

where $A \subset \Omega$ is Borel, and $\nu(u, v, A) = $ cardinality $\{(x, y) \in A\colon G(x, y) = (u, v)\}$. Since $\nu(u, v, A) = 0$ whenever $|v| > 1$, we have for the right member of (42)

$$\int_{-\infty}^{\infty} \int_{-1}^{1} \nu(u, v, A)\,dv\,du = \int_{-\infty}^{\infty} \int_{-1}^{0} (\nu(u, v, A) + \nu(u, 1+v, A))\,dv\,du.$$

To prove our assertion, it will suffice to show

$$(43)\quad \nu(u, v, A) + \nu(u, 1+v, A) = \nu(u, v, \phi^{-1} A) + \nu(u, 1+v, \phi^{-1} A)$$

for every $u \in \mathbb{R}$ and $v \in (-1, 0)$.

Consider the orbit $0(v)$ through the point $(0, 1+v)$: it consists of two diagonal segments joining the pairs $(0, 1+v)$, $(-v, 1)$ and $(-v, 0)$, $(1, 1+v)$ respectively. The left member of (43) is easily seen to be card $[0(v) \cap X^{-1}(u) \cap A]$. Next, one can check that the map ϕ is one-to-one and onto on each orbit. Hence

$$\begin{aligned} \operatorname{card}\,[0(v) \cap X^{-1}(u) \cap A] &= \operatorname{card}\,[\phi^{-1}(0(v)) \cap \phi^{-1}(X^{-1}(u)) \cap \phi^{-1}(A)] \\ &= \operatorname{card}\,[0(v) \cap X^{-1}(u) \cap \phi^{-1}(A)] \end{aligned}$$

and (43) is proven. (Essentially the same proof works for any rational γ.) Assuming $\tau(x, y)$ is suitably smooth, yet another way to prove $|\dot{X}|\,dP$ is invariant under θ_τ is via a straightforward change-of-variable argument for transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$.

Now let γ be irrational, i.e. the flow is ergodic. Then $\bar{\mathscr{A}}_\phi = \bar{X}$ since the invariant σ-field $\mathscr{A}$ is trivial, and we can ask for conditions on $X(x, y)$ which imply θ_τ preserves P, equivalently, that $|\dot{X}|$ be equal a.s. to a function of X. We are seeking functions $X(x, y)$ which satisfy

$$(44) \quad \left|\frac{\partial X}{\partial x} + \gamma \frac{\partial X}{\partial y}\right| = h(X(x, y)) \qquad \text{a.s. } (P)$$

for some Borel function h. As an example the function $X(x, y) = e^{-(x+y)}$ satisfies (44) with $h(u) = u(1+\gamma)$. More generally, let X be of the form

$$X(x, y) = \psi(x + f(y - \gamma x)),$$

with ψ, f smooth, $\psi' > 0$. In this case,

$$\frac{\partial X}{\partial x} + \gamma \frac{\partial X}{\partial y} = \psi'(x + f(y - \gamma x)),$$

and we may take $h = \psi' \circ \hat{\psi}$ in (44) where $\hat{\psi}$ is the inverse of ψ.

§ 9. Continuous Local Times

With $(\Omega, \mathscr{F}^0, P, \theta_t)$ as before, suppose now that $X_t = X \circ \theta_t$ satisfies (LT) and that $t \to \alpha_t(x, \omega)$ is *continuous* a.e. $(\pi \times P)$. Except in degenerate cases, the paths $t \to X_t(\omega)$ are no longer even approximately differentiable; indeed, the level sets are uncountable. The connection with previous sections is this. We have seen in (29) that, under the assumptions there, μ is invariant under a (random) shift of the time origin by an amount necessary to accumulate positive local time mass in "state $X(\omega)$." Of course, $\{\phi^n\}_{n \in Z}$ then defines a discrete-parameter flow over $(\Omega, \mathscr{F}^0, \mu)$. Here, we shall see that, for any $t \in \mathbb{R}$, P is preserved by shifting the time origin to accumulate t units of local time mass in "state $X(\omega)$", and corresponding to $\{\phi^n\}$ is a continuous-parameter flow over $(\Omega, \mathscr{F}^0, P)$.

We can and do assume we have a version of the local time $\alpha_t(x, \omega)$ which, aside from good measurability, is a continuous additive functional with support in $\{t\colon X_t(\omega) = x\}$ for every x, ω. Let $\hat{\alpha}_t(x, \omega)$ be the associated time-change; the equation

$$(45) \quad \hat{\alpha}_{t+s}(x, \omega) = \hat{\alpha}_t(x, \omega) + \hat{\alpha}_s(x, \theta_{\hat{\alpha}_t(x, \omega)}(\omega)) \qquad \text{for every } s, t, x, \omega$$

is then standard. Now set $\zeta_t(\omega) = \hat{\alpha}_t(X(\omega), \omega)$ and $\mathscr{V}_t = \theta_{\zeta_t}$, $t \in \mathbb{R}$.

(46) **Theorem.** *$(\mathscr{V}_t)$ is a flow over $(\Omega, \mathscr{F}^0, P)$.*

Proof. That $(\mathscr{V}_t)$ is a group (under composition) of transformations is immediate from (45) and the $(\mathscr{V}_t)$-invariance of X. (However, we can only assert that $\mathscr{V}_0(\omega) = \omega$ a.s.)

Now the assumption that (X_t) has a local time implies that

$$(47) \quad \int_{-\infty}^{\infty} f(s, X_s(\omega), \omega)\, ds = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(s, x, \omega)\, d\alpha_s(x, \omega)\, \pi(dx)$$

for any $f \geqq 0$ and $\mathscr{B} \times \mathscr{B} \times \mathscr{F}^0$-measurable. According to [11], for each t fixed, $\mathscr{V}_t$ will preserve P iff for almost every ω,

$$(48) \quad \int_{-\infty}^{\infty} g(s + \zeta_t \circ \theta_s)\, ds = \int_{-\infty}^{\infty} g(s)\, ds$$

for every Borel function $g(s) \geqq 0$. Fix g and ω and choose $f(s, x, \omega) = g(s + \hat{\alpha}_t(x, \theta_s \omega))$ in (47). Then the left members of (47) and (48) coincide. The right member of (47) becomes

$$\begin{aligned} &\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\hat{\alpha}_s(x, \omega) + \hat{\alpha}_t(x, \theta_{\hat{\alpha}_s(x, \omega)}(\omega))\, ds\, \pi(dx) \\ &\quad = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\hat{\alpha}_{s+t}(x, \omega))\, ds\, \pi(dx) \\ &\quad = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\hat{\alpha}_s(x, \omega))\, ds\, \pi(dx) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(s)\, d\alpha_s(x, \omega)\, \pi(dx) = \int_{-\infty}^{\infty} g(s)\, ds, \end{aligned}$$

using the change of variable $\alpha_s(x) \to s$ and (45). Q.E.D.

Appropriately recast, most of the results in Sections 6, 7 have "continuous" analogues, but the main one is (46) and we will leave it at that.

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Note Added in Proof

With the notation from § 1, we have now shown that (T_2) is equivalent to $(T_1 G)$, settling a question raised in § 1. Here is a simple proof that $(T_2) \Rightarrow (T_1 G)$:

For any Borel measurable $g \geqq 0$ we have

$$\int g(s)\, I_\Gamma(x(s))\, \psi(ds) = \int_\Gamma \int g(s)\, \gamma(y, ds)\, dy, \qquad \Gamma \in \mathscr{B}.$$

This extends immediately to

$$\int v(s, x(s))\, \psi(ds) = \int \int v(s, y)\, \nu(y, ds)\, dy$$

for any $\mathscr{B} \otimes \mathscr{B}$-measurable function $v \geqq 0$. Now let $\tau_n(y)$, $n = 1, 2, \ldots$ be a "measurable enumeration" of the level set $\{t\colon x(t) = y\}$, $y \in \mathbb{R}$: since $x(\cdot)$ is (T_2), this exists for a.e. y, either by a direct construction or using the material in [3] concerning graphs of stopping times. Let $0 < V \in L^1(m)$ and put $v(s, y) = 2^{-n} V(y)$ if $s = \tau_n(y)$, $v(s, y) = 1$ if $s \neq \tau_n(y)$ for the countable levels y, and $v(s, y) \equiv 1$ for the rest. Then

$$\int v(s, x(s))\, \psi(ds) = \int V(y) \sum_n 2^{-n}\, dy < \infty.$$

Since $v(s, x(s)) > 0$ for all s, the measure ψ is σ-finite and so $x(t)$ is $(T_1 G)$ by (3).