

## Continuity of Mean Recurrence Times in Denumerable Semi-Markov Processes\*

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**Summary.** For a family of semi-Markov processes where the transition matrices for the embedded Markov chains and the mean sojourn times depend continuously on a parameter, we give equivalent as well as sufficient conditions for the continuity of the mean recurrence times. The results will be used in a subsequent paper on average costs in a dynamic programming model.

### 1. Introduction and Summary

Our object of study is a family of semi-Markov processes on a denumerable state space depending on a parameter coming from a metric space. We analyse the dependence of the mean recurrence times on the parameter. Under the general assumption that the mean sojourn times and the transition matrices of the embedded Markov chains are continuous functions of the parameter, we develop some conditions equivalent to the continuity of the mean recurrence times. They give rise to a number of sufficient conditions suitable for practical applications.

Our analysis is motivated by the study of average cost criteria in dynamic programming models. For stationary strategies, the average costs can be described by means of stationary or equilibrium measures, provided the latter exist. Asking for the existence of a cost minimizing stationary strategy it is only natural to demand the continuity of these equilibrium measures. They are essentially given by the mean recurrence times  $\mu_{ii}$ . Thus, the conditions for the continuity of the  $\mu_{ii}$ 's appearing in this paper are intimately related to known conditions for the existence of average cost optimal strategies in dynamic programming models, see e.g. Hordijk [8, 9], Federgruen, Hordijk and Tijms [6], Wijngaard [15, 16]. In fact our conditions generalize all of them, cf. Deppe [5].

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It is essential for our analysis that the underlying family of Markov chains is what we call quasi-finite. This means that there is a finite subset  $K$  of the state space such that  $K$  can be reached from any starting state with probability one, for all parameters. This property allows us to exploit properties of the Markov chains induced on the finite state space  $K$  to infer that the number of recurrent classes is always upper semi-continuous, just as in the finite case. The quasi-finiteness also arises as a natural condition when we look at the relation of the continuity of the mean recurrence times to certain tightness conditions for the family of equilibrium measures. For (unchained) Markov chains such a tightness condition is known to be equivalent to the continuity of the stationary measures; see Hordijk [9] and Federgruen and Tijms [7]. We prove a similar result for semi-Markov processes. Whereas most papers in the literature on dynamic programming assume unchainedness, the quasi-finiteness assumption allows us to handle the multichain case without additional effort.

Our analysis leads to two sufficient criteria for practical applications. The one uses the tightness condition mentioned above, which was first introduced by Hordijk [8] for Markov chains. Generalizing a result by Schäl [13] we obtain a condition especially useful for applications in queuing control theory. The other sufficient criterion is of Liapunov function type. Such a condition was first used by Hordijk [9] in the context of dynamic programming. Recently, it has been extended by Federgruen et al. [6]. Our result is proved by using properties of the mean recurrence times  $\mu_{iK}$  to a fixed finite subset  $K$  of the state space.

## 2. Notations

We consider a Markov renewal process  $(Y_n, S_n)$  where  $Y_0, Y_1, \dots$  denote the sequence of states in a denumerable state space  $I$ , and  $0 = S_0 < S_1 < \dots$  denote the sequence of jump times. We write  $n(t) = \sup \{n | S_n \leq t\}$  for the index of the last jump before time  $t$ . The semi-Markov process  $Y_{n(t)}$  develops according to the transition law given by the semi-Markov kernel  $Q_{ij}(t) = P_i\{Y_1 = j, S_1 \leq t\}$ . Here,  $P_i$  is the probability measure for the process if at time  $S_0 = 0$  the process starts in state  $Y_0 = i$ . The corresponding expectation operator will be denoted by  $E_i$ .

For the embedded Markov chain  $(Y_n)$  we adopt the notation from Chung [1]. Thus,  $p_{ij} = P_i\{Y_1 = j\} = Q_{ij}(\infty)$  gives us the transition matrix,  $p_{ij}^{(n)} = P_i\{Y_n = j\}$  are the  $n$ -step transition probabilities, and  ${}_K p_{ij}^{(n)}$  is the (taboo) probability for jumping to  $j$  at the  $n$ -th step starting in  $i$  and not visiting  $K$  during the steps inbetween. Further,  ${}_K p_{ij}^* = \sum_{n=1}^{\infty} {}_K p_{ij}^{(n)}$  is the expected number of visits to  $j$  before any visit to  $K$ , starting in  $i$ . Especially,  $f_{ij}^* = {}_j p_{ij}^*$  and  $f^*(i, K) = \sum_{j \in K} {}_K p_{ij}^*$  are the probabilities for reaching  $j$  or  $K$ , resp., starting in  $i$ . The expected number of visits to  $j$  between two visits to  $i$  is denoted by  $e_{ij} = {}_i p_{ij}^*$ . The Cesaro limits  $\lim_{n \rightarrow \infty} n^{-1} \sum_{m=1}^n p_{ij}^{(m)}$  are denoted by  $\pi_{ij}$ . For the step numbers of the  $n$ -th visit to  $K$  and to  $i$  (not counting  $Y_0$ ) we write  $\tau_{Kn}$  and  $\tau_{in}$ , resp.

The corresponding time points of the semi-Markov process are  $T_{kn} = S_{\tau_{kn}}$  and  $T_{in} = S_{\tau_{in}}$ . The expected sojourn times will be denoted by  $\eta_i = E_i S_1$  and will assumed to be finite. The mean recurrence times, in both discrete and continuous time, are  $m_{iK} = E_i \tau_{K1}$ ,  $m_{ii} = E_i \tau_{i1}$ , and  $\mu_{iK} = E_i T_{K1}$ ,  $\mu_{ii} = E_i T_{i1}$ , resp. The set of recurrent states, characterized by the property  $f_{ii}^* = 1$ , will be denoted by  $D$ . Further,  $D_+ = \{i | m_{ii} < \infty\}$  and  $\tilde{D} = \{i | \mu_{ii} < \infty\}$  are the sets of positive recurrent states and states with finite mean recurrence time, resp. Any of these sets splits into classes; for the number of classes in  $D$ ,  $D_+$ , and  $\tilde{D}$  we write  $v$ ,  $v_+$ , and  $\tilde{v}$ , resp. Finally,  $D_i = \{j | f_{ij}^* > 0 \text{ and } f_{ji}^* > 0\} \cup \{i\}$  is the class of state  $i$ .

We assume that no explosions may occur, i.e.  $P_i\{\lim S_n < \infty\} = 0$  ( $i \in I$ ). Then we have a stationary probability measure  $P$  for every class  $D_i$  in  $\tilde{D}$ , characterized by the property that  $P\{Y_{n(t)} = j\}$  is independent of  $t \geq 0$  and zero for  $j \notin D_i$ . It satisfies  $P\{Y_{n(t)} = j\} = \eta_j / \mu_{jj}$  for  $j \in D_i$  (cf. Pyke and Schaufele [11]). Provided that  $\tilde{D}$  can be reached with probability one from every starting state  $i$ , the semi-Markov process will tend to an equilibrium given by a convex combination of these stationary measures according to the probabilities for reaching the different classes in  $\tilde{D}$ . Thus the equilibrium measure on the state space, corresponding to starting state  $i$ , is  $j \rightarrow f_{ij}^* \eta_j / \mu_{jj}$ . (Here, we set  $a/\infty = 0$  for any real number  $a$ .)

We analyze the dependence of these equilibrium measures on a parameter if the original data of the process depend on this parameter. To that end, we assume that the semi-Markov kernel  $Q_{ij}(t)$ , and hence all other quantities introduced above, depend on a parameter  $f$  coming from a metric space  $F$ :  $Q_{ij}^f(t)$ ,  $P_i^f$ ,  $p_{ij}(f)$ ,  $\eta_j(f)$ ,  $D(f)$ ,  $v(f)$ , .... If statements are valid for all  $f \in F$ , we will simply omit the  $f$ . Our general assumption is that the transition matrix ( $p_{ij}(f)$ ) and the expected sojourn times  $\eta_i(f)$  depend continuously on  $f$ . We then ask for conditions guaranteeing the continuity of the above equilibrium measures. As we will see in Lemma 5.4, this is equivalent to the continuity of the mean recurrence times  $\mu_{ii}$  ( $i \in I$ ) considered as functions to  $\mathbb{R} \cup \{\infty\}$ .

### 3. General Assumptions

We now summarize the general assumptions introduced in the preceding section.

*Assumption 1.* For all  $i \in I$ ,  $f^*(i, \tilde{D}) = 1$  and  $P_i\{\lim S_n = \infty\} = 1$ .

*Assumption 2.* For all  $i, j \in I$ ,  $f \rightarrow p_{ij}(f)$  and  $f \rightarrow \eta_i(f) < \infty$  are continuous functions on  $F$ .

The first part of Assumption 1 implies  $D \subset \tilde{D}$ . In fact, for every  $i \in D$  there exists a  $j \in D_i \cap \tilde{D}$ , hence  $i \in D_j \subset \tilde{D}$ . Assuming the first part of Assumption 1, its second part is equivalent to both " $\tilde{D} \subset D$ " and " $f^*(i, D) = 1$  ( $i \in I$ )". For, under this last property, it follows from Corollary 10.3.17 in Çinlar [3] that no explosions may occur; and the other two implications are easy. Thus under Assumption 1 we have  $\tilde{D} = D$  and, accordingly,  $\tilde{v} = v$ .

#### 4. Some Formulas for Mean Recurrence Times

**Lemma 4.1.** *Let  $i \in I$ ,  $\emptyset \neq K \subset J \subset I$ ,  $n \in \mathbb{N}$ . Then*

$$\begin{aligned}
 (1) \quad \mu_{iK} &= \mu_{iJ} + \sum_{j \in J \setminus K} {}_K p_{ij}^* \mu_{jJ} \\
 (2) \quad &= \eta_i + \sum_{j \notin K} {}_K p_{ij}^* \eta_j \\
 (3) \quad &= \eta_i + \sum_{m=1}^{\infty} \sum_{j \notin K} {}_K p_{ij}^{(m)} \eta_j \\
 (4) \quad &= \eta_i + \sum_{m=1}^{n-1} \sum_{j \notin K} {}_K p_{ij}^{(m)} \eta_j + \sum_{j \notin K} {}_K p_{ij}^{(n)} \mu_{jK} \\
 (5) \quad \mu_{ii} &= \sum_j e_{ij} \eta_j.
 \end{aligned}$$

*Proof.* Since  $K \subset J$  we have

$$\mu_{iK} = E_i T_{K1} = E_i T_{J1} + \sum_{m=1}^{\infty} \sum_{j \in J \setminus K} E_i([\tau_{Jm} < \tau_{K1}, Y_{\tau_{Jm}} = j] \cdot (T_{J, m+1} - T_{Jm})).$$

The last integral can be evaluated by conditioning on  $\tau_{Jm}$  which gives  $P_i\{\tau_{Jm} < \tau_{K1}, Y_{\tau_{Jm}} = j\} \cdot E_j T_{J1}$ . Thus formula (1) holds. Setting  $J = I$  we immediately obtain (2) and (3). Formula (4) can be proved by substituting the expression from (3) for  $\mu_{jK}$  in the last sum in (4). Finally, (5) is a special case of (2).  $\square$

Within a class we have

$$(6) \quad \mu_{ii} = e_{ij} \mu_{jj}.$$

This follows from (5) and the multiplicative property  $e_{ij} e_{jh} = e_{ih}$  ( $i \in D$ ,  $j \in D_i$ ,  $h \in I$ ), see Chung [1], Corollary 1 to Theorem I.9.5. By (5), (6), and  $e_{ij} = 1/e_{ji}$ , the numbers  $\eta_i/\mu_{ii}$  sum to 1 within each class of  $D$ . Therefore, and as a consequence of Assumption 1, the following two formulas hold:

$$(7) \quad v = \sum_i \eta_i / \mu_{ii}$$

$$(8) \quad \sum_{j \in I} f_{ij}^* \eta_j / \mu_{jj} = 1 \quad (i \in I).$$

#### 5. Continuity of Taboo Probabilities

**Lemma 5.1.** (Royden [12], Prop. 11.18; Hordijk [9], Lemma 4.12). *For real functions  $a_n, b_n, a, b$  on  $I$ , satisfying  $|b_n(i)| \leq a_n(i)$  ( $n \in \mathbb{N}$ ), and  $a_n(i) \rightarrow a(i)$ ,  $b_n(i) \rightarrow b(i)$  ( $i \in I$ ),  $\sum_i a_n(i) \rightarrow \sum_i a(i) < \infty$ , we have  $\sum_i b_n(i) \rightarrow \sum_i b(i)$ .  $\square$*

We are now in a position to prove the following generalization of Theorem 1(b) in Federgruen et al. [6].

**Theorem 5.2.** *Let  $i, j \in I$ ,  $J \subset I$ ,  $f_0 \in F$ ,  $n \in \mathbb{N}$ . Then*

- (a) *The function  ${}_J p_{ij}^{(n)}$  is continuous.*
- (b) *The functions  ${}_J p_{ij}^*$ ,  $f_{ij}^*$ ,  $f^*(i, J)$ , and  $e_{ij}$  are lower semicontinuous (l.s.c.).*
- (c) *If  $f^*(i, J)(f_0) = 1$  and either  $j \in J$  of  $f^*(j, J)(f_0) = 1$ , then at the point  $f_0$ , the function  ${}_J p_{ij}^*$  is finite and continuous.*
- (d) *For  $i \in D(f_0)$  and  $j \in D_i(f_0)$ ,  $e_{ij}$  is continuous at  $f_0$ .*

*Proof.* (a) follows by induction from the recursive definition of  ${}_J p_{ij}^{(n)}$  and Lemma 5.1.

(b) is a consequence of (a) by Fatou's lemma.

(c) Let first be  $j \in J$ . We show for a given  $\varepsilon > 0$  that there exist a natural number  $N \in \mathbb{N}$  and a neighbourhood  $U(f_0)$  of  $f_0$  such that

$$(9) \quad \sum_{n=N+1}^{\infty} {}_J p_{ij}^{(n)}(f) \leq \varepsilon \quad (f \in U(f_0)),$$

which proves the result by part (a). The sum in (9) is less or equal to  $P_i^f \{\tau_{J1} > N\} = \sum_{h \notin J} {}_J p_{ih}^{(N)}(f)$ , which is continuous by part (a) and Lemma 5.1.

For  $f = f_0$  this expression is smaller than  $\varepsilon/2$  for a sufficiently large  $N$ , since  $P_i^{f_0} \{\tau_{J1} < \infty\} = 1$ . Then (9) holds for a certain neighbourhood  $U(f_0)$ . For  $j \notin J$  the result follows from  ${}_J p_{ij}^* = {}_j p_{ij}^* / (1 - {}_j p_{jj}^*)$  (see Chung [1], (I.9.4.)) and the preceding argument.

(d) is a special case of (c).  $\square$

**Lemma 5.3.** *For arbitrary  $i \in I$ ,  $J \subset I$ ,  $\mu_{ij}$  is l.s.c.*

The proof follows from Eq. (2) and Theorem 5.2(b).  $\square$

**Lemma 5.4.** *The functions  $f_{ij}^* / \mu_{jj}$  are continuous for all  $i, j \in I$ , if and only if the functions  $\mu_{ii}$  are continuous for all  $i \in I$ .*

*Proof.* The if-part. Lower semi-continuity of  $f_{ij}^* / \mu_{jj}$  follows from Theorem 5.2(b). Upper semi-continuity then is a consequence of Eq. (8) and the general observation that continuity and finiteness of a sum of l.s.c. functions imply the continuity of all terms in the sum.

For the only if-part let  $i \in I$  and  $f_0 \in F$ . Upper semicontinuity of  $\mu_{ii}$  at  $f_0$  follows either from Lemma 5.3 (if  $\mu_{ii}(f_0) = \infty$ ) or from  $f_{ii}^* \leq 1$  and  $i \in D(f_0)$  (if  $\mu_{ii}(f_0) < \infty$ ).  $\square$

Lemma 5.4 shows that we may restrict ourselves to the analysis of the continuity of the  $\mu_{ii}$ 's.

## 6. Tightness Conditions and Quasi-Finiteness

For Markov chains having only one irreducible set of states it is known that the continuity of the stationary probabilities  $\pi_j$  is equivalent to the tightness of

the family  $\{\pi_j(f) | f \in F\}$  of probability measures, see Hordijk [9], Lemma 10.2, and Federgruen and Tijms [7], Theorem 2.1. In this section we show for our more general case, how the continuity of the  $\mu_{ii}$ 's is related to similar tightness conditions.

**Theorem 6.1.** *For the following statements we have the implications (10)  $\Rightarrow$  (11) and, assuming  $F$  to be compact, (11)  $\Rightarrow$  (12).*

(10)  $v$  is l.s.c., and

(10a) for all  $\varepsilon > 0$  there exists a finite set  $K \subset I$  satisfying

$$\sum_{i \in K} \eta_i(f) / \mu_{ii}(f) \geq v(f) - \varepsilon \quad (f \in F);$$

(11)  $\mu_{ii}$  is continuous ( $i \in I$ );

(12)  $v$  is l.s.c., and for all  $h \in I$

(12a) the family  $\{i \rightarrow f_{hi}^*(f) \eta_i(f) / \mu_{ii}(f) | f \in F\}$  of probability measures is tight.

*Proof.* (10)  $\Rightarrow$  (11): Let  $(f_n)$  be a sequence in  $F$  converging to an  $f_0 \in F$ . Choose a subsequence  $(f_{n'})$  such that  $\gamma_i = \lim_{n'} \eta_i(f_{n'}) / \mu_{ii}(f_{n'})$  exists for all  $i \in I$ . By Lemma 5.3 we have  $0 \leq \gamma_i \leq \eta_i(f_0) / \mu_{ii}(f_0)$ . For any  $\varepsilon > 0$  let  $K$  be as in (10a). Then

$$\begin{aligned} v(f_0) &\leq \underline{\lim} v(f_{n'}) \leq \overline{\lim} v(f_{n'}) \\ &\leq \overline{\lim} \sum_{i \in K} \eta_i(f_{n'}) / \mu_{ii}(f_{n'}) + \varepsilon \\ &= \sum_{i \in K} \gamma_i + \varepsilon \leq \sum_{i \in I} \gamma_i + \varepsilon \\ &\leq \sum_{i \in I} \eta_i(f_0) / \mu_{ii}(f_0) + \varepsilon \\ &= v(f_0) + \varepsilon, \end{aligned}$$

where the last equality follows from (7). Thus we have  $\sum_i \gamma_i = \sum_i \eta_i(f_0) / \mu_{ii}(f_0) < \infty$ , hence  $\gamma_i = \eta_i(f_0) / \mu_{ii}(f_0)$  for all  $i \in I$ . Since  $(f_{n'})$  can be chosen as a subsequence of an arbitrary subsequence of  $(f_n)$ , this shows the continuity of the  $\mu_{ii}$ 's.

(11)  $\Rightarrow$  (12), if  $F$  is compact. In fact, lower semicontinuity of  $v$  follows from (7), and (12a) is a consequence of Lemma 5.4 and (8) (apply Dini's Theorem on uniform convergence of a monotone sequence of continuous functions).  $\square$

The gap between (10) and (12) can be closed in a natural way by introducing the condition: there exists a finite set  $K$  satisfying

$$(13) \quad f^*(i, K)(f) = 1 \quad (f \in F, i \in I).$$

**Theorem 6.2.** *Condition (10a) is equivalent to any of the following conditions:*

(14) there exists a finite set  $K$  satisfying both (13) and, for all  $h \in K$ , (12a);

(15)  $\{i \rightarrow f_{hi}^*(f) \eta_i(f) / \mu_{ii}(f) | f \in F, h \in I\}$  is a tight family of probability measures on  $I$ .

*Proof.* Equivalence of (10a) and (15) is easy (use Assumption 1). (14) implies (15), since  $K$  intersects every class in  $D(f)$  for all  $f \in F$ . For the converse direction choose  $K$  such that for all  $f \in F$ ,  $h \in I$   $\sum_{i \in K} f_{hi}^*(f) \eta_i(f) / \mu_{ii}(f) > 0$ .  $\square$

Condition (13) implies that no class in  $D(f)$  may “drift to infinity” as a whole. Therefore, (13) is equivalent to  $f^*(i, K)(f) > 0$  ( $f \in F$ ,  $i \in I$ ). The real importance of (13), however, is that it allows us to consider the processes induced on the finite set  $K$ . Thus in Sect. 8–10 we can tackle the problem of finding equivalent conditions for the continuity of the  $\mu_{ii}$ ’s by looking at the (finite) Markov chains induced on  $K$ .

If there exists a finite set  $K$  satisfying (13), we will call the family of Markov chains  $\{p_{ij}(f) | f \in F\}$  quasi-finite with respect to  $K$ . It can be shown that quasi-finiteness follows from the finiteness of  $v$ , if  $F$  is compact and if the “product property” (Hordijk [9]) or “completeness” (Wijngaard [16]) can be assumed. For the remainder of the paper we assume quasi-finiteness.

*Assumption 3.* The family of Markov chains  $\{p_{ij}(f) | f \in F\}$  is quasi-finite with respect to a fixed finite set  $K \subset I$ .

The following corollary is immediate from the above discussion. It generalizes a remark in Hordijk [9], p. 84.

**Corollary 6.3.** *If  $v$  is l.s.c. and for all  $i \in K$  the family  $\{j \rightarrow f_{ij}^*(f) \cdot \eta_j(f) / \mu_{jj}(f) | f \in F\}$  of probability measures on  $I$  is tight, then  $\mu_{ii}$  is continuous for all  $i \in I$ . If  $F$  is compact, these conditions are also necessary for the continuity of the  $\mu_{ii}$ ’s.  $\square$*

## 7. Markov Chains

The condition in Corollary 6.3 is difficult to check. Therefore, we show in this section how the problem can be reduced to one of the underlying Markov chains. Generalizing a result by Schäl [13], we will give a sufficient criterion in terms of the transition probabilities  $p_{ij}(f)$ . This can be done since for Markov chains the stationary probabilities can be gained as a limit:  $\pi_{ij} = \lim_{n \rightarrow \infty} n^{-1} \sum_{m=1}^n p_{ij}^{(m)}$ . The criterion obtained is especially useful for applications in queueing control theory.

**Theorem 7.1.** *If there exist finite positive constants  $\delta$  and  $M$  such that*

$$(16) \quad \delta \leq \eta_i(f) \leq M \quad (i \in I, f \in F),$$

*then the continuity of  $\mu_{ii}$  ( $i \in I$ ) is equivalent to the continuity of  $m_{ii}$  ( $i \in I$ ).*

*Proof.* This follows from

$$\delta \cdot m_{ii} = \delta \cdot \sum_j e_{ij} \leq \sum_j e_{ij} \eta_j = \mu_{ii} \leq M \cdot \sum_j e_{ij} = M \cdot m_{ii}$$

(see Eq. (5)) by Lemma 5.1 and Lemma 5.3.  $\square$

Note that we do not need Assumptions 1 and 3 for the following result.

**Theorem 7.2.** Assume  $I = \mathbb{N}$ . Let  $v$  be l.s.c. and  $\bar{f} \in F$  a parameter satisfying

$$(17) \quad P_i^{\bar{f}}\{Y_1 \leq k\} \leq P_i^f\{Y_1 \leq k\} \quad (k, i \in I, f \in F)$$

$$(18) \quad i \rightarrow P_i^{\bar{f}}\{Y_1 \leq k\} \text{ is non-increasing } (k \in I)$$

$$(19) \quad v(\bar{f}) \text{ is finite, and the Markov chain } \{p_{ij}(\bar{f})\} \text{ is non-dissipative.}$$

Then  $\{p_{ij}(f)\}$  is non-dissipative for all  $f \in F$ , the family of Markov chains  $\{p_{ij}(f) | f \in F\}$  is quasi-finite,  $\{j \rightarrow \pi_{ij}(f) | f \in F\}$  is tight ( $i \in I$ ), and  $m_{ii}$  is continuous ( $i \in I$ ). If in addition (16) holds, then  $\mu_{ii}$  is also continuous ( $i \in I$ ).

*Proof.* (a)  $P_i^{\bar{f}}\{Y_n \leq k\}$  is non-increasing in  $i$  ( $n \in \mathbb{N}$ ,  $k \in I$ ). For, if  $n=1$ , this is (18). By induction,  $P_i^{\bar{f}}\{Y_{n+1} \leq k\} = \sum_j p_{ij}(\bar{f}) P_j^{\bar{f}}\{Y_n \leq k\}$  is also non-increasing in  $i$ .

This follows since (18) implies that  $\sum_j p_{ij}(\bar{f}) h(j)$  is non-increasing in  $i$  for all non-increasing functions  $h \geq 0$ .

(b)  $P_i^{\bar{f}}\{Y_n \leq k\} \leq P_i^f\{Y_n \leq k\}$  ( $n \in \mathbb{N}$ ,  $i, k \in I$ ,  $f \in F$ ). This can be proved by induction, using (17):

$$\begin{aligned} P_i^{\bar{f}}\{Y_{n+1} \leq k\} &= \sum_j p_{ij}(\bar{f}) P_j^{\bar{f}}\{Y_n \leq k\} \\ &\geq \sum_j p_{ij}(f) P_j^{\bar{f}}\{Y_n \leq k\} \\ &\geq \sum_j p_{ij}(\bar{f}) P_j^f\{Y_n \leq k\} \\ &= P_i^f\{Y_{n+1} \leq k\}. \end{aligned}$$

Here, the second inequality is a consequence of (17) and part (a).

(c)  $\sum_{j=1}^k \pi_{ij}(f) \geq \sum_{j=1}^k \pi_{ij}(\bar{f})$  ( $i, k \in I$ ,  $f \in F$ ). This is an immediate consequence of (b), since  $\pi_{ij}(f)$  can be gained as a limit from the quantities  $P_i^f\{Y_n \leq k\}$ .

Now, the first statement of the theorem follows from (19) and (c) by letting  $k$  tend to infinity. Moreover, we can conclude that the family  $\{j \rightarrow \pi_{ij}(f) | f \in F, i \in I\}$  is tight. By Theorem 6.2 (specialized to the Markov case) this implies the second and the third assertion. The rest follows from Corollary 6.3 and Theorem 7.1.  $\square$

The meaning of the conditions in Theorem 7.2 can be best understood if they are applied to a queuing model. Let  $Y_n$  denote the number of customers awaiting service at period  $n$ . Assume that the system can be controlled by certain (stationary) strategies  $f$ , e.g. by choosing different types of service, switching servers on or off, or by not allowing customers to enter the queue. Then (17) states that  $Y_1$  is stochastically greater under  $P_i^{\bar{f}}$  than under  $P_i^f$ , i.e.  $\bar{f}$  can be interpreted as the slowest kind of service. By (18), the event that at the next step not more than  $k$  customers are waiting for service is (under strategy  $\bar{f}$ ) the more unlikely the more customers are waiting at the current period. Finally, (19) says that with probability one we will come to a positive recurrent state: even for the slowest strategy the queue length must shrink to a limited size, no matter how big it is at the current period.



## 8. The Embedded Processes on $K$

As already mentioned, under Assumption 3 we can define embedded processes on the finite set  $K$  (cf. Çinlar [2], Lemma (2.7)). All quantities for these processes will be marked by a “^”. We have  $\hat{\eta}_i = \mu_{iK}$  and  $\hat{\mu}_{ii} = \mu_{ii}$  for  $i \in K$ . Since two states  $i, j \in K$  communicate with respect to the embedded process, if and only if they communicate with respect to the original one, we have  $\hat{v} = v$ . Thus formula (7) may be rewritten as

$$(20) \quad v = \sum_{i \in K} \mu_{iK} / \mu_{ii}.$$

Here, we define  $\infty/\infty = 0$ .

Applying (7) to the Markov chains on  $K$  we have (since  $\hat{D} = \hat{D}_+$  by the finiteness of  $K$ ):

$$v = \hat{v} = \hat{v}_+ = \sum_{i \in K} 1/\hat{m}_{ii} = \sum_{i \in K} 1/\sum_{j \in K} e_{ij}.$$

Hence we can use Theorem 5.2 (b) to infer:

$$(21) \quad v \text{ is upper semi-continuous on } F.$$

Moreover, a result by Schweitzer for finite Markov chains (cf. Schweitzer [14], Theorem 5) is valid for quasi-finite systems of Markov chains as well: continuity of  $v$  implies that for small changes of the parameter the different recurrent classes do not vary within  $K$ . Note that  $\hat{p}_{ij} = {}_K p_{ij}^*$  is continuous for  $i, j \in K$ , by Theorem 5.2 (c).

## 9. Equivalent Conditions Using $\mu_{iK}$

Formula (20) suggests that there is a relation between the continuity of  $\mu_{iK}$ ,  $\mu_{ii}$  ( $i \in K$ ), and  $v$ . In fact, we have the following central result.

**Theorem 9.1.** *The following two conditions are equivalent:*

- (22)  $\mu_{ii}$  is continuous ( $i \in I$ );
- (23)  $\forall f_0 \in F, i \in D(f_0), J \subset I$  such that  $J \cap D_i(f_0) \neq \emptyset$ :  
 $\mu_{iJ}$  is continuous and finite at  $f_0$ .

A sufficient condition for (22) and (23) is

- (24)  $v$  is l.s.c., and
- (24a)  $\mu_{iK}$  is continuous at  $f_0$  ( $f_0 \in F, i \in K \cap D(f_0)$ ),
- (24b)  $\mu_{jK}$  is bounded ( $j \in K$ ).
- (24a) is also necessary for (22) and (23).

*Proof.* (22)  $\Rightarrow$  (23): Let  $f_0 \in F, i \in D(f_0), J \subset I$ , and assume  $i \in J$ . By (1) we have  $\mu_{iJ} = \mu_{ii} - \sum_{j \in J, j \neq i} e_{ij} \mu_{jJ}$ . Hence  $\mu_{iJ}$  is continuous at  $f_0$  by Lemma 5.2 (b) and

Lemma 5.3. For  $i \notin J$ , we have (using (1))

$$\mu_{iJ} = \mu_{i, J \cup \{i\}} \cdot (1 + {}_J p_{ii}^*).$$

Thus the continuity of  $\mu_{iJ}$  at  $f_0$  follows from the continuity of  $\mu_{i, J \cup \{i\}}$ , which was proved in the first part, and from Theorem 5.2 (c).

(23)  $\Rightarrow$  (22) is obvious.

(24)  $\Rightarrow$  (22): First note that  $\mu_{iK}/\mu_{ii}$  is upper semi-continuous for all  $i \in K$  (use (24a) for  $i \in D(f_0)$  and (24b) and Lemma 5.3 for  $i \notin D(f_0)$ ). Since  $v$  is continuous by (21), this implies continuity of  $\mu_{iK}/\mu_{ii}$  ( $i \in K$ ), see (20). Again using (24a), we conclude that  $\mu_{ii}$  is continuous for all  $i \in K$ .

For arbitrary  $i \in I$  satisfying  $\mu_{ii}(f_0) < \infty$  (w.l.o.g. by Lemma 5.3), there exists a  $j \in K \cap D_i(f_0)$  (Assumption 3). By Theorem 5.2 (d)  $e_{ij}$  is continuous at  $f_0$ . Especially, for all  $f$  in a neighbourhood of  $f_0$  we have  $0 < e_{ij}(f) < \infty$ . Thus  $j \in D_i(f)$  and hence  $\mu_{ii} = e_{ij}\mu_{jj}$  on this neighbourhood (see formula (6)). This implies the continuity of  $\mu_{ii}$  at  $f_0$ .

(22)  $\Rightarrow$  (24a) can be proved by similar arguments, using (7), (21), Lemma 5.3, and (20).  $\square$

An example can be given to show that (24b) cannot be dispensed with. Also, (22) and (23) do not imply the continuity of  $\mu_{iK}$  for all  $i \in K$ .

## 10. A Liapunov Condition

The following theorem is valid without Assumption 3, which is implied by the conditions of the theorem.

**Theorem 10.1.** *Let  $F$  be locally compact and  $v$  l.s.c. Assume that there is a finite set  $K \subset I$  and a function  $y: I \rightarrow \mathbb{R}_+$  satisfying*

$$(25) \quad \sup \{ \eta_i(f) + \sum_{j \notin K} p_{ij}(f)y(j) \mid f \in F \} \leq y(i) \quad (i \in I)$$

$$(26) \quad f \rightarrow \sum_{j \notin K} p_{ij}(f)y(j) \quad \text{is continuous} \quad (i \in I)$$

$$(27) \quad \lim_{n \rightarrow \infty} \sum_{j \notin K} {}_K p_{ij}^{(n)}(f)y(j) = 0 \quad (i \in K, f \in F).$$

*Then the underlying family of Markov chains is quasi-finite with respect to  $K$ , and  $\mu_{ii}$  is continuous for all  $i \in I$ . Furthermore,*

$$y(i) \geq u^*(i) = \sup \{ \mu_{iK}(f) \mid f \in F \} \quad (i \in I)$$

*and  $u^*$  also satisfies (25)–(27).*

*Proof.* Relation (25) implies for all  $n \in \mathbb{N}$   $y(i) \geq \eta_i(f) + \sum_{m=1}^{n-1} \sum_{j \notin K} {}_K p_{ij}^{(m)}(f)\eta_j(f) + \sum_{j \notin K} {}_K p_{ij}^{(n)}(f)y(j)$ . By (3) this yields  $\mu_{iK}(f) \leq y(i) < \infty$ , hence  $f^*(i, K)(f) = 1$  and the family of Markov chains is quasi-finite. We will show

(28) for  $f_0 \in F$ ,  $i \in K \cap D(f_0)$  we have

(28a)  $f \rightarrow \sum_{j \in K} {}_K p_{ij}^{(n)}(f) \eta_j(f)$  is finite and continuous at  $f_0$  ( $n \in \mathbb{N}$ ),

(28b)  $\sum_{j \in K} {}_K p_{ij}^{(n)} \mu_{jK} \downarrow 0$  uniformly on a neighbourhood of  $f_0$ ;

this implies (24a) by formula (4), and hence the continuity of  $\mu_{ii}$  ( $i \in I$ ) by Theorem 9.1. We have

$$\begin{aligned} y(i) &\geq y_n(i, f) = \sum_{j \in K} {}_K p_{ij}^{(n)}(f) y(j) \\ &\geq \sum_{j \in K} {}_K p_{ij}^{(n)}(f) \sum_{h \in K} p_{jh}(f) y(h) \\ &= \sum_{h \in K} {}_K p_{ih}^{(n+1)}(f) y(h) = y_{n+1}(i, f). \end{aligned}$$

By (26), Lemma 5.1, and Theorem 5.2 (a), this implies the continuity of  $y_n(i, \cdot)$  ( $n \in \mathbb{N}$ ). Since  $y(i) \geq \mu_{iK}(f) \geq \eta_i(f)$ , (28a) follows (Lemma 5.1). By Dini's theorem, the convergence in (27) is uniform on compact subsets of  $F$ . This implies (28b).

It has already been shown that  $y(i) \geq u^*(i)$ . Therefore (27) as well as (26) (by Lemma 5.1) are valid for  $u^*$  substituted for  $y$ . Relation (25) holds with equality for  $u^*$ , as follows from the optimality equation in a suitable dynamic programming model; see Federgruen et al. [6], Theorem 3.  $\square$

The conditions (25)–(27) were first introduced by Hordijk [9], Theorem 5.1, where  $K$  contains only one element (which implies unchainedness). They were generalized to the present form (with  $y$  replaced by  $u^*$ ) in Federgruen et al. [6]. However, these authors demand (27) for all  $i \in I$ , and, what is more important, they assume unchainedness of the underlying Markov chains.

A condition similar to (28) was introduced in Wijngaard [15, 16]. It can be shown that (28) is also necessary for (24a), if  $F$  is locally compact and  $v(\cdot)$  is l.s.c.

In Kolonko [10] the conditions (25)–(27) are modified so that  $y$  may depend on  $f \in F$ . It is assumed that  $y_f(i)$  is continuous with respect to  $f$  for all  $i \in I$ . This generalization is also valid for the first part of our Theorem 10.1; in fact, the proof goes through without change.

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