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Infinitely Subadditive Capacities as Upper Envelopes of Measures

Bernd Anger and Jörn Lembcke

Mathematisches Institut der Universität Erlangen-Nürnberg, Bismarckstr. $1\frac{1}{2}$, D-8520 Erlangen, Federal Republic of Germany

Introduction

The theory of capacities initiated by Choquet [6] has well-known applications in potential and measure theory, as well as in probability theory including stochastic processes (cf. Dellacherie [9, 10]) and statistics (cf. Huber's programmatic article [14]).

A central result going back to Choquet [6] and Strassen [20] is the fact that strongly subadditive capacities can be represented as upper envelopes of measures (cf. Dellacherie [8], Anger [2–4], Huber and Strassen [15], Topsøe [24], Adamski [1], and Hummitzsch [16]). However, the upper envelope of a set of measures need not be strongly subadditive in general, but obviously is σ-subadditive. On the other hand, it follows from an example of Davies and Rogers [7] that there exists a σ-subadditive capacity which dominates no measure other than zero.

It therefore seems worthwhile to characterize capacities which are upper envelopes of measures by a subadditivity property, thus solving a problem posed by Choquet [6] and Fuglede [12]. The notion of multiple covers introduced in [17] by the second author led to the following conjecture proved as Theorem 2: Upper envelopes of weakly compact sets of measures are exactly those capacities c which are *infinitely subadditive*, i.e. for every finite sequence (K_i) of sets in the domain of c covering a set K n times the inequality $n\,c(K) \leqq \sum c(K_i)$ holds (Definition 1). The proof is based on a Hahn-Banach argument as carried out by the first author in [4], using in addition a measure extension theorem due to Topsøe [23].

After a preliminary first section the results are presented in Sect. 2 for an abstract setting in order to yield a representation of infinitely subadditive capacities as upper envelopes of Radon measures on locally compact spaces, of Borel measures on regular topological spaces, and of Baire measures on topological spaces.

In Sect. 3 we investigate conditions for the approximation of capacities by probability measures and show that in contrast to the strongly subadditive case the subadditivity assumption has to be strengthened (Theorem 3).

Section 4 is devoted to the representation of infinitely subadditive capacities defined on a σ-algebra. Again, the situation differs from the strongly subadditive case: The outer capacity need no longer be a Choquet capacity. This can be used for a counterexample to a conjecture of Huber [14] on the capacitability of Borel sets with respect to upper envelopes of weakly compact sets of measures. Finally, capacitability arguments lead to the representation of infinitely subadditive capacities on the Borel subsets of a Souslin space as upper envelopes of regular Borel measures (corollary of Theorem 4).

1. Basic Assumptions and Standard Examples

Let $\mathfrak{K}$ be a paving on a set X, i.e. a subset of the power set $\mathfrak{P}(X)$.

Definition 1. A family $(K_i)_{i\in I}$ of subsets of X is said to *cover* $K\subset X$ *n times*, if $K \subset \bigcup_{j\in J}\{\bigcap K_j\colon J\subset I,\ \operatorname{card} J=n\}$, i.e. $n 1_K \leqq \sum_{i\in I} 1_{K_i}$ for the corresponding indicator functions.

A set function $c\colon \mathfrak{K}\to \mathbb{R}_+$ is called *subadditive of order* $n\in\mathbb{N}$ if for every $K\in\mathfrak{K}$ and every finite family of sets $K_1,\ldots,K_m\in\mathfrak{K}$ which covers K n times, the inequality $n\,c(K)\leqq \sum_{i=1}^{m} c(K_i)$ holds. c is called *subadditive of order infinity* (abbreviated: ∞-subadditive) if c is subadditive of order n for all $n\in\mathbb{N}$.

Remark 1. If the paving $\mathfrak{K}$ is stable for finite unions and intersections, then every increasing set function $c\colon \mathfrak{K}\to\mathbb{R}_+$ which is strongly subadditive (i.e. $c(K_1\cup K_2)+c(K_1\cap K_2)\leqq c(K_1)+c(K_2)$ for $K_1,K_2\in\mathfrak{K}$) is ∞-subadditive: In fact (cf. [24], § 8, Lemma 1 (iii)), if $K\in\mathfrak{K}$ is covered by $K_1,\ldots,K_m\in\mathfrak{K}$ n times and if

$$K_i'=\bigcup_{j\in J}\{\bigcap K_j\colon J\subset\{1,\ldots,m\},\ \operatorname{card} J=i\},$$

then

$$n\,1_K\leqq \sum_{i=1}^{m} 1_{K_i}=\sum_{i=1}^{m} 1_{K_i'}$$

and hence $K\subset K_n'\subset\ldots\subset K_1'$, which yields

$$n\,c(K)\leqq \sum_{i=1}^{n} c(K_i')\leqq \sum_{i=1}^{m} c(K_i')\leqq \sum_{i=1}^{m} c(K_i).$$

The last inequality follows by induction from the strong subadditivity of c. Furthermore, every ∞-subadditive set function is subadditive (i.e. subadditive of order 1). However, the converse of these statements is false, as we will show in the following

Example 1. Let $X=\{1,2,3\}$ and $\mathfrak{K}=\mathfrak{P}(X)$. For $0\leqq\alpha\leqq 1$, define c_α on $\mathfrak{K}$ by $c_\alpha(\phi)=0$, $c_\alpha(A)=1/2$ for every one-point set A, $c_\alpha(A)=\alpha$ for every two-point set A, and $c_\alpha(X)=1$. Then it is easy to see that c_α is subadditive iff $\alpha\geqq 1/2$, c_α is ∞-subadditive iff $\alpha\geqq 2/3$, and c_α is strongly subadditive iff $\alpha\geqq 3/4$.

In the following definition (cf. [24], § 8), we introduce a useful functional analytic tool for the characterization of ∞-subadditive set functions.

Definition 2. Let $\mathscr{H}$ denote the real vector space generated by the indicator functions 1_K, $K\in\mathfrak{K}$. For a set function $c\colon \mathfrak{K}\to\mathbb{R}_+$, the functional $\hat{c}\colon \mathscr{H}\to\mathbb{R}_+$ is defined by

$$\hat{c}(h)=\inf\left\{\sum_{i=1}^{m}\alpha_i\, c(K_i)\colon m\in\mathbb{N},\ \alpha_i\in\mathbb{R}_+,\ K_i\in\mathfrak{K},\ \sum_{i=1}^{m}\alpha_i\, 1_{K_i}\geqq h\right\}$$
$$=\inf\left\{\frac{1}{n}\sum_{i=1}^{m} c(K_i)\colon m, n\in\mathbb{N},\ K_i\in\mathfrak{K},\ \frac{1}{n}\sum_{i=1}^{m} 1_{K_i}\geqq h\right\}.$$

Lemma 1. *For every set function $c\colon \mathfrak{K}\to\mathbb{R}_+$ with $c(\phi)=0$, $\hat{c}$ is an increasing sublinear functional on $\mathscr{H}$. c is subadditive of order infinity iff $c(K)=\hat{c}(1_K)$ for every $K\in\mathfrak{K}$.*

Proof. Obviously, $\hat{c}$ is increasing, sublinear, and $\hat{c}(1_K)\leqq c(K)$ for $K\in\mathfrak{K}$. If c is ∞-subadditive and if K, $K_i\in\mathfrak{K}$ and $m, n\in\mathbb{N}$ are such that $\frac{1}{n}\sum_{i=1}^{m} 1_{K_i}\geqq 1_K$, then $n\, c(K)\leqq \sum_{i=1}^{m} c(K_i)$, hence $c(K)\leqq\hat{c}(1_K)$.

If $c(K)=\hat{c}(1_K)$ for every $K\in\mathfrak{K}$ and if $K_i\in\mathfrak{K}$, $m, n\in\mathbb{N}$ are such that $n\,1_K\leqq\sum_{i=1}^{m} 1_{K_i}$, then

$$n\, c(K)=\hat{c}(n\, 1_K)\leqq\hat{c}\left(\sum_{i=1}^{m} 1_{K_i}\right)\leqq\sum_{i=1}^{m} c(K_i),$$

hence c is ∞-subadditive.

For the rest of the paper we make the following *basic assumptions:*

$\mathfrak{K}$ and $\mathfrak{G}$ are pavings on a set X, each being closed under finite unions and intersections, and containing the empty set.

We assume that for $\rho=\sigma$ and $\rho=\tau$, respectively,

(1) $\mathfrak{K}$ is closed for countable ($\rho=\sigma$) or arbitrary ($\rho=\tau$) intersections;

(2) $K\setminus G\in\mathfrak{K}$ and $G\setminus K\in\mathfrak{G}$ for $K\in\mathfrak{K}$, $G\in\mathfrak{G}$;

(3) every set of $\mathfrak{G}$ is $\mathfrak{K}$-bounded, i.e. contained in some set of $\mathfrak{K}$;

(4) for every $K\in\mathfrak{K}$ there are sets $K_i\in\mathfrak{K}$ and $G_i\in\mathfrak{G}$ with $K\subset G_i\subset K_i$ and such that $K_i\downarrow_\rho K$. ($\downarrow_\rho K$ indicates for $\rho=\sigma$ a decreasing sequence and for $\rho=\tau$ a downward directed family with intersection K.)

Let $\mathfrak{B}(\mathfrak{K})$ be the smallest σ-algebra containing all subsets A of X with $A\cap K\in\mathfrak{K}$ for every $K\in\mathfrak{K}$. $\mathscr{M}_\rho(\mathfrak{K})$ denotes the set of all σ-additive measures μ on $\mathfrak{B}(\mathfrak{K})$, which are finite on the sets of $\mathfrak{K}$, inner $\mathfrak{K}$-regular, i.e. $\mu(A)=\sup\{\mu(K)\colon K\subset A,\ K\in\mathfrak{K}\}$ for $A\in\mathfrak{B}(\mathfrak{K})$, and ρ-continuous from above on $\mathfrak{K}$, i.e. $\mu(K)=\inf\mu(K_i)$ for $K, K_i\in\mathfrak{K}$ with $K_i\downarrow_\rho K$. (For $\rho=\sigma$, the last condition is a consequence of the others.)

The *weak topology* on $\mathscr{M}_\rho(\mathfrak{K})$ is defined to be the coarsest topology for which the mappings $\mu\mapsto\mu(K)\,(K\in\mathfrak{K})$ are upper semi-continuous and the mappings $\mu\mapsto\mu(G)$ $(G\in\mathfrak{G})$ are lower semi-continuous (cf. [23], p. 197). In other words, a net (μ_i) on $\mathscr{M}_\rho(\mathfrak{K})$ converges weakly to $\mu\in\mathscr{M}_\rho(\mathfrak{K})$ iff

$$\limsup\mu_i(K)\leqq\mu(K)\qquad\text{for } K\in\mathfrak{K}$$

and

$$\liminf \mu_i(G) \geqq \mu(G) \quad \text{for } G \in \mathfrak{G}.$$

If $X \in \mathfrak{K}$ then $X \in \mathfrak{G}$ by (4), and therefore, for $K \in \mathfrak{K}$ and $G \in \mathfrak{G}$, $\complement K \in \mathfrak{G}$ and $\complement G \in \mathfrak{K}$ by (2), hence the above conditions can be replaced by each of the following:

$$\limsup \mu_i(K) \leqq \mu(K) \quad \text{for } K \in \mathfrak{K} \text{ and } \lim \mu_i(X) = \mu(X)$$

or

$$\liminf \mu_i(G) \geqq \mu(G) \quad \text{for } G \in \mathfrak{G} \text{ and } \lim \mu_i(X) = \mu(X).$$

It is easy to see that the weak topology is always Hausdorff.

The basic assumptions are satisfied for the following

Standard Examples. (1) Let X be a locally compact (Hausdorff) space, $\rho = \tau$, $\mathfrak{K}$ the paving of compact, and $\mathfrak{G}$ the paving of open, relatively compact subsets of X. Then $\mathfrak{B}(\mathfrak{K})$ is the Borel σ-algebra $\mathfrak{B}(X)$ generated by the open sets and $\mathcal{M}_\tau(\mathfrak{K})$ is the set of inner compact regular Borel measures (or Radon measures) on $\mathfrak{B}(X)$. The weak topology is the usual vague or weak $*$-topology of simple convergence on the continuous real functions with compact support.

(2) Let X be a regular (Hausdorff) space with the pavings $\mathfrak{K}$ of closed and $\mathfrak{G}$ of open sets, and $\rho = \tau$. Then $\mathfrak{B}(\mathfrak{K})$ is the Borel σ-algebra $\mathfrak{B}(X)$ and $\mathcal{M}_\tau(\mathfrak{K})$ is the set of all (automatically $\mathfrak{K}$-regular) finite Borel measures, τ-continuous from above on the paving of closed sets. The weak topology is the weak topology considered by Topsøe ([22], Theorem 8.1), which coincides with the topology of simple convergence on the bounded continuous real functions in the case of a completely regular space X.

(3) Let X be a topological space, $\rho = \sigma$, $\mathfrak{K}$ the paving of zero sets ($f^{-1}(0)$ for f continuous), and $\mathfrak{G}$ the paving of cozero sets, i.e. complements of zero sets. $\mathfrak{B}(\mathfrak{K})$ is the Baire σ-algebra $\mathfrak{B}_0(X)$ generated by the zero sets, and $\mathcal{M}_\sigma(\mathfrak{K})$ consists of all finite (automatically $\mathfrak{K}$-regular) Baire measures. (If X is perfectly normal, in particular if X is metrizable, zero sets and closed sets coincide, hence the Baire sets are the Borel sets.) The weak topology is the topology of simple convergence on the bounded continuous real functions considered by Varadarajan [25].

(4) Let $\mathfrak{A}$ be a σ-algebra on a set X, $\rho = \sigma$, and $\mathfrak{K} = \mathfrak{G} = \mathfrak{A}$. Then $\mathfrak{B}(\mathfrak{K}) = \mathfrak{A}$, and $\mathcal{M}_\sigma(\mathfrak{K})$ is the set of finite measures on $\mathfrak{A}$. The weak topology is the topology of setwise convergence considered by Gänssler [13], which coincides with the topology of simple convergence on the bounded measurable functions.

2. Approximation of Subadditive Capacities of Order Infinity

We now develope the abstract theory which can be applied to the Standard Examples.

Definition 3. A *ρ-capacity* is a set function $c: \mathfrak{K} \to \mathbb{R}_+$ with $c(\phi) = 0$, which is increasing and ρ-continuous from above: if $K, K_i \in \mathfrak{K}$ and $K_i \downarrow_\rho K$, then $c(K) = \inf c(K_i)$.

We shall frequently make use of the following result due to Topsøe ([23], Theorem 1, Lemma 1, Lemma 2 and remarks following Lemma 2).

Proposition 1 (Topsøe). *Assume* $v\colon \mathfrak{G} \to \mathbb{R}_+$ *has the following properties:*

a) v *is increasing and* $v(\phi)=0$;

b) v *is strongly additive, i.e.* $v(G_1 \cup G_2)+v(G_1 \cap G_2)=v(G_1)+v(G_2)$ *for* $G_1, G_2 \in \mathfrak{G}$;

c) $\inf\{v(G_1 \cap G_2)\colon G_i \in \mathfrak{G},\ K_i \subset G_i\}=0$ *for every pair of disjoint sets* $K_1, K_2 \in \mathfrak{K}$;

d) $v^*\colon K \mapsto \inf\{v(G)\colon K \subset G \in \mathfrak{G}\}$ *is finite on* $\mathfrak{K}$ *and* ρ*-continuous at* ϕ, *i.e.* $\inf v^*(K_i)=0$ *for* $K_i \in \mathfrak{K}$, $K_i \underset{\rho}{\downarrow} \phi$.

Then $\mu\colon A \mapsto \sup\{v^*(K)\colon K \in \mathfrak{K},\ K \subset A\}$ *is an extension of* v^* *to a measure* $\mu \in M_\rho(\mathfrak{K})$ *on* $\mathfrak{B}(\mathfrak{K})$.

The above proposition together with Lemma 1 are the tools for the proof of

Theorem 1. *Let* c *be a* ρ*-capacity which is subadditive of order infinity. Then, for every* $K_0 \in \mathfrak{K}$,

$$c(K_0)=\max\{\mu(K_0)\colon \mu \in \mathcal{M}_\rho(\mathfrak{K}),\ \mu(K) \leqq c(K) \text{ for } K \in \mathfrak{K}\}.$$

Proof. Apply the theorem of Hahn-Banach to the sublinear functional $\hat{c}$ on $\mathcal{H}$: By Lemma 1, we can find a linear form λ on $\mathcal{H}$ with $\lambda(h) \leqq \hat{c}(h)$ for $h \in \mathcal{H}$ and $\lambda(1_{K_0})=\hat{c}(1_{K_0})=c(K_0)$. λ is increasing since $\lambda(h) \leqq \hat{c}(h)=0$ for $h \in \mathcal{H}$ with $h \leqq 0$. As every $G \in \mathfrak{G}$ is $\mathfrak{K}$-bounded, we have $1_G \in \mathcal{H}$. Therefore, we may define $v\colon \mathfrak{G} \to \mathbb{R}$ by $v(G):=\lambda(1_G)$. Then $v(\phi)=0$, and v is increasing since λ is increasing. Hence v is positive real-valued and obviously strongly additive. By our basic assumption (4) on the pair $(\mathfrak{K}, \mathfrak{G})$, for $j=1,2$ and $K_j \in \mathfrak{K}$ there are sets $K_{ji} \in \mathfrak{K}$, $G_{ji} \in \mathfrak{G}$ such that $K_j \subset G_{ji} \subset K_{ji}$ and $K_{ji} \underset{\rho}{\downarrow} K_j$. As $K_{1i} \cap K_{2i} \underset{\rho}{\downarrow} K_1 \cap K_2$, the ρ-continuity of c implies

$$\begin{aligned} 0 &\leqq \inf v(G_{1i} \cap G_{2i}) \leqq \inf \lambda(1_{K_{1i} \cap K_{2i}}) \leqq \inf \hat{c}(1_{K_{1i} \cap K_{2i}}) \\ &= \inf c(K_{1i} \cap K_{2i}) = c(K_1 \cap K_2). \end{aligned}$$

If $K_1 \cap K_2=\phi$, this implies property c) of Proposition 1, and if $K_1=K_2=K$, we get $v^*(K) \leqq c(K)$ and hence property d) of Proposition 1. Let $\mu \in \mathcal{M}_\rho(\mathfrak{K})$ be an extension of v^*. Then $\mu(K) \leqq c(K)$ for $K \in \mathfrak{K}$ and $c(K_0)=\lambda(1_{K_0}) \leqq v^*(K_0) =\mu(K_0) \leqq c(K_0)$, which proves the theorem.

Remark 2. A ρ-capacity c on $\mathfrak{K}$ is ∞-subadditive iff for every $n \in \mathbb{N}$, $K \in \mathfrak{K}$ and every infinite sequence $(K_i)_{i \in \mathbb{N}}$ in $\mathfrak{K}$ which covers K n times we have $n\, c(K) \leqq \sum_{i=1}^{\infty} c(K_i)$:

Obviously, this condition implies that c is ∞-subadditive. The converse follows from Theorem 1 and the fact that $n\, 1_K \leqq \sum_{i=1}^{\infty} 1_{K_i}$ implies $n\, \mu(K) \leqq \sum_{i=1}^{\infty} \mu(K_i)$ for $\mu \in \mathcal{M}_\rho(\mathfrak{K})$.

In particular, subadditivity of order infinity implies σ-subadditivity for ρ-capacities.

The explicit formulation of the versions of Theorem 1 corresponding to the Standard Examples is left to the reader. For the following application the basic assumptions need not be satisfied. It is a simultaneous generalization of [1], Theorem 3.10.1, [4], p. 250, and [24], § 8, Theorem 2:

Corollary. *Let X be a Hausdorff space with the paving $\mathfrak{K}(X)$ of compact subsets of X, and c: $\mathfrak{K}(X) \to \mathbb{R}_+$ a τ-capacity on $\mathfrak{K}(X)$, subadditive of order infinity. Then c is the maximum on $\mathfrak{K}(X)$ of the set of inner compact regular Borel measures on X, dominated by c on $\mathfrak{K}(X)$.*

Proof. For $K_0 \in \mathfrak{K}(X)$, the restriction c_0 of c to $\mathfrak{K}(K_0)$ is an ∞-subadditive τ-capacity on $\mathfrak{K}(K_0)$. By Theorem 1 applied to c_0, there exists $\mu_0 \in \mathcal{M}_\tau(\mathfrak{K}(K_0))$ with $\mu_0(K_0) = c_0(K_0)$ and $\mu_0(K) \leqq c_0(K)$ for $K \in \mathfrak{K}(K_0)$. Define μ: $A \mapsto \mu_0(A \cap K_0)$ on $\mathfrak{B}(X)$. Then μ is an inner compact regular Borel measure on X, $\mu(K_0) = c(K_0)$ and $\mu(K) \leqq c(K)$ for $K \in \mathfrak{K}(X)$.

Proposition 2. *For every ρ-capacity, the set $M_c := \{\mu \in \mathcal{M}_\rho(\mathfrak{K})$: $\mu(K) \leqq c(K)$ for $K \in \mathfrak{K}\}$ of measures dominated by c is weakly compact. If $X \in \mathfrak{K}$, then the set M_c^1: $= \{\mu \in M_c$: $\mu(X) = 1\}$ of probability measures dominated by c is weakly compact, too.*

Proof. Let (μ_i) be a universal (ultra-) net on M_c. We have to prove that (μ_i) converges weakly on M_c. For $A \in \mathfrak{B}(\mathfrak{K})$, $(\mu_i(A))$ is a universal net on $[0, \infty]$ and hence convergent. Define ν: $\mathfrak{G} \to [0, \infty]$ by $\nu(G) := \lim \mu_i(G)$. Then ν obviously satisfies properties a) and b) of Proposition 1. For $K_1, K_2 \in \mathfrak{K}$, $\varepsilon > 0$ and $j = 1, 2$ there are (by the basic assumption (4)) sets $G_j \in \mathfrak{G}$, $K'_j \in \mathfrak{K}$ such that $K_j \subset G_j \subset K'_j$ and $c(K'_1 \cap K'_2) \leqq c(K_1 \cap K_2) + \varepsilon$. Therefore,

$$\nu(G_1 \cap G_2) \leqq \lim \mu_i(K'_1 \cap K'_2) \leqq c(K'_1 \cap K'_2) \leqq c(K_1 \cap K_2) + \varepsilon.$$

If $K_1 \cap K_2 = \phi$, this implies property c) of Proposition 1, and if $K_1 = K_2 = K$ we get $\nu^*(K) \leqq c(K)$ which implies property d) of Proposition 1. Consider the extension μ: $A \mapsto \sup\{\nu^*(K)$: $K \subset A$, $K \in \mathfrak{K}\}$ of ν^* given by Proposition 1. We have $\mu \in \mathcal{M}_\rho(\mathfrak{K})$ and $\mu(K) = \nu^*(K) \leqq c(K)$ for $K \in \mathfrak{K}$, hence $\mu \in M_c$. We claim that (μ_i) converges weakly to μ. For $G \in \mathfrak{G}$ we get

$$\mu(G) \leqq \nu(G) = \lim \mu_i(G) = \liminf \mu_i(G),$$

and for $K \in \mathfrak{K}$, $G \in \mathfrak{G}$ with $K \subset G$ we have

$$\limsup \mu_i(K) \leqq \limsup \mu_i(G) = \lim \mu_i(G) = \nu(G),$$

hence $\limsup \mu_i(K) \leqq \nu^*(K) = \mu(K)$.

Finally, if $X \in \mathfrak{K}$, then $\mu \mapsto \mu(X)$ is continuous. Thus $\{\mu \in \mathcal{M}_\rho(\mathfrak{K})$: $\mu(X) = 1\}$ is closed, which proves that M_c^1 is compact.

Theorem 2. *A set function c: $\mathfrak{K} \to \overline{\mathbb{R}}_+$ is a ρ-capacity, subadditive of order infinity, iff c is the upper envelope of a weakly compact set $M \subset \mathcal{M}_\rho(\mathfrak{K})$ of measures, in fact*

$$c(K) = \max\{\mu(K)\text{: } \mu \in M_c\} \qquad \textit{for every } K \in \mathfrak{K}.$$

Proof. By Theorem 1 and Proposition 2, it only remains to prove that for a compact set $M \subset \mathcal{M}_\rho(\mathfrak{K})$ the function c defined on $\mathfrak{K}$ by $c(K) = \sup\{\mu K)$: $\mu \in M\} = \max\{\mu(K)$: $\mu \in M\}$ is a ρ-capacity, subadditive of order infinity. Obviously, $c(\phi) = 0$, c is positive real-valued and increasing. The ρ-continuity follows from the corresponding property of the measures $\mu \in M$ by applying a minimax theorem (cf. [21], Theorem 1). If $m, n \in \mathbb{N}$, $K, K_i \in \mathfrak{K}$ are such that $n\, 1_K \leqq \sum_{i=1}^m 1_{K_i}$, then $n\,\mu(K) \leqq \sum_{i=1}^m \mu(K_i)$ and hence $n\, c(K) \leqq \sum_{i=1}^m c(K_i)$, which proves that c is ∞-subadditive.

3. Approximation of Strictly Subadditive Capacities of Order Infinity

In general, one cannot approximate a bounded capacity c, subadditive of order infinity, by measures with the same total mass, as the following examples show. In the first one, $X \in \mathfrak{K}$.

Example 2. Consider Example 1 with $\alpha = 2/3$ and let $\mathfrak{K} = \mathfrak{G} = \mathfrak{P}(X)$. $c = c_\alpha$ is an ∞-subadditive τ-capacity with $c(X) = 1$. By Theorem 1, $c = \max M_c$. But the convex combination $\mu = (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)/3$ of Dirac measures is the only probability measure in M_c, and $\mu \neq c$.

If $X \notin \mathfrak{K}$, the situation may be even worse. The following example of an ∞-subadditive τ-capacity c on a second countable locally compact space X shows that there need not be any regular Borel probability measure on X dominated by c on the paving $\mathfrak{K}$ of compact sets, if the inner capacity $c_*(X) := \{\sup c(K)$: $K \in \mathfrak{K}\} = 1$. The example is a modification of [18], Beispiel 4.11, and of the example given in [19].

Example 3. Let $A = \{\alpha_1, \alpha_2, \ldots\}$ and $B = \{\beta_1, \beta_2, \ldots\}$ be countable discrete spaces, and let $\Gamma = \prod_{k=1}^{\infty} \Gamma_k$ be the topological product of the finite discrete spaces $\Gamma_k = \{\gamma_{k0}, \ldots, \gamma_{k2^k}\}$ ($k \in \mathbb{N}$). Denote by p_k: $\Gamma \to \Gamma_k$ the projections and let $G_{ki} := p_k^{-1}(\gamma_{ki})$ ($k \in \mathbb{N}$, $i = 0, \ldots, 2^k$).

Then Γ is compact, and the topological sum X of A, B and Γ is second countable locally compact. Choose ρ, $\mathfrak{K}$, and $\mathfrak{G}$ as in Standard Example (1).

For $n \in \mathbb{N}$, define the regular Borel measure μ_n on $\mathfrak{B}(X)$ by

$$\mu_n(R) = \tfrac{1}{3}\left(\varepsilon_{\alpha_n}(R) + \sum_{\substack{k=1 \\ k \neq n}}^{\infty} \varepsilon_{\beta_k}(R)/2^k + \pi_n(R \cap \Gamma)\right),$$

where π_n denotes the product probability measure

$$\pi_n = \left(\bigotimes_{k=1}^{n-1} \varepsilon_{\gamma_{k0}}\right) \otimes \left(\sum_{i=1}^{2^n} \varepsilon_{\gamma_{ni}}/2^n\right) \otimes \left(\bigotimes_{k=n+1}^{\infty} \varepsilon_{\gamma_{k0}}\right),$$

and let $\mu_0 = \tfrac{1}{3}\left(\sum_{k=1}^{\infty} \varepsilon_{\beta_k}/2^k + \bigotimes_{k=1}^{\infty} \varepsilon_{\gamma_{k0}}\right)$.

(1) It is straightforward to check that $\mu_0 = \lim_{n\to\infty} \mu_n$ in the weak topology on $\mathcal{M}_\tau(\mathfrak{K})$, hence the set $M := \{\mu_n : n = 0, 1, \ldots\}$ *is weakly compact*. Therefore, the upper envelope c: $K \mapsto \sup\{\mu_n(K): n = 0, 1, \ldots\}$ $(K \in \mathfrak{K})$ is an ∞-subadditive τ-capacity on X.

(2) $c_*(X) = 1$, *but there is no probability measure* $\mu \in M_\tau(\mathfrak{K})$ *dominated on* $\mathfrak{K}$ *by* c.

Proof. By the regularity of the measures μ_n,

$$c_*(A) := \sup\{c(K): K \subset A, K \in \mathfrak{K}\} = \sup\{\mu_n(A): n = 0, 1, \ldots\}$$

for $A \in \mathfrak{B}(X)$. Therefore, $c_*(X) = \sup\{1 - 2^{-n}/3: n = 0, 1, \ldots\} = 1$. Now suppose that there exists some probability measure $\mu \in \mathcal{M}_\tau(\mathfrak{K})$ dominated on $\mathfrak{K}$ by c. The regularity of μ implies $\mu \leqq c_*$ on $\mathfrak{B}(X)$, hence $\mu(A)$, $\mu(\Gamma) \leqq 1/3$ and $\mu(\{\beta_k\}) \leqq 2^{-k}/3$ for $k \in \mathbb{N}$. Since

$$1 = \mu(X) = \mu(A) + \sum_{k=1}^{\infty} \mu(\{\beta_k\}) + \mu(\Gamma),$$

we get $\mu(A) = \mu(\Gamma) = 1/3$ and $\mu(\{\beta_k\}) = 2^{-k}/3$ $(k \in \mathbb{N})$.

Moreover,

$$\mu(\{\beta_k\} \cup G_{ki}) \leqq c(\{\beta_k\} \cup G_{ki}) = 2^{-k}/3 \qquad (k \in \mathbb{N},\ i = 1, \ldots, 2^k),$$

hence $\mu(G_{ki}) = 0$ for $i = 1, \ldots, 2^k$. This implies

$$\mu(G_{k0}) = \mu(\Gamma) - \mu\left(\bigcup_{i=1}^{2^k} G_{ki}\right) = 1/3.$$

On the other hand, $\mu(\{\alpha_k\} \cup G_{k0}) \leqq c(\{\alpha_k\} \cup G_{k0}) = 1/3$, hence $\mu(\{\alpha_k\}) = 0$ for $k \in \mathbb{N}$, which contradicts $\mu(A) = 1/3$.

In view of the above examples, we now assume $X \in \mathfrak{K}$ and strengthen the subadditivity requirements for the capacities.

Definition 4. Let $X \in \mathfrak{K}$. A set function c: $\mathfrak{K} \to \mathbb{R}_+$ is called *strictly subadditive of order infinity*, if for every $k \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$, $K \in \mathfrak{K}$, and for every finite sequence of sets $K_1, \ldots, K_m \in \mathfrak{K}$ such that $\bigcup_{i=1}^{m} K_i$ covers $K\, n+k$ times and $\complement K$ k times, i.e. $k + n 1_K \leqq \sum_{i=1}^{m} 1_{K_i}$, the inequality $k c(X) + n c(K) \leqq \sum_{i=1}^{m} c(K_i)$ holds.

Remark 3. Obviously, strict subadditivity of order infinity implies subadditivity of order infinity. Example 2 shows that the converse is false. On the other hand, if $X \in \mathfrak{K}$, it follows as in Remark 1 that every strongly subadditive increasing function c: $\mathfrak{K} \to \mathbb{R}_+$ is even strictly subadditive of order infinity. Again, the converse implication is false.

Without loss of generality, we may restrict the following considerations to capacities with total mass 1, and probability measures.

Theorem 3. *Let* $X \in \mathfrak{K}$. *A set function* c: $\mathfrak{K} \to \overline{\mathbb{R}}_+$ *is a* ρ*-capacity with* $c(X) = 1$, *strictly subadditive of order infinity, iff* c *is the upper envelope of a weakly compact set* $M \subset \mathcal{M}_\rho(\mathfrak{K})$ *of probability measures*, i.e. $c = \max M_c^1$.

Proof. We prove that $c=\max M_c^1$ for every ρ-capacity c with $c(X)=1$, strictly subadditive of order infinity. The proof of the converse implication follows as in Theorem 2.

Define $\tilde{c}$: $\mathscr{H}\to\overline{\mathbb{R}}$ by

$$\tilde{c}(h)=\inf\left\{-\alpha+\sum_{i=1}^{m}\alpha_i\, c(K_i)\colon \alpha, \alpha_i\in\mathbb{R}_+,\ K_i\in\mathfrak{K},\ m\in\mathbb{N},\ -\alpha+\sum_{i=1}^{m}\alpha_i\, 1_{K_i}\geqq h\right\}$$
$$=\inf\left\{-\frac{k}{n}+\frac{1}{n}\sum_{i=1}^{m} c(K_i)\colon m, n\in\mathbb{N},\ k\in\mathbb{Z}_+,\ -\frac{k}{n}+\frac{1}{n}\sum_{i=1}^{m} 1_{K_i}\geqq h\right\}.$$

Then $\tilde{c}\leqq\hat{c}<\infty$ and $\tilde{c}(0)\geqq 0$, as c is ∞-subadditive with $c(X)=1$, hence $\tilde{c}(0)=0$. This implies that $\tilde{c}$ is real-valued, increasing, and sublinear. Obviously, $\tilde{c}(-1)\leqq -1$. As c is strictly ∞-subadditive we have $\tilde{c}(1_K)\geqq c(K)$, hence $\tilde{c}(1_K)=c(K)$ for $K\in\mathfrak{K}$. By the theorem of Hahn-Banach, for every $K_0\in\mathfrak{K}$ there is a linear form λ on $\mathscr{H}$ with $\lambda(h)\leqq\tilde{c}(h)$ for $h\in\mathscr{H}$ and $\lambda(1_{K_0})=\tilde{c}(1_{K_0})=c(K_0)$. λ is increasing as $\lambda(h)\leqq\tilde{c}(h)\leqq 0$ for $h\leqq 0$, and $\lambda(1)=1$ as

$$\lambda(1)\leqq\tilde{c}(1)=c(X)=1\leqq -\tilde{c}(-1)\leqq -\lambda(-1)=\lambda(1).$$

Define v: $\mathfrak{G}\to\mathbb{R}$ by $v(G):=\lambda(1_G)$. As $\lambda\leqq\tilde{c}\leqq\hat{c}$, it follows from the proof of Theorem 1 that v has the properties required in Proposition 1 and that $v^*(K)\leqq c(K)$ for $K\in\mathfrak{K}$. Let $\mu\in\mathscr{M}_\rho(\mathfrak{K})$ be an extension of v^* to $\mathfrak{B}(\mathfrak{K})$. Then $\mu(K)\leqq c(K)$ for $K\in\mathfrak{K}$, $\mu(X)=v^*(X)=v(X)=\lambda(1)=1$, and $c(K_0)=\lambda(1_{K_0})\leqq v^*(K_0)\leqq c(K_0)$. Hence $c(K_0)=\max\{\mu(K_0)\colon \mu\in M_c^1\}$. As M_c^1 is weakly compact by Proposition 2, this proves the theorem.

4. Approximation of Choquet Capacities

We now turn to the problem of approximating set functions defined on a paving containing $\mathfrak{K}$, for which the restriction to $\mathfrak{K}$ is a ρ-capacity, subadditive of order infinity. In contrast to strongly subadditive capacities, the outer capacity corresponding to a ρ-capacity, (strictly) subadditive of order infinity, need not be a Choquet capacity. A counterexample is given in the following

Remark 4. For $\mu\in\mathscr{M}_\rho(\mathfrak{K})$, the inner and the outer measure μ_* and μ^*, respectively, are defined on the power set $\mathfrak{P}(X)$ by

$$\mu_*(A):=\sup\{\mu(B)\colon B\subset A,\ B\in\mathfrak{B}(\mathfrak{K})\}=\sup\{\mu(K)\colon K\subset A,\ K\in\mathfrak{K}\}$$

and

$$\mu^*(A):=\inf\{\mu(B)\colon A\subset B\in\mathfrak{B}(\mathfrak{K})\}.$$

If c is a ρ-capacity, subadditive of order infinity, and $M\subset\mathscr{M}_\rho(\mathfrak{K})$ is compact with $c=\sup M$, then

$$c'\colon A\mapsto\sup\{\mu^*(A)\colon \mu\in M\}\qquad (A\subset X)$$

is an extension of c to a *Choquet $\mathfrak{K}$-capacity* on $\mathfrak{P}(X)$, i.e. c': $\mathfrak{P}(X)\to\overline{\mathbb{R}}_+$ is increasing, σ-continuous from below on $\mathfrak{P}(X)$ and σ-continuous from above on

$\mathfrak{K}$ (cf. [11], chap. III, déf. 27). For $A \in \mathfrak{B}(\mathfrak{K})$, $c'(A)$ is equal to the *inner capacity*

$$c_*(A) := \sup\{c(K)\colon K \in \mathfrak{K},\ K \subset A\}.$$

The *outer capacity* c^* is defined on $\mathfrak{P}(X)$ by

$$c^*(A) := \inf\{c_*(G)\colon A \subset G \in \mathfrak{G}\}.$$

Obviously $c' \leqq c^*$, but equality does not hold in general as the following slight modification of an example due to Fuglede ([12], Example 5.6) shows:

Let $X = [0,1] \times [0,1]$ be endowed with the Euclidean topology and let $\mathfrak{K}$ and $\mathfrak{G}$ be the pavings of compact and open subsets of X, respectively. Consider the set $M := \{\lambda_x\colon x \in [0,1]\}$ of one-dimensional Lebesgue measures $\lambda_x = \varepsilon_x \otimes \lambda \in \mathscr{M}_\tau(\mathfrak{K})$. As the mapping $x \mapsto \lambda_x$ is continuous, M is weakly compact. Let A denote the union of the compact sets

$$K_n = (\{0\} \times [1/2, 1]) \cup \left(\bigcup_{i=1}^{n} (\{1/i\} \times [0, 1/2])\right).$$

Then $c_*(A) = c'(A) = 1/2$, whereas $c^*(A) = 1$. As c is τ-continuous on $\mathfrak{K}$ and X is compact, the outer capacity c^* is an extension of c. But c^* is not σ-continuous from below and hence no Choquet $\mathfrak{K}$-capacity.

A conjecture of Huber ([14], p. 88, final remark of section 3) states that in a second countable locally compact space X, $c_*(A) = c^*(A)$ (i.e. A is c^*-capacitable) if A is a Borel subset of X and if c is the upper envelope of a weakly compact set of probability measures on $\mathfrak{B}(X)$. By the above example, this conjecture is false.

Lemma 2. *Let $\mathfrak{B}$ be a paving on X, stable for countable unions and countable intersections. Then every increasing set function $\ell\colon \mathfrak{B} \to \overline{\mathbb{R}}$ which is σ-continuous from below can be extended to an increasing set function $\bar{\ell}\colon \mathfrak{P}(X) \to \overline{\mathbb{R}}$ which is σ-continuous from below.*

Proof. Define $\bar{\ell}\colon \mathfrak{P}(X) \to \overline{\mathbb{R}}$ by $\bar{\ell}(A) := \inf\{\ell(B)\colon A \subset B \in \mathfrak{B}\}$. Then $\bar{\ell}$ is increasing. Let (A_n) be an increasing sequence of subsets of X with $A := \bigcup_{n=1}^{\infty} A_n$ and suppose $\sup \bar{\ell}(A_n) < \alpha$ for some $\alpha \in \mathbb{R}$. For $n \in \mathbb{N}$, there exist sets $B_n \in \mathfrak{B}$ with $A_n \subset B_n$ and $\ell(B_n) < \alpha$. Then $B'_n := \bigcap_{m=n}^{\infty} B_m$ defines an increasing sequence in $\mathfrak{B}$. As $A_n \subset B'_n \subset B_n$ for $n \in \mathbb{N}$, we have $A \subset B' := \bigcup_{n=1}^{\infty} B'_n$ and therefore

$$\bar{\ell}(A) \leqq \ell(B') = \sup \ell(B'_n) \leqq \sup \ell(B_n) \leqq \alpha.$$

Since $\sup \bar{\ell}(A_n) \leqq \bar{\ell}(A)$, this proves the lemma.

Theorem 4. *Let $\mathfrak{B}$ be a paving on X with $\mathfrak{K} \subset \mathfrak{B}$, stable for countable unions and countable intersections [with $X \in \mathfrak{K}$]. Let $\ell\colon \mathfrak{B} \to \overline{\mathbb{R}}_+$ be an increasing set function, σ-continuous from below, such that the restriction c of ℓ to $\mathfrak{K}$ is a ρ-capacity, [strictly] subadditive of order infinity. If every $B \in \mathfrak{B}$ is $\mathfrak{K}$-analytic (in the sense of* [11], chap. III, déf. 7), *then for $B \in \mathfrak{B}$*

$$\ell(B) = \sup\{\mu_*(B)\colon \mu \in \mathscr{M}_\rho(\mathfrak{K}) \text{ [and } \mu(X) = 1],\ \mu_*(A) \leqq \ell(A) \text{ for } A \in \mathfrak{B}\}.$$

Proof. By Lemma 2 and the assumptions on ℓ, there exists a Choquet $\mathfrak{K}$-capacity $\bar{\ell}$ extending ℓ to $\mathfrak{P}(X)$. By the capacitability theorem of Choquet (cf. [11], chap. III, théorème 28), every $B \in \mathfrak{B}$ is $\bar{\ell}$-capacitable (in the sense of [11], chap. III, déf. 27) and hence, by Theorem 1 [Theorem 3],

$$\begin{aligned}\ell(B) = \bar{\ell}(B) &= \sup\{\bar{\ell}(K):\ K \subset B,\ K \in \mathfrak{K}\} = c_*(B)\\ &= \sup\{\mu(K):\ K \subset B,\ K \in \mathfrak{K},\ \mu \in M_c [\mu \in M_c^1]\}\\ &= \sup\{\mu_*(B):\ \mu \in M_c [\mu \in M_c^1]\}.\end{aligned}$$

As $\mu \in M_c$ is equivalent to $\mu_*(A) \leqq c_*(A) = \ell(A)$ for $A \in \mathfrak{B}$, the theorem is proved.

Remark 5. The above proof shows that the conclusion of Theorem 4 remains valid if the sets of $\mathfrak{B}$ are capacitable for every Choquet $\mathfrak{K}$-capacity.

Example 4. The assumptions on $\mathfrak{B}$ in Theorem 4 are satisfied in the following situations:

a) $\mathfrak{B}$ is the paving of $\mathfrak{K}$-analytic subsets of X (cf. [11], chap. III, théorème 8).

b) X is a topological space as in Standard Example (3) with the σ-algebra $\mathfrak{B} = \mathfrak{B}_0(X)$ of Baire sets and the paving $\mathfrak{K}$ of zero sets. In fact, every Baire set is $\mathfrak{K}$-analytic as every cozero set is the intersection of a sequence of zero sets and hence $\mathfrak{K}$-analytic (cf. [11], chap. III, théorèmes 8 et 12).

c) X is a regular topological space as in Standard Example (2) which in addition is assumed to be Souslin (in the sense of [5], chap. IX, § 6, déf. 2). $\mathfrak{B} = \mathfrak{B}(X)$ is the Borel σ-algebra and $\mathfrak{K}$ is the paving of closed sets. Then every Borel set is $\mathfrak{K}$-analytic, but it may be more convenient to note that the Borel sets are capacitable for all Choquet $\mathfrak{K}$-capacities on X (cf. [5], chap. IX, § 6, propositions 10 et 15), and to apply Remark 5.

By essentially the same argument, replacing Theorem 1 by its corollary in the proof of Theorem 4, one gets the following modification of the last example:

Corollary. *Let X be a (Hausdorff) Souslin space with the pavings $\mathfrak{F}(X)$ and $\mathfrak{K}(X)$ of closed and compact subsets of X, respectively. Let ℓ: $\mathfrak{B}(X) \to \mathbb{R}_+$ be an increasing set function with $\ell(\phi) = 0$, σ-continuous from below on the Borel σ-algebra $\mathfrak{B}(X)$, such that its restriction to $\mathfrak{K}(X)$ is finite and subadditive of order infinity. Suppose that ℓ is σ-continuous from above on $\mathfrak{F}(X)$ or that ℓ is continuous on the right, i.e. for every compact set K in X*

$$\ell(K) = \inf\{\ell(U):\ K \subset U,\ U \text{ open}\}.$$

Then ℓ is the upper envelope of the set of inner compact regular Borel measures on X, dominated by ℓ.

Proof. As in the proof of Theorem 4, we have to show that

$$\ell(B) = \sup\{\ell(K):\ K \subset B,\ K \in \mathfrak{K}(X)\} \qquad \text{for } B \in \mathfrak{B}(X)$$

and

$\ell(K_0) = \sup\{\mu(K_0)$: μ inner compact regular Borel measure on X with $\mu \leqq \ell\}$ for $K_0 \in \mathfrak{K}(X)$.

The first assertion follows from the capacitability theorem (cf. [5], chap. IX, § 6, théorème 6, propositions 10 et 15). As every compact subset K_0 of X is metrizable (cf. [5], chap. IX, app. I, cor. 2), the restriction c_0 of $\mathscr{C}$ to $\mathfrak{K}(K_0)$ is an ∞-subadditive τ-capacity in both cases. Therefore, the second assertion follows as in the proof of the corollary to Theorem 1.

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Note Added in Proof. Infinitely subadditive set functions have also been investigated by G.G. Lorentz (Can. J. Math. **4**, 455–462 (1952)) and J. Moulin Ollagnier, D. Pinchon (Bull. Soc. Math. France **110**, 259–277 (1982)) under the names of multiply subadditive and completely subadditive set functions, respectively.