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Affine Normability of Partial Sums of I.I.D. Random Vectors: A Characterization*

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Summary. Let $X, X_1, X_2, \ldots$ be i.i.d. d-dimensional random vectors with partial sums S_n. We identify the collection of random vectors X for which there exist non-singular linear operators T_n and vectors $v_n \in \mathbb{R}^d$ such that $\{\mathscr{L}(T_n(S_n - v_n)), n \geqq 1\}$ is tight and has only full weak subsequential limits. The proof is constructive, providing a specific sequence $\{T_n\}$. The random vector X is said to be in the generalized domain of attraction (GDOA) of a necessarily operator-stable law γ if there exist $\{T_n\}$ and $\{v_n\}$ such that $\mathscr{L}(T_n(S_n - v_n)) \to \gamma$. We characterize the GDOA of every operator-stable law, thereby extending previous results of Hahn and Klass; Hudson, Mason, and Veeh; and Jurek. The characterization assumes a particularly nice form in the case of a stable limit. When γ is symmetric stable, all marginals of X must be in the domain of attraction of a stable law. However, if γ is a non-symmetric stable law then X may be in the GDOA of γ even if no marginal is in the domain of attraction of any law.

§ 1. Introduction

Let $X, X_1, X_2, \ldots$ be i.i.d. random vectors with values in $\mathbb{R}^d$, $d > 1$. $\mathscr{L}(X)$ is assumed to be *full*, i.e. the support of $\mathscr{L}(X)$ is not contained in any $d-1$ dimensional hyperplane. Let $S_n = \sum_{j \leqq n} X_j$.

Classically, one is interested in the distribution of S_n. One way this distribution can be approximated is via a weak limit theorem of the form $\mathscr{L}(f_n(S_n - v_n)) \to \gamma$ provided f_n is invertible, $v_n \in \mathbb{R}^d$ and γ is full. Traditionally, this is accomplished

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only for X-distributions for which f_n can be chosen to be a scalar transformation $f_n(x)=x/a_n$ with $a_n>0$. In this case, X is said to be in the *domain of attraction* (DOA) of γ. Fullness of the limit is essential for conveying information about the joint distribution of the components of S_n. But scalar norming is needlessly restrictive. The partial sums from a strictly larger class of distributions can be approximated by allowing f_n to be linear. See Hahn and Klass (1980a, 1981a, b).

In principle, an approximation is possible if $\mathscr{L}(f_n(S_n-v_n))$ is close to some family of full laws whose distributions can be considered known and approximable. Of course, the elements of this family should be closed with respect to weak limits. A tight family of laws on $\mathbb{R}^d$ will be called *f-tight* if all of its weak subsequential limits are full.

We are naturally led to the following two basic problems:

Problem 1. When do there exist linear transformations T_n and centering vectors v_n such that the sequence $\{\mathscr{L}(T_n(S_n-v_n)), n\geqq 1\}$ is f-tight?

Problem 2. When do there exist linear transformations T_n and centerings v_n such that $\mathscr{L}(T_n(S_n-v_n))$ tends to a limit? Here the limit can be left unspecified or it can be specified.

Problem 1 will be treated first, in Sect. 2. Let $S^{d-1}=\{x\in\mathbb{R}^d\colon \|x\|=1\}$ be the unit sphere in $\mathbb{R}^d$ and let $X^s=X-\bar{X}$, with $\bar{X}$ an independent copy of X, denote the symmetrization of X. The X-distributions solving Problem 1 are characterized by the condition

$$\lim_{c\to\infty}\overline{\lim_{t\to\infty}}\sup_{\theta\in S^{d-1}}\frac{t^2P(|\langle X^s,\theta\rangle|>ct)}{E(\langle X^s,\theta\rangle^2\wedge t^2)}=0. \tag{1.1}$$

Our proof gives an explicit method of constructing the norming operators T_n. Because convergence to a single law is not required, the norming operators so constructed need not be sensitive to any such limit law. Therefore, they can be determined from the X-distribution itself in the following manner: Define the *canonical* 1*-dimensional norming constants*

$$a_n(\theta)=\sup\{a\geqq 0\colon nE(\langle X,\theta\rangle^2\wedge a^2)\geqq a^2\}. \tag{1.2}$$

They induce a *minimal preferred orthonormal basis* (minimal PONB) $\theta_{n1},\ldots,\theta_{nd}$ on $\mathbb{R}^d$ by

$$\begin{aligned} &a_n(\theta_{n1})=\inf\{a_n(\theta)\colon \theta\in S^{d-1}\}\\ &a_n(\theta_{nj})=\inf\{a_n(\theta)\colon \theta\in S^{d-1}, \langle\theta,\theta_{ni}\rangle=0 \text{ for } i=1,\ldots,j-1\}\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } j=2,\ \ldots,d. \end{aligned} \tag{1.3}$$

The existence of θ_{nj} follows from the continuity of $\theta\to a_n(\theta)$ for n sufficiently large (cf. Hahn-Klass (1980b) or (1981a)). Now T_n may be chosen to be

$$T_nx=\sum_{j=1}^{k}(\langle x,\theta_{nj}\rangle/a_n(\theta_{nj}))\theta_{nj}. \tag{1.4}$$

This is exactly the technique used in Hahn and Klass (1980a, 1980b) for operator-norming partial sums which converge to a spherically symmetric stable law.

The limit laws γ which can arise in Problem 2 are called *operator-stable laws*. M. Sharpe (1969) characterized them as the full laws for which there exists a nonsingular linear operator B on $\mathbb{R}^d$, and vectors $b(t)\in\mathbb{R}^d$ such that $\gamma^{*t} = t^B\gamma * \delta_{b(t)}$ where $t^B=\exp(B\ln t)$ and $t^B\gamma(E)=\gamma(t^{-B}E)$. The collection of random vectors X for which there exist norming operators T_n and centering vectors v_n such that

$$\mathscr{L}(T_n(S_n-v_n)) \underset{w}{\to} \gamma$$

is called the *generalized domain of attraction* (GDOA) *of* γ.

Unfortunately, the T_n in (1.4) are sometimes inappropriate for operator norming to a limit law γ which is not spherically symmetric stable. In fact, even if some T_n normalizes S_n^s, it may not normalize S_n-v_n (for any v_n) for convergence to any law. See Examples 4.1, 4.13, and 4.14.

Section 3 tackles the problem of selecting linear operators to achieve weak convergence of the partial sums to a limit law. Theorem 3.13 provides a characterization of GDOA(γ), for any operator-stable law γ. Three conditions on the projections $\langle X,\theta\rangle$ are required to hold uniformly in θ. Condition (I) is a tail condition, Condition (II) involves the behavior of the 1-dimensional norming constants $a_n(\theta)$ relative to the limit law and Condition (III) governs centerings by truncated first moments. Uniformity is essential. The proof is constructive, providing the norming linear operators.

Remark 3.19 shows that for stable limits Condition (I) can be replaced (in the presence of Condition (II)) by two more familiar looking conditions. Finally, Remark 3.25 discusses the independence of Conditions (I)–(III). Such independence is governed by the limit law γ.

Convergence to a symmetric stable law requires the most regularity from the 1-dimensional projections of X. In fact, if $X\in$GDOA of a symmetric stable law then $\langle X,\theta\rangle$ is in the DOA of a symmetric stable law for each θ. However, each operator-stable law γ which is not symmetric stable has a random vector X in its GDOA for which $\langle X,\theta\rangle$ is not in the DOA of any law for any $\theta\in S^{d-1}$. See Example 4.1. Consequently GDOA(γ) is substantially larger than DOA(γ) in a perhaps unexpected way.

§ 2. F-tightness of Affinely Normed Partial Sums

Let $X, X_1, X_2, \ldots$ be i.i.d. full random vectors on $\mathbb{R}^d$ and let $S_n=X_1+\ldots+X_n$. By 1-dimensional results, for each fixed $\theta\in S^{d-1}$,

(i) $$\{\langle S_n,\theta\rangle/a_n(\theta)\} \quad \text{is shift-}f\text{-tight}$$

iff

(ii) $$\lim_{c\to\infty}\limsup_{t\to\infty} t^2P(|\langle X,\theta\rangle|>ct)/E(\langle X,\theta\rangle^2\wedge t^2)=0.$$

(Here $a_n(\theta)$ is defined in (1.2).) Extending this result, Theorem 2.1 below will show that (i) holds uniformly in θ iff (ii) does also.

The main objective is to determine when there exist linear operators T_n and vectors v_n in $\mathbb{R}^d$ such that $\{\mathscr{L}(T_n(S_n-v_n)), n\geqq 1\}$ is f-tight. Passing to marginal distributions, $\{\mathscr{L}(T_n(S_n-v_n)), n\geqq 1\}$ is f-tight iff for every $\psi_n\in S^{d-1}$, $\{\mathscr{L}(\langle T_n(S_n-v_n), \psi_n\rangle), n\geqq 1\}$ is f-tight (in $\mathbb{R}^1$). Letting $\theta_n\in S^{d-1}$ satisfy $\psi_n = T_n^{*-1}\theta_n/\|T_n^{*-1}\theta_n\|$ and noting that $\langle T_n(S_n-v_n), \psi_n\rangle = \langle S_n-v_n, \theta_n\rangle/\|T_n^{*-1}\theta_n\|$, $\{\mathscr{L}(T_n(S_n-v_n)), n\geqq 1\}$ is f-tight iff for all $\theta_n\in S^{d-1}$, $\{\mathscr{L}(\langle S_n-v_n, \theta_n\rangle/\|T_n^{*-1}\theta_n\|), n\geqq 1\}$ is f-tight. In view of (i), this can occur only if

$$\begin{aligned} 0 &< \liminf_{n\to\infty} \inf_{\theta\in S^{d-1}} \|T_n^{*-1}\theta\|/a_n(\theta) \\ &\leqq \limsup_{n\to\infty} \sup_{\theta\in S^{d-1}} \|T_n^{*-1}\theta\|/a_n(\theta) < \infty. \end{aligned}$$

Observe that $\{\theta\|T_n^{*-1}\theta\|: \theta\in S^{d-1}\}$ describes an ellipsoid in $\mathbb{R}^d$. Hence $\{\theta a_n(\theta): \theta\in S^{d-1}\}$ must also be roughly ellipsoidal. There is a natural set of principal axes determined by $a_n(\theta)$; namely. $\theta_{n1} a_n(\theta_{n1}), \ldots, \theta_{nd} a_n(\theta_{nd})$ where the θ_{nj} are defined as in (1.3). This suggests using $A_n^{*-1}x = \sum_{j=1}^{d} \langle x, \theta_{nj}\rangle \theta_{nj} a_n(\theta_{nj})$ for $n\geqq 1$ as a sequence of linear operators to make $S_n - v_n$ f-tight (whenever f-tightness is possible). Indeed, this is the case. Less obvious is the fact that the uniformized version of (ii) is sufficient by itself to make $\{\theta a_n(\theta): \theta\in S^{d-1}\}$ ellipsoidal and hence insure that $\{\mathscr{L}(A_n(S_n-v_n)), n\geqq 1\}$ is f-tight for some $v_n\in\mathbb{R}^d$. $\Big($The vectors $v_n = \sum_{j=1}^{d} \theta_{nj} nE\langle X, \theta_{nj}\rangle I(|\langle X, \theta_{nj}\rangle| \leqq a_n(\theta_{nj}))$ will do.$\Big)$

Condition (ii) is not the only condition on the X-distribution which is equivalent to (i). Le Cam (1965) uses the condition

(iii) $$\lim_{c\to\infty} \limsup_{n\to\infty} nE\left\{\left(\frac{\langle X, \theta\rangle}{c a_n(\theta)}\right)^2 \wedge 1\right\} = 0$$

for tightness. In fact, (iii) gives f-tightness. It is also possible to use

(iv) $$\lim_{c\to\infty} \overline{\lim_{n}}\, nP(|\langle X, \theta\rangle| > c a_n(\theta)) = 0.$$

Uniformized versions of the these statements are equivalent. Before so proving, we make a simple reduction: If $\{\mathscr{L}(T_n S_n^s), n\geqq 1\}$ is f-tight there must exist $v_n\in\mathbb{R}^d$ such that $\{\mathscr{L}(T_n(S_n-v_n)), n\geqq 1\}$ is f-tight, and conversely (cf. Araujo and Giné (1980)). Hence it suffices to prove the result for X symmetric. Moreover, since $P(|\langle X^s, \theta\rangle| > 2t) \leqq 2P(|\langle X, \theta\rangle| > t)$ for $t>0$ and $P(|\langle X^s, \theta\rangle| > t) \geqq P(|\langle X, \theta\rangle| \geqq 2t)$ for $t \geqq |\mathrm{med}\langle X, \theta\rangle|$, the uniformized version of (ii) holds for X iff it holds for X^s. Therefore, (1.1) is an immediate consequence of the following theorem.

(2.1) **Theorem.** Let $X, X_1, X_2, \ldots$ be i.i.d. full symmetric random vectors on $\mathbb{R}^d$. Put $S_n = X_1 + \ldots + X_n$. Define $a_n(\theta)$ as in (1.2). The following are equivalent:

(A) $\lim_{c\to\infty} \limsup_{t\to\infty} \sup_{\theta\in S^{d-1}} \dfrac{t^2 P(|\langle X,\theta\rangle| > ct)}{E(\langle X,\theta\rangle^2 \wedge t^2)} = 0$;

(B) $\lim_{c\to\infty} \limsup_{n\to\infty} \sup_{\theta\in S^{d-1}} nE\left\{\left(\dfrac{\langle X,\theta\rangle}{c a_n(\theta)}\right)^2 \wedge 1\right\} = 0$;

(C) For any sequence $\theta_n \in S^{d-1}$, $\{\mathscr{L}(\langle S_n,\theta_n\rangle/a_n(\theta_n)), n\geqq 1\}$ is f-tight (in $\mathbb{R}^1$);

(D) If $A_n x = \sum_{j=1}^{d} (\langle x,\theta_{nj}\rangle/a_n(\theta_{nj}))\theta_{nj}$ with θ_{nj} defined as in (1.3), then $\{\mathscr{L}(A_n S_n), n\geqq 1\}$ is f-tight;

(E) There exist linear operators $\{T_n, n\geqq 1\}$ such that $\{\mathscr{L}(T_n S_n), n\geqq 1\}$ is f-tight.

Proof. *(A)⇒(B).* Assume (A). Let $g(\theta) = E(\langle X,\theta\rangle^2 \wedge 1)$. $g(\cdot)$ is a strictly positive, continuous function on S^{d-1}. Hence there exists $\theta_0 \in S^{d-1}$ at which $g(\theta_0)$ achieves its minimal value. Clearly, for $n\geqq 1/g(\theta_0)$, $a_n(\theta)\geqq 1$ for all $\theta\in S^{d-1}$. In particular this means that for all such n and θ,

$$\begin{aligned} n^{-1} &= E((\langle X,\theta\rangle^2/a_n^2(\theta)) \wedge 1) \\ &\geqq (1/a_n^2(\theta))\, g(\theta) \geqq g(\theta_0)/a_n^2(\theta) \end{aligned}$$

so that $a_n(\theta)\geqq \sqrt{n g(\theta_0)}$. As a result, $\inf_{\theta\in S^{d-1}} a_n(\theta)\to\infty$. Consequently, for any $\theta_n\in S^{d-1}$,

$$\begin{aligned} 0 &= \lim_{c\to\infty} \limsup_{n\to\infty} \sup_{\theta\in S^{d-1}} \frac{nP(|\langle X,\theta\rangle| > c a_n(\theta_n))}{nE\left(\dfrac{\langle X,\theta\rangle^2}{a_n^2(\theta_n)} \wedge 1\right)} \\ &\geqq \lim_{c\to\infty} \limsup_{n\to\infty} nP(|\langle X,\theta_n\rangle| > c a_n(\theta_n)). \end{aligned}$$

Invoking (A.3) of Lemma A.1 from the Appendix, (B) holds.

(B)⇒(C). Assume (B) and take any $\theta_n\in S^{d-1}$. Part (A.2) of Lemma A.1 of the Appendix implies that $\{\mathscr{L}(\langle S_n,\theta_n\rangle/a_n(\theta_n)), n\geqq 1\}$ is tight. To obtain non-constancy of the weak subsequential limits, we appeal to the Kolmogorov-Rogozin-Esseen inequality, using a bound obtained by Le Cam (1965):

$$P(|\langle S_{9n},\theta_n\rangle| > 2^{-1} a_n(\theta_n)) > 1 - 3^{-1}\sqrt{2\pi}.$$

Writing $\langle S_{9n},\theta_n\rangle$ as the sum of 9 (nine) i.i.d. random variables, each having law $\mathscr{L}(\langle S_n,\theta_n\rangle)$, it follows that

$$P(|\langle S_n,\theta_n\rangle| > (18)^{-1} a_n(\theta_n)) > 9^{-1}(1 - 3^{-1}\sqrt{2\pi}). \tag{2.2}$$

As a consequence of (2.2), no subsequential weak limit of $\langle S_n,\theta_n\rangle/a_n(\theta_n)$ is identically zero. Thus, by symmetry, each must be non-constant. Hence (B) ⇒(C).

(C)⇒(D). Tightness of $\{\mathscr{L}(A_n S_n), n\geqq 1\}$ follows immediately from the orthonormality of θ_{nj} and the fact that for each $1\leqq j\leqq d$, $\{\mathscr{L}(\langle A_n S_n,\theta_{nj}\rangle), n\geqq 1\}$ is clearly tight. Fullness of the subsequential weak limits remains.

We will proceed by constructing an auxiliary tight sequence $\{\mathscr{L}(A_n S_{nn}), n \geqq 1\}$. Fullness of subsequential limits of this sequence will imply fullness of all subsequential limits for $\{\mathscr{L}(A_n S_n), n \geqq 1\}$. It will be shown that if $\mathscr{L}(A_{n'} S_{n'n'}) \to \mathscr{L}(Z)$ then both

$$E\langle A_{n'} S_{n'n'}, \theta\rangle^2 \to E(\langle Z, \theta\rangle)^2$$

and

$$\liminf_{n\to\infty} \inf_{\theta \in S^{d-1}} E\langle A_n S_{nn}, \theta\rangle^2 > 0.$$

By condition (C), there exists $c_0 > 1$ such that

$$nP(|\langle X, \theta\rangle| > c_0 a_n(\theta)) < (2d)^{-1} \quad \text{for all } \theta \in S^{d-1}. \tag{2.3}$$

To see this, suppose (2.3) fails. Then there exist $\theta_n \in S^{d-1}$, $c_n \to \infty$, and an infinite (sub)set Q of positive integers such that

$$nP(|\langle X, \theta_n\rangle| > c_n a_n(\theta_n)) \geqq (2d)^{-1} \quad \text{for } n \in Q.$$

By symmetry together with a conditioning argument,

$$\begin{aligned} &2 \liminf_{n\to\infty} P(|\langle S_n, \theta_n\rangle| > c_n a_n(\theta_n)) \\ &\geqq \liminf_{n\to\infty} P(\max_{1\leqq j \leqq n} |\langle X_j, \theta_n\rangle| > c_n a_n(\theta_n)) > 0. \end{aligned}$$

Therefore $\{\mathscr{L}(\langle S_n, \theta_n\rangle / a_n(\theta_n)), n \in Q\}$ is *not* tight, which contradicts (C). Hence (2.3) must hold.

Let $X_{ni} = \sum_{j=1}^{d} \theta_{nj} \langle X_i, \theta_{nj}\rangle I(|\langle X_i, \theta_{nj}\rangle| \leqq c_0 a_n(\theta_{nj}))$ and $S_{nn} = \sum_{i=1}^{n} X_{ni}$. Conditional on $X_{n1}, \ldots, X_{nn}$, $S_n - S_{nn}$ is symmetric. Hence not only is $\{\mathscr{L}(A_n S_{nn}), n \geqq 1\}$ tight, but if all its subsequential weak limits are full, the same must be true of $\{\mathscr{L}(A_n S_n), n \geqq 1\}$. Suppose $\mathscr{L}(Z) = \lim_{n'\to\infty} \mathscr{L}(A_{n'} S_{n'n'})$. Since Z is symmetric, fullness will follow if $0 < E\langle Z, \theta\rangle^2 < \infty$ for all $\theta \in S^{d-1}$. Observe that

$$\begin{aligned} E\langle A_n S_{nn}, \theta\rangle^4 &\leqq d^3 \sum_{j=1}^{d} \langle \theta, \theta_{nj}\rangle^4 E\langle A_n S_{nn}, \theta_{nj}\rangle^4 \\ &\leqq d^3 \sum_{j=1}^{d} \left\{ \frac{nE\langle X_{n1}, \theta_{nj}\rangle^4}{a_n^4(\theta_{nj})} + 3\left(\frac{nE\langle X_{n1}, \theta_{nj}\rangle^2}{a_n^2(\theta_{nj})}\right)^2 \right\}. \end{aligned}$$

Since $\langle X_{n1}, \theta_{nj}\rangle^2 \leqq c_0^2(\langle X_{n1}, \theta_{nj}\rangle^2 \wedge a_n^2(\theta_{nj}))$,

$$\begin{aligned} E\langle A_n S_{nn}, \theta\rangle^4 &\leqq d^3 \sum_{j=1}^{d} \left(c_0^4 nE\left\{ \left(\frac{\langle X_{n1}, \theta_{nj}\rangle}{a_n(\theta_{nj})}\right)^2 \wedge 1 \right\} + 3\left(c_0^2 nE\left\{ \left(\frac{\langle X_{n1}, \theta_{nj}\rangle}{a_n(\theta_{nj})}\right)^2 \wedge 1 \right\}\right)^2 \right) \\ &\leqq 4d^4 c_0^4. \end{aligned}$$

By uniform integrability, for all $\theta \in S^{d-1}$,

$$E\langle Z, \theta\rangle^2 = \lim_{n'\to\infty} E\langle A_{n'} S_{n'n'}, \theta\rangle^2. \tag{2.4}$$

It now suffices to show that the second moments of $\langle A_n S_{nn}, \theta\rangle$ are uniformly bounded away from zero. We construct another operator B_n for which this is guaranteed and then relate A_n and B_n.

First define an ONB $\gamma_{n1}, \ldots, \gamma_{nd}$ by

$$E\langle S_{nn}, \gamma_{n1}\rangle^2 = \inf\{E\langle S_{nn}, \theta\rangle^2 \colon \theta \in S^{d-1}\}$$

$$E\langle S_{nn}, \gamma_{nj}\rangle^2 = \inf\{E\langle S_{nn}, \theta\rangle^2 \colon \theta \in S^{d-1}, \langle \theta, \gamma_{ni}\rangle = 0 \text{ for } 1 \leqq i \leqq j-1\},$$
$$\text{for } j = 2, \ldots, d.$$

It follows by Hahn-Klass (1980a), p. 271 that for $i \neq j$,

$$E\langle S_{nn}, \gamma_{ni}\rangle\langle S_{nn}, \gamma_{nj}\rangle = 0. \tag{2.5}$$

Now let B_n be the linear operators defined by

$$B_n \gamma_{nj} = \gamma_{nj} / \sqrt{2E\langle S_{nn}, \gamma_{nj}\rangle^2}. \tag{2.6}$$

Upon writing $S_{nn} = \sum_{j=1}^{d} \langle S_{nn}, \gamma_{nj}\rangle \gamma_{nj}$, (2.6) and (2.5) imply for all $\theta \in S^{d-1}$

$$E\langle B_n S_{nn}, \theta\rangle^2 = \tfrac{1}{2} \tag{2.7}$$

and

$$\|B_n^{-1}\theta\|^2 = 2E\langle S_{nn}, \theta\rangle^2. \tag{2.8}$$

If we can establish the existence of $K < \infty$ with

$$\|A_n^{*-1}\theta\| \leqq K \|B_n^{-1}\theta\| \qquad \text{for all } \theta \in S^{d-1}, \tag{2.9}$$

then

$$\begin{aligned}\inf_{\theta\in S^{d-1}} E\langle A_n S_{nn}, \theta\rangle^2 &= \inf_{\theta\in S^{d-1}} E\langle A_n S_{nn}, A_n^{*-1}\theta/\|A_n^{*-1}\theta\|\rangle^2 \\ &= \inf_{\theta\in S^{d-1}} E(\langle S_{nn}, \theta\rangle/\|A_n^{*-1}\theta\|)^2 \\ &= \inf_{\theta\in S^{d-1}} 2^{-1}\|B_n^{-1}\theta\|^2/\|A_n^{*-1}\theta\|^2 > 0.\end{aligned}$$

Consequently, the subsequential limits of $\mathscr{L}(A_n S_{nn})$ and hence of $\mathscr{L}(A_n S_n)$ will be full, thereby completing the proof of (C)⇒(D).

The following properties of B_n together with Lemma 2.15 below yield (2.9) with $K = (c_0\sqrt{2d})^{-1}(1 + c_0\sqrt{2d})^d$.

(2.10) *Properties of* B_n $(\theta \in S^{d-1})$

(i) $\|B_n^{-1}\theta\| \geqq a_n(\theta)$

(ii) $\|B_n^{-1}\theta\| \leqq c_0\sqrt{2d}\, \|A_n^{*-1}\theta\|$

(iii) $\|B_n^{-1}\theta\| \geqq \|A_n^{*-1}\theta_{nj}\|$ for all $\theta \Theta \theta_{n1}, \ldots, \theta_{nj-1}$.

Proof. (i) Let $F=\bigcap_{j=1}^{d}\{|\langle X,\theta_{nj}\rangle|\leqq c_0 a_n(\theta_{nj})\}$. Observe that $nP(F^c)<\frac{1}{2}$ by (2.3). Hence

$$\begin{aligned}a_n^2(\theta)&=nE(\langle X,\theta\rangle^2\wedge a_n^2(\theta))\\&\leqq n a_n^2(\theta)P(F^c)+nE\langle X_{n1},\theta\rangle^2\\&\leqq a_n^2(\theta)2^{-1}+E\langle S_{nn},\theta\rangle^2\\&=(a_n^2(\theta)+\|B_n^{-1}\theta\|^2)2^{-1}\quad\text{by (2.8).}\end{aligned}$$

Thus (i) holds.

(ii)
$$\begin{aligned}\|B_n^{-1}\theta_{nj}\|&=2nE\langle X_{n1},\theta_{nj}\rangle^2\\&\leqq 2nEc_0^2\{\langle X,\theta_{nj}\rangle^2\wedge a_n^2(\theta_{nj})\}\\&=2c_0^2a_n^2(\theta_{nj})\\&=2c_0^2\|A_n^{*-1}\theta_{nj}\|.\end{aligned}$$

Hence

$$\begin{aligned}\|B_n^{-1}\theta\|^2&=\left\|\sum_{j=1}^{d}\langle\theta,\theta_{nj}\rangle B_n^{-1}\theta_{nj}\right\|^2\\&\leqq d\sum_{j=1}^{d}\langle\theta,\theta_{nj}\rangle^2\|B_n^{-1}\theta_{nj}\|^2\\&\leqq 2c_0^2 d\sum_{j=1}^{d}\langle\theta,\theta_{nj}\rangle^2\|A_n^{*-1}\theta_{nj}\|^2\\&=2c_0^2d\|A_n^{*-1}\theta\|^2\end{aligned}$$

which yields (ii).

(iii) Using (i) and (1.3),

$$\|B_n^{-1}\gamma_{n1}\|\geqq a_n(\gamma_{n1})\geqq a_n(\theta_{n1})=\|A_n^{*-1}\theta_{n1}\|.$$

Moreover, for $2\leqq j\leqq d$ and $\theta\perp\theta_{n1},\ldots,\theta_{n\,j-1}$

$$\|B_n^{-1}\theta\|\geqq a_n(\theta)\geqq a_n(\theta_{nj})=\|A_n^{*-1}\theta_{nj}\|,$$

which is precisely (iii).

$(D)\Rightarrow(E)$. This is trivial.

$(E)\Rightarrow(A)$. Assume $\{\mathscr{L}(T_nS_n),n\geqq 1\}$ is f-tight. In particular, T_nS_n is stochastically bounded and hence so is $\max_{1\leqq j\leqq n}\|T_nX_j\|$. Thus

$$\begin{aligned}0&=\lim_{c\to\infty}\overline{\lim_{n\to\infty}}\sup_{\theta\in S^{d-1}}P(\max_{1\leqq j\leqq n}|\langle T_nX_j,\theta\rangle|>c)\\&=\lim_{c\to\infty}\overline{\lim_{n\to\infty}}\sup_{\theta\in S^{d-1}}[1-(1-P(|\langle T_nX_1,\theta\rangle|>c))^n].\end{aligned}$$

Consequently,

$$\lim_{c\to\infty}\overline{\lim_{n\to\infty}}\sup_{\theta\in S^{d-1}}nP(|\langle X,\theta\rangle|>c\|T_n^{*-1}\theta\|)=0. \tag{2.11}$$

The following fact allows $\|T_n^{*-1}\theta\|$ to be replaced by $a_{n/\varepsilon}(\theta)$ in (2.11).

(2.12) *Fact.* There exists $\varepsilon>0$ such that $\|T_n^{*-1}\theta\|\leqq a_{n/\varepsilon}(\theta)$ for all θ and n sufficiently large.

Proof. If not, there exist $\theta_n\in S^{d-1}$, $\varepsilon_n\searrow 0$ and $(n')\subseteq(n)$ such that

$$\|T_{n'}^{*-1}\theta_{n'}\|>a_{n'/\varepsilon_{n'}}(\theta_{n'}).$$

Take a subsequence $(n'')\subset(n')$ such that $\varphi_{n''}\equiv T_{n''}^{*-1}\theta_{n''}/\|T_{n''}^{*-1}\theta_{n''}\|\to\theta^*$ for some θ^* and $\mathscr{L}(T_{n''}S_{n''})\to\mathscr{L}(Z)$ for some full Z. Hence, considering $\langle T_{n''}S_{n''},\varphi_{n''}\rangle$,

$$\mathscr{L}(\langle S_{n''},\theta_{n''}\rangle/\|T_{n''}^{*-1}\theta_{n''}\|)\to\mathscr{L}(\langle Z,\theta^*\rangle).$$

Note that $a_{cn}\geqq\sqrt{c}\,a_n$ for $c\geqq 1$. Utilizing this fact together with Lemma A.1 of the Appendix,

$$\begin{aligned}
P(|\langle S_{n''},\theta_{n''}\rangle|>\varepsilon_{n''}^{1/4}\|T_{n''}^{*-1}\theta_{n''}\|)&\leqq P(|\langle S_{n''},\theta_{n''}\rangle|>\varepsilon_{n''}^{1/4}a_{n''/\varepsilon_{n''}}(\theta_{n''}))\\
&\leqq P(|\langle S_{n''},\theta_{n''}\rangle|>a_{n''(\varepsilon_{n''})^{-1/2}}(\theta_{n''}))\\
&\quad\text{since } \frac{a_n(\theta)}{\sqrt{n}} \text{ increases in } n\\
&\leqq n''E((\langle X,\theta_{n''}\rangle^2/a^2_{n''(\varepsilon_{n''})^{-1/2}}(\theta_{n''}))\wedge 1)\\
&\quad\text{by (A.2)}\\
&=\sqrt{\varepsilon_{n''}}\to 0.
\end{aligned}$$

Hence, $\langle S_{n''},\theta_{n''}\rangle/\|T_{n''}^{*-1}\theta_{n''}\|\xrightarrow{Pr}0$, which contradicts the assumption that $\langle Z,\theta^*\rangle$ is non-constant. □

Now to prove that (A) holds, suppose the contrary. Thus there exists $\lambda>0$, $t_n\to\infty$, $\theta_n\in S^{d-1}$ and $c_n\to\infty$ such that

$$\frac{t_n^2P(|\langle X,\theta_n\rangle|>c_nt_n)}{E(\langle X,\theta_n\rangle^2\wedge t_n^2)}>\lambda. \tag{2.13}$$

Let n' satisfy

$$a_{n'/\varepsilon}(\theta_n)\leqq t_n<a_{1+n'/\varepsilon}(\theta_n). \tag{2.14}$$

Now (2.14) together with (2.13) and the monotonicity of $E(Y^2t^{-2}\wedge 1)$ in t (for any Y) entail

$$\begin{aligned}
\lambda&<(1+n'/\varepsilon)P(|\langle X,\theta_n\rangle|>c_nt_n)\\
&\leqq(1+n'/\varepsilon)P(|\langle X,\theta_n\rangle|>c_na_{n'/\varepsilon}(\theta_n))\\
&\to 0\quad\text{as}\quad n'\to\infty\qquad\text{by (2.11) and Fact 2.12.}
\end{aligned}$$

This contradiction establishes (A). □

The proof of Theorem 2.1 is now complete, modulo the following lemma. The lemma is needed to establish (2.9).

(2.15) **Lemma.** Let $\tilde{A}$ and $\tilde{B}$ be linear operators on $\mathbb{R}^d$. Let $\theta_1, \ldots, \theta_d$ be an ONB for $\mathbb{R}^d$ and $0 < a_1 \leqq a_2 \leqq \ldots \leqq a_d$. Suppose

(i) $\|\tilde{A}\theta\|^2 = \sum_{j=1}^{d} \langle \theta, \theta_j \rangle^2 a_j^2$;

(ii) $\|\tilde{B}\theta\| \leqq c\sqrt{d}\,\|\tilde{A}\theta\|$ for all $\theta \in S^{d-1}$ and some $1 \leqq c < \infty$;

(iii) $\|\tilde{A}\theta_1\| \leqq \|\tilde{B}\theta\|$ for all $\theta \in S^{d-1}$;

(iv) $\|\tilde{A}\theta_j\| \leqq \|\tilde{B}\theta\|$ for all $\theta \in S^{d-1}$ with $\theta \perp \theta_1, \ldots, \theta_{j-1}$.

Then

$$\|\tilde{A}\theta\| \leqq (c\sqrt{d})^{-1}(1 + c\sqrt{d})^d \|\tilde{B}\theta\|.$$

Proof. There exist reals $0 \leqq b_1 \leqq \ldots \leqq b_d$ and an ONB $\gamma_1, \ldots, \gamma_d$ such that $\|\tilde{B}\theta\|^2 = \sum_{j=1}^{d} \langle \theta, \gamma_j \rangle^2 b_j^2$. The b_j's are the principal axes of the ellipse determined by $\|\tilde{B}\theta\|$. We first assert that these axes dominate the principle axes $a_1, \ldots, a_d$ determined by $\tilde{A}$, i.e.

$$a_1 \leqq b_1, \ldots, a_d \leqq b_d. \tag{2.16}$$

This is true if $j=1$ by (iii). Suppose it is true for $1 \leqq i \leqq j-1$. By a dimensionality argument, there exists a unit vector ϕ_j in the span of $\gamma_1, \ldots, \gamma_j$ which is perpendicular to $\theta_1, \ldots, \theta_{j-1}$. Since ϕ_j is in the span of $\gamma_1, \ldots, \gamma_j$, $b_j \geqq \|\tilde{B}\phi_j\|$. By property (iv), $\|\tilde{B}\phi_j\| \geqq a_j$. Hence $b_j \geqq a_j$ and (2.16) holds by induction.

Now let U be the unitary operator satisfying $U\theta_j = \gamma_j$.

$$\begin{aligned}\|\tilde{A}\theta\|^2 &\leqq \sum_{j=1}^{d} \langle \theta, \theta_j \rangle^2 b_j^2 \quad \text{(since } a_j \leqq b_j\text{)}\\ &= \sum_{j=1}^{d} \langle U\theta, \gamma_j \rangle^2 \|\tilde{B}\gamma_j\|^2\\ &= \|\tilde{B}U\theta\|^2.\end{aligned}$$

We need to upper-bound $\|\tilde{B}U\theta\|$ in terms of $\|\tilde{B}\theta\|$. Surprisingly, we first derive a reverse bound.

$$\begin{aligned}\|\tilde{B}\theta\|^2 &\leqq c^2 d\,\|\tilde{A}\theta\|^2\\ &\leqq c^2 d\,\|\tilde{B}U\theta\|^2.\end{aligned}$$

Hence

$$\|\tilde{B}U^{-2}\theta\| \leqq c\sqrt{d}\,\|\tilde{B}U^{-1}\theta\|$$

and so by iteration

$$\|\tilde{B}U^{-k}\theta\| \leqq (c^2 d)^{(k-1)/2} \|\tilde{B}U^{-1}\theta\|.$$

Let $p(x) = \sum_{k=0}^{d} \lambda_k x^k$ denote the (monic) characteristic polynomial of U^{-1}. Thus $\lambda_d = 1$ and $(-1)^{d-k}\lambda_k$ = sum of all products of collections of $d-k$ of the eigenvalues of U^{-1}. Since these eigenvalues all have modulus 1,

$$|\lambda_k| \leqq \binom{d}{k} \quad \text{and} \quad |\lambda_0|^{-1} = 1.$$

Using $I = -\lambda_0^{-1} \sum_{k=1}^{d} \lambda_k U^{-k}$,

$$\begin{aligned}\|\tilde{B}\theta\| &= \left\|\tilde{B}\left(-\lambda_0^{-1} \sum_{k=1}^{d} \lambda_k U^{-k}\theta\right)\right\| \\ &\leqq \sum_{k=1}^{d} |\lambda_0^{-1}\lambda_k| \, \|\tilde{B}U^{-k}\theta\| \\ &\leqq \sum_{k=1}^{d} \binom{d}{k} (c^2 d)^{(k-1)/2} \|\tilde{B}U^{-1}\theta\| \\ &\leqq (c\sqrt{d})^{-1}(1+c\sqrt{d})^d \|\tilde{B}U^{-1}\theta\|.\end{aligned}$$

Replacing θ by $U\theta$,

$$\|\tilde{B}U\theta\| \leqq (c\sqrt{d})^{-1}(1+c\sqrt{d})^d \|\tilde{B}\theta\|.$$

Thus for all $\theta \in S^{d-1}$,

$$\|\tilde{A}\theta\| \leqq (c\sqrt{d})^{-1}(1+c\sqrt{d})^d \|\tilde{B}\theta\|. \quad \square$$

§ 3. Characterization of GDOA(γ)

Our objective is to characterize the set of X-distributions for which there exist linear operators T_n and vectors $v_n \in \mathbb{R}^d$ such that $\mathscr{L}(T_n(S_n - v_n))$ converges weakly to a given operator-stable law γ. Of course, such X-distributions must satisfy (1.1). It may seem natural to first consider symmetric random vectors directly and then prove a desymmetrization lemma. This is not feasible because $\mathscr{L}(T_n S_n^s) \to \gamma^s$ does *not* imply the existence of $b_n \in \mathbb{R}^d$ such that $\mathscr{L}(T_n S_n + b_n) \to \gamma$ (see Example 4.14). Consequently, our approach will be to consider the partial sums S_n directly and identify the quantities that should be utilized for normalization. GDOA(γ) will then be identified in terms of the tail behavior relative to the pertinent quantities together with several regularity constraints.

Throughout we assume that $E\langle X, \theta\rangle = 0$ whenever $E|\langle X, \theta\rangle| < \infty$.[1] To identify the appropriate norming method, several equivalent formulations of the desired weak convergence are useful. We require the following notation:

(3.0) (1) $\rho(\mu, \nu)$ is the Prohorov distance between μ and ν.

(2) An infinitely divisible law η has Levy representation $\eta \sim [a, \Phi, \mu]$ where $a \in \mathbb{R}^d$, Φ is a covariance operator, and μ is a Levy measure (i.e. $\int (1 \wedge \|x\|^2)\, d\mu(x) < \infty$ and $\mu(\{0\}) = 0$).

(3) Let η_θ be defined for each Borel set A of $\mathbb{R}$ by $\eta_\theta(A) = \eta(\{x \in \mathbb{R}^d : \langle x, \theta\rangle \in A\})$. If $\eta \sim [a, \Phi, \mu]$, then the projections $\eta_\theta \sim [b_\theta, \Phi_\theta, \mu_\theta]$ are again infinitely divisible with

[1] With this assumption

$$\lim_{t\to\infty} \sup_{\theta \in S^{d-1}} (E\langle X, \theta\rangle I_{(|\langle X,\theta\rangle| \leqq t)})^2 / E\langle X, \theta\rangle^2 I_{(|\langle X,\theta\rangle| \leqq t)} = 0$$

where $S^{d-1} = \{x \in \mathbb{R}^d : \|x\| = 1\}$. Therefore truncated variances and truncated second moments of $\langle X, \theta\rangle$ are uniformly asymptotically equivalent.

$$b_\theta = \langle a, \theta\rangle + \int \langle u, \theta\rangle ((1+\langle u, \theta\rangle^2)^{-1} - (1+\|u\|^2)^{-1})\, \mu(du);$$

$$\Phi_\theta(t, s) = \Phi(t\theta, s\theta) \text{ for } s, t \in \mathbb{R}; \text{ and } \mu_\theta(B) = \mu\{x \in \mathbb{R}^d: \langle x, \theta\rangle \in B\backslash\{0\}\}.$$

(4) For any random vector X and $\delta > 0$, let $(X)_\delta$ denote the truncated random vector $(X)_\delta = X I_{(\|X\| \leqq \delta)}$.

(3.1) *Definition.* X satisfies

(i) the *uniform* $(d_{n,\theta}, \mu_{n,\theta})$*-tail condition* if for every $y > 0$,

$$\lim_{n\to\infty} \sup_{\theta \in S^{d-1}} |nP(\langle X, \theta\rangle \geqq d_{n,\theta}\, y) - \mu_{n,\theta}([y, \infty))| = 0;$$

(ii) the *uniform* $(d_{n,\theta}, \Phi_{n,\theta})$*-variance condition* if

$$\lim_{\delta \downarrow 0} \limsup_{n\to\infty} \sup_{\theta \in S^{d-1}} |n \operatorname{Var}(\langle X, \theta\rangle / d_{n,\theta})_\delta - \Phi_{n,\theta}(1, 1)| = 0;$$

(iii) the *uniform* $(d_{n,\theta}, \mu_{n,\theta}, v_n, b_{n,\theta})$*-centering condition* if

$$\lim_{n\to\infty} \sup_{\theta \in S^{d-1}} |nE(\langle X, \theta\rangle / d_{n,\theta})_1 - \langle v_n, \theta\rangle / d_{n,\theta} - a_1(\mu_{n,\theta}, b_{n,\theta})| = 0$$

where

$$a_1(\mu_{n,\theta}, b_{n,\theta}) = b_{n,\theta} + \int_{|u| \leqq 1} \frac{u^3}{1+u^2}\, d\mu_{n,\theta}(u) + \int_{|u|>1} \frac{u}{1+u^2}\, d\mu_{n,\theta}(u).$$

If X satisfies (i)–(iii), then we say X satisfies the *uniform* $(d_{n,\theta}, \mu_{n,\theta}, \Phi_{n,\theta}, v_n, b_{n,\theta})$*-central convergence criterion.*

(3.2) **Proposition.** Let γ be a full operator-stable law on $\mathbb{R}^d$ with Lévy representation $[a, \Phi, \mu]$. Assume $X, X_1, X_2, \ldots$ are i.i.d. with $S_n = \sum_{j \leqq n} X_j$. The following four conditions are equivalent to $X \in \text{GDOA}(\gamma)$: There exist linear operators T_n and vectors $v_n \in \mathbb{R}^d$ such that if $\xi_n(\theta) := T_n^{*-1}\theta / \|T_n^{*-1}\theta\|$, then

(A) $\lim_{n\to\infty} \mathscr{L}(T_n(S_n - v_n)) = \gamma$;

(B) $\lim_{n\to\infty} \sup_{\theta \in S^{d-1}} \rho(\mathscr{L}(\langle T_n(S_n - v_n), \theta\rangle), \gamma_\theta) = 0$;

(C) $\lim_{n\to\infty} \sup_{\theta \in S^{d-1}} \rho(\mathscr{L}(\langle S_n - v_n, \theta\rangle / \|T_n^{*-1}\theta\|), \gamma_{\xi_n(\theta)}) = 0$;

(D) X satisfies the uniform $(\|T_n^{*-1}\theta\|, \mu_{\xi_n(\theta)}, \Phi_{\xi_n(\theta)}, v_n, b_{\xi_n(\theta)})$-central convergence criterion, where the relevant quantities are defined in (3.0)(3).

Proof. (A)$\Leftrightarrow$(B). By the Cramer-Wald device, (A) is equivalent to

$$\lim_{n\to\infty} \rho(\mathscr{L}(\langle T_n(S_n - v_n), \theta\rangle), \gamma_\theta) = 0 \quad \text{for all } \theta \in S^{d-1}. \tag{3.3}$$

Lemma 1 of Hahn-Klass (1980b) states that (3.3) is equivalent to the seemingly stronger condition (B), by utilizing a Lemma of Rao.

(B)$\Leftrightarrow$(C). Since γ is full, T_n may be assumed to be invertible for all n. For (B)$\Rightarrow$(C) replace θ by $\xi_n(\theta) := T_n^{*-1}\theta / \|T_n^{*-1}\theta\|$ and isolate $\langle S_n - v_n, \theta\rangle$. For (C) $\Rightarrow$(B) replace θ by $T_n^*\theta / \|T_n^*\theta\|$.

(C)$\Leftrightarrow$(D). Due to a theorem of M. Sharpe (see Hudson (1980), Theorem 1), a full operator-stable law γ is absolutely continuous. Therefore each γ_θ is a continuous probability law on $\mathbb{R}$. The desired result follows immediately from considering subsequences $\theta_n \in S^{d-1}$, using the tightness of $\{\gamma_{\xi_n(\theta_n)}, n \geqq 1\}$ and the 1-dimensional central convergence criterion for triangular arrays. □

(3.4) *Remark.* In view of the equivalence of (B) and (3.3), one might expect (C) to be equivalent to the same condition with the uniformity in θ omitted. Example 4 of Hahn-Klass (1980a) shows this is not the case; the uniformity is essential. Similarly, the uniformity is essential in (D).

Notice that the conditions provided by (D) for $X \in \mathrm{GDOA}(\gamma)$ involve $\|T_n^{*-1}\theta\|$ and $\xi_n(\theta)$ which are not determined by X and γ in any obvious manner. To eliminate this dependence upon T_n, the role of $\|T_n^{*-1}\theta\|$ must be clarified.

Specialize momentarily to γ spherically symmetric, so that all $\gamma_{\xi_n(\theta)}$ are equal to the same law on $\mathbb{R}$ which is necessarily stable. Condition (D) then says that the 1-dimensional central convergence criterion must hold in each direction θ at a uniform rate. Moreover, $\|T_n^{*-1}\theta\|$ must behave like a 1-dimensional norming constant for $\langle X, \theta\rangle$. Our next lemma establishes that $\|T_n^{*-1}\theta\|$ generally behaves like a "1-dimensional norming constant for $\langle X, \theta\rangle$ tailored to $\gamma_{\xi_n(\theta)}$".

The quantity $a_n(\theta)$ defined in (1.2) is a canonical norming constant for $\langle X, \theta\rangle$ which disregards any specificity of limit law or limiting sequence. Define

(3.5) $$w_\theta := \sup\{w \geqq 0 \colon \Phi_\theta(1,1) + \int_{\mathbb{R}} (y^2 \wedge w^2)\, d\mu_\theta(y) \geqq w^2\}.$$

Since γ is full, either $\inf\{\Phi_\theta(1,1)\colon \theta \in S^{d-1}\} > 0$ or $\mu_\gamma(\mathbb{R}^d) = \infty$. Consequently, $\inf\{w_\theta \colon \theta \in S^{d-1}\} > 0$. The quantity $w_{\xi_n(\theta)}$ relates to the multiple by which $a_n(\theta)$ must be adjusted to be compatible with $\gamma_{\xi_n(\theta)}$, as indicated by the following lemma.

(3.6) **Lemma.** If $X \in \mathrm{GDOA}(\gamma)$ with norming linear operators T_n then

$$\lim_{n\to\infty} \sup_{\theta \in S^{d-1}} |a_n(\theta)/(w_{\xi_n(\theta)} \|T_n^{*-1}\theta\|) - 1| = 0.$$

Proof. By Proposition 3.2, (D) holds. The uniform $(\|T_n^{*-1}\theta\|, \mu_{\xi_n(\theta)})$-tail condition and uniform $(\|T_n^{*-1}\theta\|, \Phi_{\xi_n(\theta)})$-variance condition together with footnote (1) imply

(3.7) $$\lim_{n\to\infty} \sup_{\theta \in S^{d-1}} |nP(\langle X, \theta\rangle \geqq y \|T_n^{*-1}\theta\|) - \mu_{\xi_n(\theta)}([y, \infty))| = 0 \quad \text{for all} \quad y \in \mathbb{R}^+$$

and

(3.8) $$\lim_{\delta \downarrow 0} \limsup_{n\to\infty} \sup_{\theta \in S^{d-1}} |nE(\langle X, \theta\rangle^2 I_{(|\langle X, \theta\rangle| \leqq \delta \|T_n^{*-1}\theta\|)})/\|T_n^{*-1}\theta\|^2 - \Phi_{\xi_n(\theta)}(1,1)| = 0.$$

Let θ_n be an arbitrary sequence in S^{d-1}. Take $0 < \varepsilon < 1$ and define

$$U^\varepsilon = \{n \colon a_n(\theta_n)/\|T_n^{*-1}\theta_n\| > (1+\varepsilon)\, w_{\xi_n(\theta_n)}\}$$
$$U_\varepsilon = \{n \colon a_n(\theta)/\|T_n^{*-1}\theta_n\| < (1-\varepsilon)\, w_{\xi_n(\theta_n)}\}.$$

We will show that U^ε and U_ε each have only finitely many points. Suppose that U^ε has infinitely many points $\{n'\}$. There is a subsequence $(n'')\subset(n')$ for which $\xi_{n''}(\theta_{n''})$ converges to say $\xi^*\in S^{d-1}$. Then

$$\begin{aligned}
1&=n''E((\langle X,\theta_{n''}\rangle/a_{n''}(\theta_{n''}))^2\wedge 1)\\
&=\lim_{n''\to\infty} n''E((\langle X,\theta_{n''}\rangle/a_{n''}(\theta_{n''}))^2\wedge 1)\\
&=\lim_{n''\to\infty} n''\int_0^\infty ((y/a_{n''}(\theta_{n''}))^2\wedge 1)\,dP(|\langle X,\theta_{n''}\rangle|\leqq y)\\
&=\lim_{n''\to\infty} n''\int_0^\infty ((y\,\|T_{n''}^{*-1}\theta_{n''}\|/a_{n''}(\theta_{n''}))^2\wedge 1)\,dP(|\langle X,\theta_{n''}\rangle|\leqq y\,\|T_{n''}^{*-1}\theta_{n''}\|)\\
&\leqq\lim_{n''\to\infty} n''\int_0^\infty ((y/((1+\varepsilon)\,w_{\xi_{n''}(\theta_{n''})}))^2\wedge 1)\,dP(|\langle X,\theta_{n''}\rangle|\leqq y\,\|T_{n''}^{*-1}\theta_{n''}\|)\\
&=\lim_{\delta\downarrow 0}\lim_{n''\to\infty} n''\left[\int_0^\delta+\int_\delta^\infty\right]((y/((1+\varepsilon)\,w_{\xi_{n''}(\theta_{n''})}))^2\wedge 1)\,dP(|\langle X,\theta_{n''}\rangle|\leqq y\,\|T_{n''}^{*-1}\theta_{n''}\|)\\
&\leqq\lim_{\delta\downarrow 0}\limsup_{n''\to\infty}[(1+\varepsilon)\,w_{\xi_{n''}(\theta_{n''})}]^{-2}\,n''E((\langle X,\theta_{n''}\rangle/\|T_{n''}^{*-1}\theta_{n''}\|)_\delta)^2\\
&\quad+\int_{-\infty}^\infty ((y/((1+\varepsilon)\,w_{\xi^*}))^2\wedge 1)\,d\mu_{\xi^*}(y)
\end{aligned}$$

by Billingsley ((1968) Theorem 5.5), (3.7), and the fact that $w_{\xi^*}\neq 0$

$$=[(1+\varepsilon)\,w_{\xi^*}]^{-2}\,\Phi_{\xi^*}(1,1)+\int_{-\infty}^\infty ((y/((1+\varepsilon)\,w_{\xi^*}))^2\wedge 1)\,d\mu_{\xi^*}(y)\qquad\text{by (3.8)}$$

<1 by the definition of w_{ξ^*} (3.5).

This is impossible, which verifies that U^ε has only finitely many points. A similar argument works for U_ε. Consequently, we may conclude that

$$\lim_{n''\to\infty}|a_{n''}(\theta_{n''})/(w_{\xi_{n''}(\theta_{n''})}\,\|T_{n''}^{*-1}\theta_{n''}\|)-1|=0.$$

By the subsequence principle, the lemma follows. □

(3.9) **Proposition.** Let $X\in\mathrm{GDOA}(\gamma)$ with nonsingular norming linear operators $\{T_n, n\geqq 1\}$. Let $\gamma_{n1},\dots,\gamma_{nd}$ denote a basis of unit vectors for which

$$T_n^{*-1}\gamma_{nj}=\|T_n^{*-1}\gamma_{nj}\|\,e_j.$$

If linear operators R_n are defined by $R_n^{*-1}\gamma_{nj}=(a_n(\gamma_{nj})/w_{e_j})\,e_j$, then

$$\lim_{n\to\infty}\|T_nR_n^{-1}-I\|=0.$$

Consequently,

$$\lim_{n\to\infty}\mathscr{L}(R_n(S_n-v_n))=\lim_{n\to\infty}\mathscr{L}(T_nR_n^{-1}R_n(S_n-v_n))=\lim_{n\to\infty}\mathscr{L}(T_n(S_n-v_n))=\gamma.$$

Proof. By Lemma 3.6

$$\|T_n^{*-1}\gamma_{nj}\|\sim a_n(\gamma_{nj})/w_{\xi_n(\gamma_{nj})}=a_n(\gamma_{nj})/w_{e_j}=\|R_n^{*-1}\gamma_{nj}\|.$$

To complete the proof, apply the following general lemma with $A_n = R_n^{*-1}$ and $B_n = T_n^{*-1}$.

(3.10) **Lemma.** Let A_n and B_n be invertible linear operators on $\mathbb{R}^d$. Let $e_1, \ldots, e_d$ be the standard orthonormal basis and let $\gamma_{n1}, \ldots, \gamma_{nd}$ be d linearly independent unit vectors. Suppose for each $1 \leqq j \leqq d$ and $n \geqq 1$ that

(a) $$A_n \gamma_{nj}/\|A_n \gamma_{nj}\| = e_j = B_n \gamma_{nj}/\|B_n \gamma_{nj}\|$$

and

(b) $$\lim_{n\to\infty} \|A_n \gamma_{nj}\|/\|B_n \gamma_{nj}\| = 1.$$

Then

$$\lim_{n\to\infty} \|A_n B_n^{-1} - I\| = 0,$$

and

$$\lim_{n\to\infty} \|B_n^{*-1} A_n^* - I\| = 0.$$

Proof. Assume (a) and (b).

$$\begin{aligned}\lim_{n\to\infty} \|A_n B_n^{-1} - I\| &= \lim_{n\to\infty} \sup_{\theta\in S^{d-1}} \|(A_n B_n^{-1} - I)\theta\| \\ &\leqq \lim_{n\to\infty} \sum_{j=1}^{d} \|(A_n B_n^{-1} - I) e_j\| \\ &= \lim_{n\to\infty} \sum_{j=1}^{d} |\|A_n \gamma_{nj}\|/\|B_n \gamma_{nj}\| - 1| \quad \text{by (a)} \\ &= 0 \quad \text{by (b).}\end{aligned}$$

The final assertion follows from the facts that $\|(A_n B_n^{-1} - I)^*\| = \|A_n B_n^{-1} - I\|$ and that I is self-adjoint so $(A_n B_n^{-1} - I)^* = B_n^{*-1} A_n^* - I$. □

The fact that $\{R_n, n \geqq 1\}$ is a suitable sequence of norming linear operators suggests a characterization theorem which improves upon Proposition 3.2(D) by specifying the tail, variance and centering behavior of X through normalization via the quantities $a_n(\theta)$ and w_θ. To state the theorem, we first introduce the following terminology.

(3.11) *Definition.* Let $m_{n,\,l_n(\theta)}(\theta) := nE\langle X, \theta\rangle I_{(|\langle X,\theta\rangle| \leqq a_n(\theta)/w_{l_n(\theta)})}$ and $h_{n,\,l_n(\theta)}(\theta) := m_{n,\,l_n(\theta)}(\theta) - a_1(\mu_{l_n(\theta)}, b_{l_n(\theta)})\, a_n(\theta)/w_{l_n(\theta)}$ with $a_1(\cdot,\cdot)$ defined in Definition 3.1. The truncated means $m_{n,\,l_n(\theta)}$ are said to be vector-like with respect to orthonormal bases $\{\zeta_{n1}, \ldots, \zeta_{nd}\}$ if

(3.12) $$\lim_{n\to\infty} \sup_{\theta\in S^{d-1}} \left| h_{n,\,l_n(\theta)}(\theta) - \sum_{j=1}^{d} h_{n,\,l_n(\zeta_{nj})}(\zeta_{nj}) \langle \theta, \zeta_{nj}\rangle \right| / a_n(\theta) = 0.$$

(3.13) **Theorem.** Let $X_1, X_2, \ldots$ be i.i.d. random vectors on $\mathbb{R}^d$ with $S_n = X_1 + \ldots + X_n$. Assume $E\langle X, \theta\rangle = 0$ if $E|\langle X, \theta\rangle| < \infty$. Let $\gamma \sim [a, \Phi, \mu]$ be a full operator-stable law on $\mathbb{R}^d$. There exist affine transformations (T_n, v_n) such that $\mathscr{L}(T_n(S_n - v_n)) \to \gamma$ iff there exist bases of unit vectors $\{\gamma_{nj}, j = 1, \ldots, d\}_{n\geqq 1}$ such that with $R_n^{*-1} \gamma_{nj} := (a_n(\gamma_{nj})/w_{e_j})\, e_j$ and $g_n(\theta) := R_n^{*-1}\theta/\|R_n^{*-1}\theta\|$.

(I) X satisfies a uniform $(a_n(\theta)/w_{g_n(\theta)}, g_n(\theta))$-tail condition;

(II) $\lim_{n\to\infty} \sup_{\theta\in S^{d-1}} |a_n(\theta)/(w_{g_n(\theta)} \|R_n^{*-1}\theta\|)-1|=0;$

and

(III) the truncated means

$$m_n(\theta):=m_{n,\,g_n(\theta)}(\theta)$$

are vector-like with respect to the orthonormal basis $\{\zeta_{n1}, \ldots, \zeta_{nd}\}$ determined by the polar decomposition of R_n^{*-1} in the form

$$R_n^{*-1}x=\sum_{j=1}^{d} \langle x, \zeta_{nj}\rangle \|R_n^{*-1}\zeta_{nj}\| \psi_{nj},$$

where $\{\psi_{n1}, \ldots, \psi_{nd}\}$ is also an ONB.

Note. This theorem remains valid if n is restricted to an infinite subset of the positive integers.

Proof (Necessity). Assume $X\in\text{GDOA}(\gamma)$. By Proposition 3.9, there exists a basis $\{\gamma_{n1}, \ldots, \gamma_{nd}\}$ and R_n defined through it (as above) such that $\mathscr{L}(R_n(S_n-v_n))\to\gamma$. Replacing T_n by R_n in Proposition 3.2 and Lemma 3.6 yields (I) and (II) respectively. Moreover, Proposition 3.2(D) says that

$$\lim_{n\to\infty} \sup_{\theta\in S^{d-1}} |(m_n(\theta)-\langle v_n, \theta\rangle)/(a_n(\theta)/w_{g_n(\theta)})-a_1(\mu_{g_n(\theta)}, b_{g_n(\theta)})|=0. \tag{3.14}$$

Let $h_n(\theta):=h_{n,\,g_n(\theta)}(\theta)$. Since $w_{g_n(\theta)}$ is bounded away from 0 and ∞, the linearity of $\langle v_n, \cdot\rangle$ permits the equivalence of (3.14) and (III). To see this note that (3.14) can be expressed as

$$h_n(\theta)=\sum_{j=1}^{d} h_n(\zeta_{nj})\langle\zeta_{nj}, \theta\rangle+o(a_n(\theta))+\sum_{j=1}^{d} \langle\zeta_{nj}, \theta\rangle\, o(a_n(\zeta_{nj}))$$

uniformly in θ. Now since $\|R_n^{*-1}\theta\|^2=\sum_{j=1}^{d} \langle\theta, \zeta_{nj}\rangle^2 \|R_n^{*-1}\zeta_{nj}\|^2$, by Cauchy-Schwarz we have

$$\begin{aligned}\left|\sum_{j=1}^{d} \langle\zeta_{nj}, \theta\rangle\, o(a_n(\zeta_{nj}))\right| &= o\left(\sqrt{\sum_{j=1}^{d} \langle\zeta_{nj}, \theta\rangle^2 a_n^2(\zeta_{nj})}\right)\\ &=o(\|R_n^{*-1}\theta\|)\\ &=o(a_n(\theta)).\end{aligned}$$

Hence, (III) holds if (3.14) holds (and conversely, by trivial argument). Thus the proof of necessity is complete. □

Before giving the proof of sufficiency we isolate a lemma which allows the uniform tail condition to be taken uniformly over y in a compact subset of $(0, \infty)$ as well as uniformly in θ.

(3.15) **Lemma.** If condition (I) holds, then for any $\varepsilon>0$,

$$\lim_{n\to\infty} \sup_{\varepsilon<y\leqq 1/\varepsilon} \sup_{\theta\in S^{d-1}} |nP(\langle X, \theta\rangle\geqq y a_n(\theta)/w_{g_n(\theta)})-\mu_{g_n(\theta)}([y, \infty))|=0. \tag{3.16}$$

Proof. Take $0<\varepsilon<1$. Let $y_n\in[\varepsilon, 1/\varepsilon]$ and $\theta_n\in S^{d-1}$ be arbitrary sequences. Take any $(n')\subset(n)$. By compactness there exist θ^* and g^* in S^{d-1}, $y^*\in[\varepsilon, 1/\varepsilon]$, and a subsequence $(n'')\subset(n')$ such that $\theta_{n''}\to\theta^*$, $g_{n''}(\theta_{n''})\to g^*$ and $y_{n''}\to y^*$. Let $0<\delta<\varepsilon$. Then

$$\begin{aligned}\overline{\lim_{n''\to\infty}}\, n'' P(\langle X, \theta_{n''}\rangle > y_{n''}\, a_{n''}(\theta_{n''})/w_{g_{n''}(\theta_{n''})})\\ \leqq \overline{\lim_{n''\to\infty}}\, n'' P(\langle X, \theta_{n''}\rangle > (y^*-\delta)\, a_{n''}(\theta_{n''})/w_{g_{n''}(\theta_{n''})})\\ = \mu_{g^*}([y^*-\delta, \infty)) \quad \text{by (I)}.\end{aligned}$$

Similarly,

$$\underline{\lim_{n''\to\infty}}\, n'' P(\langle X, \theta_{n''}\rangle > y_{n''}\, a_{n''}(\theta_{n''})/w_{g_{n''}(\theta_{n''})}) \geqq \mu_{g^*}([y^*+\delta, \infty)).$$

Letting $\delta\to 0$ and using the facts that $\eta\to\mu_\eta$ is continuous and each μ_η is a continuous σ-finite measure (Lemma A.4 in the Appendix) yields the result. □

Proof (Sufficiency). Assume the existence of bases $\{\gamma_{nj},\ j=1, \dots, d\}$ of unit vectors which satisfy (I)–(III). We show that $\mathscr{L}(R_n(S_n - v_n))\to\gamma$ by verifying (D) of Proposition 3.2 with $T_n = R_n$. (I) and (II) imply the required tail condition while (III), which is equivalent to (3.14), and (II) imply the required centering condition. It only remains to verify that X satisfies the uniform $(\|R_n^{*-1}\theta\|, \Phi_{g_n(\theta)})$-variance condition.

Let

$$\begin{aligned} A_{n,\theta,\delta} &:= nE\langle X, \theta\rangle^2 I_{(|\langle X,\theta\rangle|\leqq\delta a_n(\theta)/w_{g_n(\theta)})}/(a_n^2(\theta)/w_{g_n(\theta)}^2)\\ B_{n,\theta,\delta} &:= nE(\langle X, \theta\rangle^2\wedge a_n^2(\theta))\, I_{(|\langle X,\theta\rangle|\geqq\delta a_n(\theta)/w_{g_n(\theta)})}/(a_n^2(\theta)/w_{g_n(\theta)}^2)\\ C_{n,\theta,\delta} &:= E\langle X, \theta\rangle\, I_{(|\langle X,\theta\rangle|\leqq\delta a_n(\theta)/w_{g_n(\theta)})}/(a_n(\theta)/w_{g_n(\theta)})\\ D_{n,\theta} &:= \int_{-\infty}^{\infty} (y^2\wedge w_{g_n(\theta)}^2)\, d\mu_{g_n(\theta)}(y).\end{aligned}$$

The quantity $\delta_0 := \inf_{\eta\in S^{d-1}} w_\eta$ is strictly positive. Thus, for $\delta\leqq\delta_0\wedge 1$, the definitions of $w_{g_n(\theta)}$, δ_0 and $a_n(\theta)$ imply that

$$\begin{aligned}\Phi_{g_n(\theta)}(1,1) &= w_{g_n(\theta)}^2 - \int (y^2\wedge w_{g_n(\theta)}^2)\, d\mu_{g_n(\theta)}(y)\\ &= w_{g_n(\theta)}^2\, \frac{nE(\langle X,\theta\rangle^2\wedge a_n^2(\theta))}{a_n^2(\theta)} - \int (y^2\wedge w_{g_n(\theta)}^2)\, d\mu_{g_n(\theta)}(y)\\ &= B_{n,\theta,\delta} + A_{n,\theta,\delta} - D_{n,\theta}.\end{aligned}$$

Denote by $o_u(1)$ a quantity which goes to 0 uniformly in $\theta\in S^{d-1}$. Utilizing Lemma 3.15,

$$\begin{aligned} B_{n,\theta,\delta} &= n\delta^2 P(|\langle X,\theta\rangle| > \delta a_n(\theta)/w_{g_n(\theta)})\\ &\quad + \frac{n}{a_n^2(\theta)/w_{g_n(\theta)}^2} \int_{\delta a_n(\theta)/w_{g_n(\theta)}}^{a_n(\theta)} 2y P(|\langle X,\theta\rangle| > y)\, dy\\ &= o_u(1) + \delta^2 \mu_{g_n(\theta)}([\delta,\infty)) + n\int_{\delta}^{w_{g_n(\theta)}} 2y P(|\langle X,\theta\rangle| > y a_n(\theta)/w_{g_n(\theta)})\, dy\\ &= o_u(1) + \delta^2 \mu_{g_n(\theta)}([\delta,\infty)) + 2\int_{\delta}^{w_{g_n(\theta)}} y\mu_{g_n(\theta)}([y,\infty))\, dy\\ &= o_u(1) + \int (y^2\wedge w_{g_n(\theta)}^2)\, I_{(|y|>\delta)}\, d\mu_{g_n(\theta)}(y).\end{aligned}$$

Consequently,

$$|B_{n,\theta,\delta}-D_{n,\theta}|\leqq|o_u(1)|+\int y^2 I_{(|y|\leqq\delta)}\,d\mu_{g_n(\theta)}(y)$$
$$\leqq|o_u(1)|+\int_{\mathbb{R}^d}(\|x\|^2\wedge\delta^2)\,d\mu(x).$$

Therefore

$$\lim_{\delta\downarrow 0}\overline{\lim_{n\to\infty}}\sup_{\theta\in S^{d-1}}|A_{n,\theta,\delta}-\Phi_{g_n(\theta)}(1,1)|\leqq\lim_{\delta\downarrow 0}\int_{\mathbb{R}^d}(\|x\|^2\wedge\delta^2)\,d\mu(x)=0 \tag{3.17}$$

by the dominated convergence theorem and the fact that μ is a Lévy measure.

This is precisely the uniform $(\|R_n^{*-1}\theta\|, \Phi_{g_n(\theta)})$-variance condition of Proposition 3.2(D) since either $E\langle X,\theta\rangle=0$ or $E|\langle X,\theta\rangle|^2=\infty$ so $nC^2_{n,\theta,\delta}=o_u(A_{n,\theta,\delta})$ (see footnote 1). Therefore, all the hypotheses of Proposition 3.2(D) are satisfied, so $\mathscr{L}(R_n(S_n-v_n))\to\gamma$. $\square$

(3.18) *Remark* (on construction of the bases $\{\gamma_{n1},\ldots,\gamma_{nd}\}$). The above proof shows that $\{R_n, n\geqq 1\}$ is a suitable sequence of norming operators. The bases on which R_n depends can be constructed from 1-dimensional information as follows: Let $\mathscr{B}$ denote the collection of all bases $\lambda=\{\lambda_1,\ldots,\lambda_d\}$ for $\mathbb{R}^d$. For each basis $\lambda\in\mathscr{B}$ define

$$R^{*-1}_{n,\lambda}(\lambda_j)=(a_n(\lambda_j)/w_{e_j})\,e_j \quad\text{and}\quad g_{n,\lambda}(\theta)=R^{*-1}_{n,\lambda}\theta/\|R^{*-1}_{n,\lambda}\theta\|.$$

Let

$$\begin{aligned}\varepsilon_n(\lambda,v)\equiv&\sup_{\theta\in S^{d-1}}|a_n(\theta)/(w_{g_{n,\lambda}(\theta)}\|R^{*-1}_{n,\lambda}\theta\|)-1|\\&+\int_0^\infty\sup_{\theta\in S^{d-1}}\{|nP(\langle X,\theta\rangle\geqq y a_n(\theta)/w_{g_{n,\lambda}(\theta)})\\&\qquad-\mu_{g_{n,\lambda}(\theta)}([y,\infty))|\wedge 1\}\,e^{-y}\,dy\\&+\sup_{\theta\in S^{d-1}}|(m_{n,g_{n,\lambda}(\theta)}(\theta)-\langle v,\theta\rangle)/(a_n(\theta)/w_{g_{n,\lambda}(\theta)})\\&\qquad-a_1(\mu_{g_{n,\lambda}(\theta)},b_{g_{n,\lambda}(\theta)})|.\end{aligned}$$

Then we claim that $X\in\mathrm{GDOA}(\gamma)$ iff

$$\lim_{n\to\infty}\inf_{\lambda\in\mathscr{B}}\inf_{v\in\mathbb{R}^n}\varepsilon_n(\lambda,v)=0, \tag{IV}$$

in which case a suitable basis is obtained by choosing any $\gamma_n^*=\{\gamma_{n1}^*,\ldots,\gamma_{nd}^*\}\in\mathscr{B}$ and $v_n^*\in\mathbb{R}^d$ such that $\varepsilon_n(\gamma_n^*,v_n^*)\to 0$.

Indeed, (I)–(III) and (3.14) clearly imply (IV). Moreover, if (IV) is satisfied, then (II) and (3.14), hence (III) are immediate. For (I), rewrite the integral in the definition of $\varepsilon_n(\lambda,v)$ as

$$\int_0^\infty h_n(y)\,e^{-y}\,dy.$$

Condition (IV) says that $h_n(y)\to 0$ in measure (relative to $e^{-y}\,dy$). If (n') is any subsequence of (n) then there exists a subsubsequence $(n'')\subset(n')$ with $h_{n''}(y)\to 0$ a.e. (relative to $e^{-y}\,dy$), and so a.e. with respect to Lebesgue measure. Since in actuality it suffices to verify condition (I) for all y in a dense subset of $\mathbb{R}^+$, we have convergence to γ along this subsequence. By the subsequence principle the entire sequence converges.

(3.19) *Remark* (on stable limits). In the case of a stable non-normal limit, condition (I) can be replaced by two more familiar looking conditions. Recall that an infinitely divisible law γ is stable of index α, $0<\alpha<2$, iff all its projections are stable of index α. Thus, $\gamma \sim [a, 0, \mu]$ iff $\gamma_\theta \sim [a_\theta, 0, \mu_\theta]$, $\forall \theta \in S^{d-1}$. The form of μ_θ is

(3.20) $$\mu_\theta([y, \infty)) = \frac{2-\alpha}{2} c_1(\theta) y^{-\alpha} \quad \text{for } y>0$$

and

$$\mu_\theta((-\infty, -y]) = \frac{2-\alpha}{2} c_1(-\theta) y^{-\alpha} \quad \text{for } y>0.$$

Let

(3.21) $$c(\theta) = c_1(\theta) + c_1(-\theta).$$

Then $c(\theta)$ is related to w_θ as follows:

$$\begin{aligned} w_\theta^2 &= \int (y^2 \wedge w_\theta^2)\, d\mu_\theta(y) \\ &= \int_0^{w_\theta} 2y \mu_\theta(x: |x| \geqq y)\, dy \\ &= (2-\alpha) c(\theta) \int_0^{w_\theta} y^{1-\alpha}\, dy \\ &= c(\theta) w_\theta^{2-\alpha}. \end{aligned}$$

Since w_θ and $c(\theta)$ *are positive*,

(3.22) $$w_\theta = c^{1/\alpha}(\theta).$$

In the presence of (II), (I) can be replaced by

(I′) $$\lim_{t\to\infty} \sup_{\theta \in S^{d-1}} \left| \frac{t^2 P(|\langle X, \theta\rangle| > t)}{E(\langle X, \theta\rangle^2 \wedge t^2)} - \frac{2-\alpha}{2} \right| = 0,$$

(I″) $$\lim_{n\to\infty} \sup_{\theta \in S^{d-1}} \left| \frac{P(\langle X, \theta\rangle > y a_n(\theta))}{P(|\langle X, \theta\rangle| > y a_n(\theta))} - \frac{c_1(g_n(\theta))}{c(g_n(\theta))} \right| = 0.$$

(For justification, see below.)

If γ is symmetric stable the conditions (I′) and (II′) obviously can be replaced by the single condition

(I‴) $$\lim_{t\to\infty} \sup_{\theta \in S^{d-1}} \left| \frac{t^2 P(\langle X, \theta\rangle > t)}{E(\langle X, \theta\rangle^2 \wedge t^2)} - \frac{2-\alpha}{4} \right| = 0.$$

Moreover, $h_n(\theta)$ in (III′) is asymptotic to $\tilde{m}_n(\theta) = nE\langle X, \theta\rangle I_{(|\langle X, \theta\rangle| \leqq a_n(\theta))}$. (For γ spherically symmetric, this was shown in Hahn-Klass (1980b).)

If γ is the standard multivariate normal, condition (I′) is necessary and sufficient for $X \in \mathrm{GDOA}(\gamma)$ (cf. Hahn-Klass (1980a)).

For any spherically symmetric limit γ (γ is necessarily stable), $\{\gamma_{n1}, \ldots, \gamma_{nd}\}$ may be chosen to be the minimal PONB constructed in (1.3). In particular, this means that R_n assumes the form given in (1.4). Thus, for a spherically symmetric limit law, the same linear operators constructed to solve Problem 1 in Sect. 2 actually give convergence (c.f. Hahn-Klass (1980b), (1981a)).

Justification that (I) can be replaced by (I') and (I'') if $\gamma=\mathscr{L}(Z)$ is stable of index α, $0<\alpha<2$.

First assume (I)–(III). Then $\mathscr{L}(R_n(S_n-v_n))\to\gamma\equiv\mathscr{L}(Z)$ and consequently, $\mathscr{L}(R_n S_n^s)\to\gamma^s\equiv\mathscr{L}(Z^s)$ where $Z^s=Z-Z'$ with Z and Z' i.i.d. Since Z^s is symmetric stable, the form of the Levy measure implies that $\mathscr{L}(\langle Z^s,\theta\rangle) =\mathscr{L}(c^{1/\alpha}(\theta)\langle Z,e_1\rangle/c^{1/\alpha}(e_1))$. Thus, (B) of Proposition 3.2 and properties of the Prohorov distance (Fact 4.4 of Hahn-Klass (1981a)) imply that

$$\text{(3.23)}\quad \lim_{n\to\infty}\sup_{\theta\in S^{d-1}}\rho(\mathscr{L}(\langle S_n^s,\theta\rangle/(c^{1/\alpha}(g_n(\theta))\,\|R_n^{*-1}\theta\|)),\ \mathscr{L}(\langle Z^s,e_1\rangle/c^{1/\alpha}(e_1)))=0.$$

As in Hahn-Klass ((1980b), pp. 64–70), this leads to

$$\lim_{t\to\infty}\sup_{\theta\in S^{d-1}}\left|\frac{t^2P(\langle X^s,\theta\rangle>t)}{E(\langle X^s,\theta\rangle^2\wedge t^2)}-\frac{2-\alpha}{4}\right|=0.$$

Condition (I') now follows immediately from the observation that uniformly in θ

$$P(\langle X^s,\theta\rangle>t)\sim P(\langle X,\theta\rangle>t)+P(\langle -X,\theta\rangle>t)=P(|\langle X,\theta\rangle|>t).$$

To obtain condition (I''), apply condition (I) with y replaced by $yc^{1/\alpha}(g_n(\theta))$ (which is legitimate by Lemma 3.15) to obtain

$$\text{(3.24)}\qquad \lim_{n\to\infty}\sup_{\theta\in S^{d-1}}\left|nP(\langle X,\theta\rangle>ya_n(\theta))-y^{-\alpha}\frac{2-\alpha}{2}\frac{c_1(g_n(\theta))}{c(g_n(\theta))}\right|=0.$$

Now replacing $\langle X,\theta\rangle$ by $|\langle X,\theta\rangle|$ in (3.24) replaces $c_1(\cdot)$ by $c(\cdot)$. Since $c(\theta)$ is bounded away from 0 and ∞, (I') together with (3.24) imply (I'').

For the converse, assume (I') and (I''). On page 72 of Hahn-Klass (1980b), it is shown that (I') implies

$$\lim_{n\to\infty}\sup_{\theta\in S^{d-1}}\left|nP(|\langle X,\theta\rangle|>ya_n(\theta))-\frac{2-\alpha}{2}y^{-\alpha}\right|=0.$$

Lemma 3.15 together with (I'') yield (I) as desired. □

(3.25) *Remark.* The independence of conditions (I)–(III) is governed by the limit law γ. If γ is the standard multivariate normal, condition (I) implies both (II) and (III) (cf. Hahn-Klass (1980a)). If γ is spherically symmetric stable of index $\alpha\neq 1$, (I) and (II) imply (III). This implication fails when $\alpha=1$ (Theorem 4.6 of Hahn-Hahn-Klass (1983)). More generally, any X satisfying (I) and (II) will automatically satisfy (III) if γ has the property $PTP(\mathscr{S})$ defined in Hahn-Hahn-Klass (1983).

(3.26) *Definition.* γ has $PTP(\mathscr{S})$ if whenever there exists an infinitely divisible law η and a function $r: S^{d-1}\to\mathbb{R}$ with

$$\gamma_\theta=\eta_\theta*\delta_{r(\theta)}$$

then there also exists a vector $b\in\mathbb{R}^d$ with

$$\gamma=\eta*\delta_b.$$

(3.27) **Corollary.** Any random vector on $\mathbb{R}^d$ satisfying (I) and (II) also satisfies (III) if γ has $PTP(\mathscr{I})$.

Proof. Assume first that $\gamma\sim[a,\Phi,\mu]$ has $PTP(\mathscr{I})$. Let X satisfy (I) and (II). Now

$$(3.28)\qquad \begin{aligned}&\lim_{n\to\infty}\sup_{\theta\in S^{d-1}}|nP(\langle R_n X,\theta\rangle\geqq y)-\mu_\theta([y,\infty))|\\ &=\lim_{n\to\infty}\sup_{\theta\in S^{d-1}}|nP(\langle X,\theta\rangle\geqq y\,\|R_n^{*-1}\theta\|)-\mu_{g_n(\theta)}([y,\infty))|\\ &=0\quad \text{by (I).}\end{aligned}$$

Conditions (I) and (II) imply (3.17), so

$$(3.29)\qquad \begin{aligned}&\lim_{\delta\downarrow 0}\lim_{n\to\infty}\sup_{\theta\in S^{d-1}}|n\operatorname{Var}\langle R_n X,\theta\rangle\, I_{(|\langle R_n X,\theta\rangle|\leqq\delta)}-\Phi(\theta,\theta)|\\ &=\lim_{\delta\downarrow 0}\lim_{n\to\infty}\sup_{\theta\in S^{d-1}}|n\operatorname{Var}(\langle X,\ \theta\rangle\, I_{(|\langle X,\theta\rangle|\leqq\delta\,\|\mathbb{R}_n^{*-1}\theta\|)})/\|R_n^{*-1}\theta\|\\ &\quad-\Phi_{g_n(\theta)}(1,1)|\\ &=0\quad \text{by (3.17) and (II).}\end{aligned}$$

Thus conditions $(\mathrm{I})_{\mu(\theta)}$ and $(\mathrm{II})_{\Phi(\theta)}$ of Hahn-Hahn-Klass (1983) hold uniformly in θ. Hence, by Theorem 2.12 of that paper, the fact that γ has $PTP(\mathscr{I})$ implies the existence of centering vectors $v_n\in\mathbb{R}^d$ such that $\mathscr{L}(R_n(S_n-v_n))$ converges weakly to γ. Consequently, Theorem 3.13(I)–(III) must hold, from which we deduce the conclusion that (I) and (II) imply (III). □

Theorem 3.5 and Proposition 3.6 of Hahn-Hahn-Klass (1983) can be used to determine whether γ has $PTP(\mathscr{I})$.

§ 4. Examples

Let Z be any full *non-symmetric* stable law on $\mathbb{R}^d$. We will construct in Example 4.1 an $X\in\mathrm{GDOA}(Z)$ with the property that $\langle X,\theta\rangle$ is not in the DOA of any law for any $\theta\in S^{d-1}$. This example will then be modified to establish two additional points. First, there exist Z and $X\in\mathrm{GDOA}(Z)$ such that the PB need *not* be orthogonal no matter which affine modification of Z attracts the normalized partial sums. Second, if $\mathscr{L}(A_n S_n^s)$ converges weakly, where S_n^s is the symmetrization of S_n, then there need not exist $b_n\in\mathbb{R}^d$ such that $\mathscr{L}(A_n S_n+b_n)$ converges weakly (see Example 4.14). Consequently, a proof of Theorem 3.13 based first on symmetric random vectors cannot easily be de-symmetrized.

The construction of our first example is based on the observation that for any non-symmetric stable law Z there exist two directions θ_1 and θ_2 with genuinely different Levy measures. In fact, θ_1 and θ_2 can be chosen so that

$\langle Z, \theta_1 \rangle$ is not symmetric and $\langle Z, \theta_2 \rangle$ is symmetric. Indeed, θ_1 exists since Z is not symmetric. For notational simplicity assume $\theta_1 = e_1$. Then if μ is the Levy measure for Z, $\mu_{e_1}([1,\infty)) \neq \mu_{-e_1}([1,\infty))$. Moreover, the function

$$f(y) = \mu_{e_1 \cos y + e_2 \sin y}([1,\infty)) - \mu_{-e_1 \cos y - e_2 \sin y}([1,\infty))$$

is continuous with $f(0) f(\pi) < 0$. Hence, there exists $0 < y^* < \pi$ such that $f(y^*) = 0$. Let $\theta_2 = e_1 \cos y^* + e_2 \sin y^*$.

The idea is to construct an X-distribution all of whose marginals are in the domain of partial attraction of both $\langle Z, \theta_1 \rangle$ and $\langle Z, \theta_2 \rangle$. The fact that X must also be in the GDOA of Z suggests constructing X directly from Z using various operators and truncations.

(4.1) *Example.* Let Z be a stable, non-symmetric law of index α, $0 < \alpha < 2$. Let θ_1 and θ_2 be two unit vectors with $\langle Z, \theta_1 \rangle$ non-symmetric and $\langle Z, \theta_2 \rangle$ symmetric. Choose sequences of integers $0 = c_0 < c_1 < c_2 < \ldots$ and unitary operators A_j with the following three properties:

(i) $\lim_{j \to \infty} \|A_j^{-1} A_{j-1} - I\| + \|A_j^{-1} A_{j+1} - I\| = 0$

(ii) $\lim_{j \to \infty} c_{j+1}/c_j = \infty$

(iii) there exists a subsequence j_k such that $A_{j_k}\theta_1$ and $A_{j_k}\theta_2$ are both dense in S^{d-1}.

Then

$$X = \sum_{j=0}^{\infty} A_j Z I_{(\|Z\| \in (c_j^{1/\alpha}, c_{j+1}^{1/\alpha}])} \tag{4.2}$$

is in the GDOA of Z while for each $\theta \in S^{d-1}$, $\langle X, \theta \rangle$ is in the domain of partial attraction (DOPA) of $\langle Z, \theta_1 \rangle$ and $\langle Z, \theta_2 \rangle$. Hence $\langle X, \theta \rangle$ is in no DOA.

Proof. Let $X_1, X_2, \ldots$ be i.i.d. $\mathscr{L}(X)$ and $S_n = X_1 + \ldots + X_n$. For $c_j \leqq n < c_{j+1}$ let

$$T_n = n^{-1/\alpha} A_j^{-1} \quad \text{and} \quad v_n = nEXI_{(\|Z\| \leqq n^{1/\alpha})}.$$

We first show that there exists $v \in \mathbb{R}^d$ such that

$$\mathscr{L}(T_n(S_n - v_n)) \to \mathscr{L}(Z + v). \tag{4.3}$$

Since there exists $v \in \mathbb{R}^d$ for which

$$\mathscr{L}\left(n^{-1/\alpha} \sum_{i=1}^{n} (Z_i - EZI_{(\|Z\| \leqq n^{1/\alpha})})\right) \to \mathscr{L}(Z + v), \tag{4.4}$$

it suffices to compare $T_n X_i$ and $n^{-1/\alpha} Z_i$.

To avoid double subscripts think of $n \in [c_j, c_{j+1})$. If γ_j is any sequence of real numbers increasing to ∞, then $\lim_{j \to \infty} nP(\|Z\| > (n\gamma_j)^{1/\alpha}) = 0$. Hence

$$n^{-1/\alpha} \sum_{i=1}^{n} Z_i'' \xrightarrow{pr} 0 \quad \text{and} \quad T_n \sum_{i=1}^{n} X_i'' \xrightarrow{pr} 0 \tag{4.5}$$

where for Y_i equal to Z_i or X_i let $Y_i'' \equiv Y_i I_{(\|Z_i\| > (n\gamma_i)^{1/\alpha})}$. Let $Y_i' = Y_i - Y_i''$. We thus restrict our comparison to $T_n X_i'$ and $n^{-1/\alpha} Z_i'$.

To do this there are four relevant sets to consider:

$$\begin{aligned}
B_i &= \{\|Z_i\| \leqq c_{j-1}^{1/\alpha}\}\\
D_i &= \{c_{j-1}^{1/\alpha} < \|Z_i\| \leqq c_j^{1/\alpha}\}\\
F_i &= \{c_j^{1/\alpha} < \|Z_i\| \leqq c_{j+1}^{1/\alpha}\}\\
G_i &= \{c_{j+1}^{1/\alpha} < \|Z_i\| \leqq (n\gamma_j)^{1/\alpha} \vee c_{j+1}^{1/\alpha}\}.
\end{aligned}$$

(4.6) *Restricting Z_i to F_i*: $T_n X_i I_{(Z_i \in F_i)} = n^{-1/\alpha} Z_i I_{(Z_i \in F_i)}$.

To treat the remaining sets, define

$$R = \sup_n n^{1-2/\alpha} E\|Z\|^2 I_{(\|Z\| \leqq n^{1/\alpha})} + nP(\|Z\| > n^{1/\alpha}) < \infty.$$

(4.7) *Restricting Z_i to B_i*:

$$\begin{aligned}
E\left\| T_n \sum_{i=1}^n (X_i I_{(Z_i \in B_i)} - EX_i I_{(Z_i \in B_i)})\right\|^2 &\leqq nE\|T_n X_1 I_{(Z_1 \in B_1)}\|^2\\
&= n^{1-2/\alpha} E\|Z_1 I_{(Z_1 \in B_1)}\|^2\\
&\leqq c_j^{1-2/\alpha} E\|Z_1 I_{(Z_1 \in B_1)}\|^2\\
&\leqq R(c_j/c_{j-1})^{1-2/\alpha} \to 0 \quad \text{as} \quad n \to \infty\\
&\quad \text{by (ii).}
\end{aligned}$$

Similarly,

$$E\left\| n^{-1/\alpha} \sum_{i=1}^n (Z_i I_{(Z_i \in B_1)} - EZ_i I_{(Z_i \in B_i)})\right\|^2 \to 0.$$

(4.8) Restricting Z_i *to* D_i:

$$\begin{aligned}
&E\left\| \sum_{i=1}^n (T_n(X_i I_{(Z_i \in D_i)} - EX_i I_{(Z_i \in D_i)}) - n^{-1/\alpha}(Z_i I_{(Z_1 \in D_i)} - EZ_i I_{(Z_i \in D_i)}))\right\|^2\\
&\quad = n^{1-2/\alpha} E\|(A_j^{-1} A_{j-1} - I)(Z_1 I_{(Z \in D_1)} - EZ_1 I_{(Z_1 \in D_1)})\|^2\\
&\quad \leqq c_j^{1-2/\alpha} \|A_j^{-1} A_{j-1} - I\|^2 E\|Z_1 I_{(Z_1 \in D_1)}\|^2\\
&\quad \leqq R\|A_j^{-1} A_{j-1} - I\|^2 \to 0 \quad \text{by (i).}
\end{aligned}$$

At this point we take $\gamma_j \to \infty$ so slowly that

$$\gamma_j^{1/\alpha} \|A_j^{-1} A_{j-1} - I\| \to 0. \tag{4.9}$$

(4.10) *Restricting Z_i to G_i*:

$$\begin{aligned}
E\left\| \sum_{i=1}^n (T_n X_i I_{(Z_i \in G_i)} - n^{-1/\alpha} Z_i I_{Z_i \in G_i)})\right\| &\leqq n^{1-1/\alpha} E\|(A_j^{-1} A_{j+1} - I) Z_1 I_{(Z_1 \in G_1)}\|\\
&\leqq \|A_j^{-1} A_{j+1} - I\| n\gamma_j^{1/\alpha} P(\|Z\| > n^{1/\alpha})\\
&\leqq R\gamma_j^{1/\alpha} \|A_j^{-1} A_{j+1} - I\| \to 0\\
&\quad \text{by (4.9).}
\end{aligned}$$

Combining (4.5)–(4.8) and (4.10) shows that

$$\lim_{n\to\infty} \mathscr{L}(T_n(S_n - v_n)) = \lim_{n\to\infty} \mathscr{L}\left(n^{-1/\alpha}\sum_{i=1}^{n}(Z_i - EZ_i I_{(\|Z_i\| \leq n^{1/\alpha})})\right) = \mathscr{L}(Z+v)$$

by (4.4), which establishes (4.3). Therefore, X is indeed in the GDOA of Z.

It remains to show that for any fixed $\theta \in S^{d-1}$, $\langle X, \theta\rangle \in$ DOPA of both $\langle Z, \theta_1\rangle$ and $\langle Z, \theta_2\rangle$. Let τ denote an arbitrary element of S^{d-1}. Property (iii) ensures the existence of a sequence of integers $f(j)\to\infty$ such that

$$A_{f(j)}\tau \to \theta.$$

Take any integer $n_j = n_j(\tau)$ near $\sqrt{c_{f(j)}c_{f(j)+1}}$ and note that for any fixed $y>0$

$$\lim_{j\to\infty} |P(\langle X, \theta\rangle > y n_j^{1/\alpha})/P(\langle A_{f(j)} Z, \theta\rangle > y n_j^{1/\alpha}) - 1| = 0.$$

Therefore,

$$\begin{aligned}
(4.11) \qquad \lim_{j\to\infty} n_j P(\langle X, \theta\rangle \geqq y n_j^{1/\alpha}) &= \lim_{j\to\infty} n_j P(\langle A_{f(j)} Z, \theta\rangle \geqq y n_j^{1/\alpha}) \\
&= \lim_{j\to\infty} n_j P(\langle Z, A^*_{f(j)}\theta\rangle \geqq y n_j^{1/\alpha}) \\
&= \lim_{j\to\infty} n_j (y n_j^{1/\alpha})^{-\alpha} \mu_{A^*_{f(j)}\theta}([1,\infty)) \\
&= y^{-\alpha}\mu_\tau([1,\infty)).
\end{aligned}$$

Similarly,

$$\lim_{j\to\infty} n_j P(\langle X, \theta\rangle \leqq -y n_j^{1/\alpha}) = y^{-\alpha}\mu_{-\tau}([1,\infty)).$$

Moreover,

$$\begin{aligned}
(4.12) \qquad &\lim_{\delta\downarrow 0}\lim_{j\to\infty} n_j^{1-2/\alpha} E\langle X, \theta\rangle^2 I_{(|\langle X,\theta\rangle| \leqq \delta n_j^{1/\alpha})} \\
&\leqq \lim_{\delta\downarrow 0}\lim_{j\to\infty} \sup_{\gamma\in S^{d-1}} n_j^{1-2/\alpha} E\langle Z, \gamma\rangle^2 I_{(|\langle Z,\gamma\rangle| \leqq \delta n_j^{1/\alpha})} \\
&= 0.
\end{aligned}$$

Consequently, (4.11) and (4.12) for $\tau = \theta_1$ and θ_2 imply that $\langle X, \theta\rangle$ is indeed in the DOPA of both $\langle Z, \theta_1\rangle$ and $\langle Z, \theta_2\rangle$. Hence $\langle X, \theta\rangle$ is not in the DOA of any law. □

(4.13) *Example.* Example 4.1 can be broadened to include A_j which are not all unitary. For example, replace (ii) by

(ii′) $\lim_{j\to\infty}(c_{j-1}/c_j)^{2/\alpha - 1}\|A_j^{-1}\|^2 \max_{1\leqq i<j}\|A_i\|^2 = 0$

and assume that A_{j_k} in (iii) have the property that $\{\|A_{j_k}^{*-1}\|\}$ is uniformly bounded. Assume Z has the property that among all affine transformations only the identity leaves the law of Z invariant. It then follows that the affine transformations (T_n, v_n) used in Example 4.1 are unique up to multiplication by $\delta_n \to I$. Hence, the PB is not orthogonal whenever

$$\liminf_{n\to\infty} |\langle(\delta_n A_j^{-1})^* e_i, (\delta_n A_j^{-1})^* e_k\rangle| > 0 \quad \text{for some } i \neq k \quad \text{and all} \quad \delta_n \to I,$$

or equivalently, if for any weak subsequential limit A^{-1} of A_j^{-1},

$$|\langle A^{*-1} e_i, A^{*-1} e_k\rangle| > 0 \quad \text{for some } i \neq k.$$

If there is more than one such limit A^{-1} then X does not have a PONB when its partial sums are normalized for convergence to any affine modification of Z. □

(4.14) *Example.* Let Γ be a finite measure on S^{d-1} satisfying the following three properties:

(i) Γ is not symmetric, i.e. $\bar{\Gamma} \neq \Gamma$ where $\bar{\Gamma}(A) \equiv \Gamma(-A)$ for any Borel set A in S^{d-1};

(ii) $\Gamma + \bar{\Gamma}$ is uniformly distributed;

(iii) no element of the orthogonal group leaves Γ invariant.

Construct a stable random vector Z with Levy measure given in polar form by

$$\mu(dx) = \Gamma(du) \times dr/r^{1+\alpha}, \qquad r = \|x\|, \qquad u = x/\|x\| \in S^{d-1}.$$

Z is a non-symmetric full stable random vector of index α with only the identity affine transformation leaving the law of Z invariant. Moreover, if Z, Z' are i.i.d., then $Z^s := Z - Z'$ is spherically symmetric stable of index α.

Construct $X \in \mathrm{GDOA}(Z)$ according to Example 4.13. Since $X^s \in \mathrm{GDOA}(Z^s)$ and Z^s is spherically symmetric, there exist norming transformations A_n constructed using the minimal PONB's given in (1.3) such that

$$\mathscr{L}\left(A_n \sum_{i=1}^{n} X_i^s\right) \to \mathscr{L}(Z^s).$$

However, the conclusion of Example 4.13 is that if

$$\mathscr{L}\left(T_n \sum_{i=1}^{n} X_i\right) \to \mathscr{L}(Z)$$

then T_n is not constructable from the minimal PONB. Consequently, T_n and A_n do not differ merely by a translational (centering) component. □

Appendix

(A.1) **Lemma.** Let $Y_1, \ldots, Y_n$ be i.i.d. symmetric random variables. Let $S_n = Y_1 + \ldots + Y_n$. Define $a_y = \sup\{a : yE(Y_1^2 \wedge a^2) \geqq a^2\}$. Then for any $y \geqq 1$

(A.2) $$P(|S_n| \geqq y a_n) \leqq nE((Y_1^2/y^2 a_n^2) \wedge 1)$$

(A.3) $$nE((Y_1^2/(y a_n)^2) \wedge 1) \leqq y^{-1} + nP(|Y_1| > \sqrt{y}\, a_n).$$

Proof. We prove (A.3) first. Note that

$$(Y_1/y a_n)^2 \wedge 1 \leqq y^{-1}((Y_1/a_n)^2 \wedge 1) + I(|Y_1| > \sqrt{y}\, a_n).$$

Multiplying by n, taking expectations, and using the definition of a_n gives (A.3).

We now verify (A.2). Fix n and let $Y_j'=Y_j I_{(|Y_j|\leqq ya_n)}$ and $Y_j''=Y_j I_{(|Y_j|>ya_n)}$. Then

$$\begin{aligned}
P(|S_n|>ya_n) &\leqq P\left(\left\{\left|\sum_{j=1}^{n} Y_j'\right|>ya_n\right\} \text{ or } \bigcup_{j=1}^{n}\{Y_j''\neq 0\}\right)\\
&\leqq P\left(\left|\sum_{j=1}^{n} Y_j'\right|>ya_n\right)+nP(|Y|>ya_n)\\
&\leqq E\left(\sum_{j=1}^{n} Y_j'/ya_n\right)^2+nP(|Y|>ya_n)\\
&=nEY^2 I_{(|Y|\leqq ya_n)}/y^2a_n^2+nP(|Y|>ya_n)\\
&=nE\{(Y/ya_n)^2\wedge 1\}. \quad \square
\end{aligned}$$

(A.4) **Lemma.** Let η be an operator-stable law on $\mathbb{R}^d$ with Lévy measure μ. For each $\theta\in S^{d-1}$, μ_θ is a continuous σ-finite measure on $\mathbb{R}$.

Proof. Let $\eta\sim[a,\Phi,\mu]$ be operator-stable and let B be an exponent of η. Sharpe (1969) noticed that μ is a mixture of Lévy measures concentrated on single orbits of t^B. We utilize the following explicit representation, given by Hudson-Mason ((1981), Theorem 2) (also see Jurek (1978))

$$\mu(A)=\int_L\int_0^\infty I_A(t^Bu)\,t^{-2}\,dt\,dv(u) \tag{A.5}$$

where v is a finite Borel measure on S^{d-1} and $L=\{u\in S^{d-1}$: for all $t>1$, $|t^Bu|>1\}$.

The continuity of μ_θ requires showing $\mu_\theta(\{a\})=0$ for each $a\in\mathbb{R}$. By definition $\mu_\theta(\{0\})=0$. Let $a\in\mathbb{R}\setminus\{0\}$. By (A.5),

$$\mu_\theta(\{a\})=\int_L\int_0^\infty I_{\{x\in\mathbb{R}^d:\,\langle x,\theta\rangle=a\}}(t^Bu)\,t^{-2}\,dt\,dv(u).$$

The inner integral will be 0 if each $K_{u,\theta}\equiv\{t\colon \langle t^Bu,\theta\rangle=a\}$ is at most countable. We appeal to the analyticity of $f_{u,\theta}(z)\equiv\langle z^Bu,\theta\rangle$ on $\Omega=\mathbb{C}\setminus(-\infty,0]$. Just notice that

$$\begin{aligned}
\langle z^Bu,\theta\rangle=\langle e^{B\ln z}u,\theta\rangle &=\sum_{j=1}^{d}\left\langle\sum_{k=0}^{\infty}\frac{B^ku(\ln z)^k}{k!},e_j\right\rangle\langle\theta,e_j\rangle\\
&=\sum_{j=1}^{d}\sum_{k=0}^{\infty}\langle B^ku,e_j\rangle\frac{(\ln z)^k}{k!}\langle\theta,e_j\rangle.
\end{aligned}$$

Since $|\langle B^ku,e_j\rangle|\leqq\|B\|^k$, the above series converges uniformly for all z in any compact subset of Ω. Thus, $f_{u,\theta}(z)$ is analytic by the Weierstrass M-test. Now if $K_{u,\theta}$ contains uncountably many $t\in\mathbb{R}^+$, then $f_{u,\theta}(z)=a$ on a set of points containing an accumulation point in Ω, in which case these two analytic functions must agree on all of Ω, i.e. $f_{u,\theta}(z)\equiv a$. However, $f_{u,\theta}(z)$ is not identically some non-zero constant, for suppose the contrary. Differentiating with respect to t k-times, we find by induction that

$$\langle B^k t^B u, \theta\rangle \equiv 0 \quad \text{for } k \geqq 1.$$

So putting $t=1$, $\langle B^k u, \theta\rangle = 0$ for $k \geqq 1$. Let $q(x)$ be the minimal monic polynomial for B. So there exists $m \leqq d$ and constants $c_0, c_1, \ldots, c_{m-1}, c_m$ with $c_m = 1$ such that

$$q(x) = c_m x^m + c_{m-1} x^{m-1} + \ldots + c_1 x + c_0.$$

Since Z is full, B is invertible and hence $c_0 \neq 0$. Since $q(B) = 0$,

$$0 = \langle q(B)u, \theta\rangle / c_0 = \sum_{j=0}^{m} c_j \langle B^j u, \theta\rangle / c_0 = \langle u, \theta\rangle = f_{u,\theta}(1) = a,$$

a contradiction. Hence, $K_{u,\theta}$ contains at most countably many $t \in \mathbb{R}^+$, which in turn means that $\mu_\theta(\{a\}) = 0$. □

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