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Limit Laws for Mixtures with Applications to Asymptotic Theory of Extremes

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1. Introduction

For a sequence $X_1, X_2, \ldots$ of random variables, consider the events $A_j(x) = \{X_j \geqq x\}$, $j = 1, 2, \ldots$, where x is an arbitrary real number. Putting $v_n(x)$ for the number of $A_1(x), A_2(x), \ldots, A_n(x)$ which occur, the event $\{v_n(x) = 0\}$ reduces to $\{Z_n < x\}$, where $Z_n = \max\{X_1, X_2, \ldots, X_n\}$. Here n can be a given integer or a random variable itself. This research has, in fact, started with the aim of unifying techniques for proving limit laws for the extremes when (i) the X's are independent and n is a random variable, independently distributed of the X's and when (ii) the X's are from an infinite sequence of exchangeable random variables and n is a fixed integer. The common property of these two cases is that the distribution of $v_n(x)$ can be written in the form

$$P(v_n(x) = k) = E[f_k(n, y(x))] \tag{1.1}$$

where $f_k(n, y(x))$, $k = 0, 1, 2, \ldots$, is a probability distribution with parameters n and y. As a matter of fact, in the two quoted cases,

$$f_k(n, y) = \binom{n}{k} y^k (1-y)^{n-k}, \tag{1.2}$$

where in case (i), n is a random variable and $y = y(x)$ is a given function, while in case (ii), n is given and $y = y(x)$ is a random variable. Our aim in the present paper is to investigate the limiting properties of (1.1) under assumptions on $f_k(n, y)$, which includes, but is not limited to, (1.2). We shall then apply the results to asymptotic theory of extremes. These results extend those of Berman [2] and [3], Benczur [1], and the limit theorem in Kendall [7] as well as classical results on the asymptotic theory of extremes.

Section 2 gives the general limit law together with some examples, while in Section 3, we discuss extreme value theory.

2. A Limit Theorem on Mixtures

Let $f_k(\lambda, \alpha)$ be a two-parameter (λ, α) family of discrete distributions on the non-negative integers $k = 0, 1, 2, \ldots$. We assume that $f_k(\lambda, \alpha)$ tends to a one-parameter family $g_k(a)$ of distributions as λ and α go through certain sequences

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$\{\lambda_n\}$ and $\{\alpha_n\}$ of real numbers, where, as $n\to+\infty$,

$$\lim f_k(\lambda_n, a\alpha_n)=g_k(a) \tag{2.1}$$

for any $a>0$. We further assume that the sequences $\{\lambda_n\}$ and $\{\alpha_n\}$ determine one another in the following sense: given $\{\lambda_n\}$, if for each k, as $n\to+\infty$,

$$\lim f_k(\lambda_n, \alpha_n)=\lim f_k(\lambda_n, \alpha_n^*),$$

then $\lim \alpha_n/\alpha_n^*=1$, and conversely, given $\{\alpha_n\}$, a similar property is satisfied by the sequence $\{\lambda_n\}$.

We say that a distribution $g_k(a)$, $k=0,1,2,\ldots$, generates an identifiable mixture of distributions if a distribution function $U(a)$, $U(0)=0$, is uniquely determined by the sequence

$$p_k=\int_0^{+\infty} g_k(a)\,dU(a), \qquad k=0,1,2,\ldots. \tag{2.2}$$

We now have the following result.

Theorem 2.1. *Let the two-parameter family $f_k(\lambda,\alpha)$, $k=0,1,\ldots$, of distributions satisfy the preceding assumptions and we further assume that its limit $g_k(a)$ generates an identifiable mixture of distributions. Let $U_n(a)$ be a sequence of proper distribution functions with $U_n(0)=0$ for each n. Then, for each k, as $n\to+\infty$,*

$$\lim \int_0^{+\infty} f_k(\lambda_n, a\alpha_n)\,dU_n(a)=p_k \tag{2.3}$$

exists and $\{p_k\}$ is a distribution if, and only if, as $n\to+\infty$,

$$\lim U_n(a)=U(a) \tag{2.4}$$

exists at each continuity point a of $U(a)$ and $U(a)$ is a proper distribution function. When they exist $\{p_k\}$ and $U(a)$ satisfy (2.2).

Remarks. 2.1. Naturally, the roles of the two parameters λ an α can be freely interchanged and thus the above statement can be repeated when λ_n takes the role of α_n.

2.2. We shall discuss applications of Theorem 2.1 to special choices of $f_k(\lambda,\alpha)$ in detail after the proof. We mention, however, already here that several other choices of $f_k(\lambda,\alpha)$ can lead to interesting applications as indeed several important discrete distributions satisfy our assumptions, see Johnson and Kotz [6], in particular pp. 31–48, 76–79, 104–114, 137–138 and 248–253 as well as Chapters 8 and 9. For identifiability of mixtures, see Teicher [10] and its references.

Proof of Theorem 2.1. Sufficiency. Put

$$p_{k,n}=\int_0^{+\infty} f_k(\lambda_n, a\alpha_n)\,dU_n(a). \tag{2.5}$$

Multiplying (2.5) by e^{itk} and adding up with respect to k from 0 to $+\infty$, we get

$$\varphi_n(t)=\sum_{k=0}^{+\infty} p_{k,n}\, e^{itk}=\int_0^{+\infty} \psi(t;\lambda_n, a\alpha_n)\, dU_n(a), \tag{2.6}$$

where $\psi(t;\lambda,\alpha)$ is the characteristic function of the distribution $\{f_k(\lambda,\alpha)\}$.

By assumption, $\psi(t;\lambda_n, a\alpha_n)$ tends to the characteristic function $\xi(t;a)$ of the distribution $\{g_k(a)\}$. Hence, by the Helly-Bray lemma ([8], p. 180) and by (2.4), for any fixed A, as $n\to+\infty$

$$\lim\int_0^A \psi(t;\lambda_n, a\alpha_n)\, dU_n(a)=\int_0^A \xi(t;a)\, dU(a). \tag{2.7}$$

On the other hand, for A with $2(1-U(A))<\varepsilon$, and for n sufficiently large,

$$\left|\int_A^{+\infty} \psi(t;\lambda_n, a\alpha_n)\, dU_n(a)\right|\leqq 1-U_n(A)\leqq 2(1-U(A))<\varepsilon,$$

and thus, as $n\to+\infty$,

$$\lim\int_0^{+\infty} \psi(t;\lambda_n, a\alpha_n)\, dU_n(a)=\int_0^{+\infty} \xi(t;a)\, dU(a). \tag{2.8}$$

Since $\xi(t;a)$ is a characteristic function in t, the right hand side of (2.8) is continuous at $t=0$ and thus, by the continuity theorem of characteristic functions, (2.3) holds and the limit is a proper distribution by $U(a)$ being so. The sufficiency part of the proof is thus completed.

Necessity. Assume that (2.3) holds and that $\{p_k\}$ is a distribution. By the compactness of distribution functions ([8], p. 179), $U_n(a)$ has a subsequence $U_{n(j)}(a)$, such that $U_{n(j)}(a)$ tends weakly to a distribution function $U^*(a)$ (possibly not proper, as yet). Repeating the first part of the proof for this subsequence (replacing $1-U(A)<\varepsilon$ by $U^*(+\infty)-U^*(A)<\varepsilon$), we have from (2.8)

$$p_k=\int_0^{+\infty} g_k(a)\, dU^*(a).$$

Since $U_n(0)=0$ for each n, $U^*(0)=0$. On the other hand, from the assumption of $\{p_k\}$ being a distribution, we get that $U^*(+\infty)=1$, that is, $U^*(a)$ is a proper distribution function. But since $g_k(a)$ generates an identifiable mixture of distributions, $\{p_k\}$ uniquely determines $U^*(a)$. Therefore, any weakly convergent subsequence of the distributions $U_n(a)$ has the same limit $U^*(a)$, that is, $U_n(a)$ is weakly convergent, which proves the necessity part of our theorem. Since by the uniqueness theorem of characteristic function, (2.8) and (2.2) are equivalent, Theorem 2.1 is established.

Let us give some examples for application.

Example 2.1. Let

$$f_k(\lambda,\alpha)=\binom{\lambda}{k}\alpha^k(1-\alpha)^{\lambda-k}, \qquad k=0,1,\ldots,\lambda,$$

where λ is a positive integer and $0<\alpha<1$. If $\lambda_n \to +\infty$ and for $\alpha_n \sim a/\lambda_n$,

$$f_k(\lambda_n, \alpha_n) \to g_k(a) = a^k e^{-a}/k!,$$

and thus our assumptions are evidently satisfied (the fact that the Poisson distribution generate an identifiable mixture of distributions is well known).

Theorem 2.1 therefore implies

Corollary 2.1. *For each k, as* $\lambda \to +\infty$,

$$\lim \int_0^1 \binom{\lambda}{k} \alpha^k (1-\alpha)^{\lambda-k} \, dU_\lambda(\alpha) = p_k$$

exists and $\{p_k\}$ *is a distribution if, and only if,* $U_\lambda^*(a) = U_\lambda(a/\lambda)$ *converges weakly to a distribution function* $U(a)$. *The limits* p_k *satisfy*

$$p_k = \frac{1}{k!} \int_0^{+\infty} a^k e^{-a} \, dU(a).$$

Since the Poisson distribution generates an identifiable mixture of distributions, Corollary 2.1 yields

Corollary 2.2. *As* $\lambda \to +\infty$,

$$\lim \binom{\lambda}{k} \int_0^1 \alpha^k (1-\alpha)^{\lambda-k} \, dU_\lambda(\alpha) = p_k$$

is a Poisson distribution if, and only if, $U_\lambda(a/\lambda) = U_\lambda^*(a)$ *tends to* $U(a)$, *degenerated at a positive number.*

Interchanging the roles of λ and α in Example 2.1, we get from Theorem 2.1

Corollary 2.3. *Let* $U_\alpha(\lambda)$, *for fixed* α, *be a distribution function having jumps only at non-negative integers. Then, as* $\alpha \to 0$,

$$\lim \int_0^{+\infty} \binom{\lambda}{k} \alpha^k (1-\alpha)^{\lambda-k} \, dU_\alpha(\lambda) = p_k$$

exists for each k and $\{p_k\}$ *is a distribution if, and only if,* $U_\alpha^*(a) = U_\alpha(\alpha a)$ *converges weakly to a distribution function* $U(a)$. *The limits* p_k *satisfy*

$$p_k = \frac{1}{k!} \int_0^{+\infty} a^k e^{-a} \, dU(a).$$

It is interesting to note that the limit distributions $\{p_k\}$ in Corollaries 2.1 and 2.3 are formally the same. In the next section, where we apply these corollaries, we shall return to this remark.

Example 2.2. For integers M, N and n, let

$$h_k(M, N, n) = \binom{M}{k} \binom{n-M}{n-k} \Big/ \binom{N}{n}.$$

Putting

$$f_k(\lambda, \alpha) = h_k(\lambda N, N, \alpha), \qquad N > \alpha,$$

where N can vary independently of (λ, α), except that $N > \alpha$ should be satisfied, we have from Theorem 2.1

Corollary 2.4. *For fixed* α *and* N, *with* $\alpha < N$, *let* $U_\alpha(\lambda)$ *be a distribution function with jumps at the values* k/N, $0 \leqq k \leqq N$. *Then, as* $\alpha \to +\infty$,

$$\lim \int_0^1 \binom{\lambda N}{k} \binom{N-\lambda N}{\alpha - k} \Big/ \binom{N}{\alpha} \, dU_\alpha(\lambda) = p_k$$

exists for each k *and* $\{p_k\}$ *is a distribution if, and only if,* $U_\alpha^*(a) = U_\alpha(a/\alpha)$ *converges weakly to a distribution function* $U(a)$. *The limits* p_k *are necessarily of the form*

$$p_k = \frac{1}{k!} \int_0^{+\infty} a^k e^{-a} \, dU(a).$$

Example 2.3. Let $S(t)$ be the number of occurances in the interval $(0, t)$ in a Poisson process with intensity one, that is,

$$P\big(S(t) = k\big) = \frac{t^k e^{-t}}{k!}.$$

Let $X_1, X_2, \ldots, X_\lambda$ be independent uniform variates on the interval $(0, \alpha)$ and let the X's be independent of the Poisson process. Put

$$f_k(\lambda, \alpha) = P\big(S(W_\lambda) = k\big)$$

where $W_\lambda = \min(X_1, X_2, \ldots, X_\lambda)$. Since

$$P(W_\lambda < x) = 1 - \left(1 - \frac{x}{\alpha}\right)^\lambda, \qquad 0 \leqq x \leqq \alpha,$$

by our assumptions

$$f_k(\lambda, \alpha) = \frac{1}{k!} \int_0^\alpha x^k e^{-x} \frac{\lambda}{\alpha} \left(1 - \frac{x}{\alpha}\right)^{\lambda - 1} dx,$$

which, by the dominated convergence theorem, has a limit as $\lambda \to \infty$ and $\alpha \sim \lambda/a$ with a fixed $\alpha > 0$,

$$f_k(\lambda, \alpha) \to g_k(a) = \frac{1}{k!} \int_0^{+\infty} x^k e^{-x} a e^{-ax} \, dx = \frac{a}{(1+a)^{k+1}},$$

a geometric distribution. Since, as one can easily see, all of our assumptions are satisfied, $S(W_\lambda)$ with random sample size λ has a limit, as $\alpha \to +\infty$, if, and only if, λ/α has a limiting distribution $U(a)$. In this case,

$$\lim P\big(S(W_\lambda) = k\big) = \int_0^{+\infty} \frac{a}{(1+a)^{k+1}} \, dU(a).$$

The only problem may be here to see that $g_k(a) = a(1+a)^{-k-1}$ generates an identifiable mixture of distributions. This however easily follows by considering the transformation $y = 1/(1+a)$, by which the mixture of $g_k(a)$ and $U(a)$ becomes the k-th moment of a distribution bounded by 0 and 1. Such a distribution is known to be determined by its moments.

We included this last example for illustration of Theorem 2.1 being applicable to a variety of problems and in particular, to give an example where $g_k(a)$ is not a Poisson distribution. For additional examples, see Remark 2.2.

Corollaries 2.1 and 2.4 are significant in terms of exchangeable events. We say that the events $A_1, A_2, \ldots, A_N$ are exchangeable if the probabilities

$$w_k = P(A_{i_1} A_{i_2} \ldots A_{i_k}), \qquad 1 \leqq i_1 < i_2 < \cdots < i_k < N+1 \tag{2.9}$$

depend on k but not on the actual subscripts i_j, $1 \leqq j \leqq k$. There is a significant difference between situations when $N = +\infty$ and when N can not be increased to infinity. In case of $N = +\infty$, the now classical theorem of DeFinetti [4] says that there is a random variable ξ such that $0 \leqq \xi \leqq 1$ and

$$w_k = E(\xi^k). \tag{2.10}$$

Therefore, if v_n denotes the number of $A_1, A_2, \ldots, A_n$ which occur, we have

$$P(v_n = k) = \binom{n}{k} E\left(\xi^k (1-\xi)^{n-k}\right) \tag{2.11}$$

and Corollary 2.1 therefore gives a necessary and sufficient condition for the existence of a limiting distribution $\{p_k\}$ of v_n when, for each n, $A_j = A_{j,n}$, $1 \leqq j \leqq n$, is a segment of an infinite sequence of exchangeable events. In this setting, Corollary 2.1 is essentially due to Benczur [1].

Let us put $w_{k,n}$ and ξ_n for w_k and ξ respectively when we have sequences $\{A_{j,n}\}$, $n = 1, 2, \ldots$ of infinite sequences of exchangeable events. Then, if, as $n \to +\infty$,

$$n w_{1,n} \to a > 0 \quad \text{and} \quad n^2 w_{2,n} \to a^2, \tag{2.12}$$

$\xi_n/n \to a$ in probability by the Chebishev inequality, which in turn implies that ξ_n/n has a degenerate limit law. Corollary 2.2 thus yields that v_n has a limit law and it is Poisson, which is the limit theorem obtained by Kendall [7]. Though Kendall's limit theorem has been reobtained in the literature a number of times, the present setting may give some new light to its significance. First of all, it gives the best possible result in terms of Poisson limits of v_n if only the sequences $\{w_{k,n}\}$ but not the variables ξ_n are known. Secondly, it immediately extends to a simple criterion for obtaining $g_k(a)$ as limit law in the notations of Theorem 2.1.

If the sequence A_j, $1 \leqq j \leqq n$, is from a finite set of exchangeable events then DeFinetti's Theorem does not apply. It was shown by Kendall [7] that in this case,

$$P(v_n = k) = \int_0^N \binom{x}{k} \binom{N-x}{n-k} \bigg/ \binom{N}{n} \, dU_N(x) \tag{2.13}$$

where $U_N(x)$ is a distribution with jumps only at non-negative integers and $U_N(N) = 1$. Corollary 2.4 is therefore giving a criterion for v_n to have a limit law. The limit law will again be Poisson if, and only if, the limit distribution $U(a)$ is degenerate. A sufficient condition for this case, similar to Kendall's result for infinite sequences, was obtained by Ridler-Rowe [9]. We remark here that it was obtained in my paper [5] that exchangeability is not important here: the distribution of the number of those occurring in a given sequence of n events can always be reduced to exchangeable ones. In principle, therefore, Corollary 2.4

is applicable to v_n when the sequence $A_{1,n}, A_{2,n}, \dots, A_{n,n}$ is arbitrary. This possibility is, however, limited by lack of knowledge on N when only $w_1, w_2, \dots, w_n$ of (2.9) are available (this same difficulty also arises when the A's are known to be exchangeable).

3. Extreme Value Theory: Exchangeable Variables and Random Sample Size

For a triangular array $X_{j,n}$, $1 \leqq j \leqq N(n)$, $n=1,2,\dots$, of random variables, we consider the order statistics

$$X^*_{1,n} \leqq X^*_{2,n} \leqq \cdots \leqq X^*_{N,n}, \qquad N=N(n).$$

For fixed t, as $n \to +\infty$, $X^*_{N-t,n}$ are called the extreme values or extreme order statistics. In this section, we investigate conditions for the existence of normalizing sequences $\{a_n\}$ and $\{b_n\}$ such that

$$P_n(X^*_{N-t,n} < a_n x + b_n) = F_{n,t}(x) \tag{3.1}$$

tends to a limit for each t, as $n \to +\infty$, for two systems of $X_{j,n}$: (i) for each n, $X_{j,n}$, $1 \leqq j \leqq n$, is a segment of an infinite sequence of exchangeable variables and (ii) the X's are independent and identically distributed and the size $N(n)$ of the n-th row is a random variable which is independently distributed of the X's. In both cases we assume that the sequences $\{a_n\}$ and $\{b_n\}$ are, in a sense, characteristic sequences of the extremes. Let us put this requirement into a definition. We say that $\{a_n\}$ and $\{b_n\}$ are characteristic sequences of the extremes if for at least one fixed t, as $n \to +\infty$,

$$\lim F_{n,t}(x) = F_t(x) > 0 \qquad \text{for } x > x_0, \tag{3.2}$$

and for any function $t(n) \to +\infty$ as $n \to +\infty$,

$$\lim F_{n,t(n)}(x) = 1, \qquad x > x_0. \tag{3.3}$$

Lemma 3.1. *$\{a_n\}$ and $\{b_n\}$ are characteristic sequences of the extremes if, and only if, the limits $F_t(x)$ in (3.2) satisfy the condition that, as $t \to +\infty$,*

$$\lim F_t(x) = 1, \qquad x > x_0. \tag{3.4}$$

Proof. Evidently, for $t_1 < t_2$, $F_{n,t_2}(x) \geqq F_{n,t_1}(x)$ for any x. Hence, for n sufficiently large and for any fixed t,

$$F_{n,t(n)}(x) \geqq F_{n,t}(x),$$

if $t(n) \to +\infty$ with n. Therefore (3.4) implies (3.3). Conversely, assume that (3.4) fails. Since $F_t(x)$ is monotonic in t, $\lim F_t(x) = q(x)$ always exists as $t \to +\infty$. Hence, if $q(x) < 1$, there is an N such that for all fixed t, and for all $n \geqq N$, $F_{n,t}(x) \leqq q^*(x) < 1$. Thus, for any $n \geqq N$, we can have with a $T \geqq t(n)$,

$$F_{n,t(n)}(x) \leqq F_{n,T}(x) \leqq q^*(x).$$

Since the extreme right hand inequality does not depend on n, (3.3) also fails in view of the choice of $q^*(x)$. The proof is completed.

Before formulating our results, let us quote the definition of exchangeabilty of random variables. A sequence $Y_1, Y_2, \dots$ of random variables is called exchangeable if the distribution of the vector $(Y_{i_1}, Y_{i_2}, \ldots, Y_{i_k})$ for distinct subscripts i_j depends on k only, that is, it does not depend on the actual subscripts i_j, $1 \leqq j \leqq k$. In particular, the events $\{Y_j \geqq x\}$ are exchangeable in the sense of the previous section and we can therefore apply DeFinetti's Theorem as expressed in (2.10) and (2.11). Let us put $\xi_n(x)$ for the random variable in (2.10) and (2.11) for the sequence $A_{j,n}(x) = \{X_{j,n} \geqq x\}$, $1 \leqq j \leqq n$, and let $v_n(x)$ be the number of those among $A_{j,n}(x)$, which occur. Then evidently

$$\{v_n(x)=0\} = \{X^*_{n,n} < x\} \tag{3.5}$$

and, in general,

$$\{v_n(x) \leqq t\} = \{X^*_{n-t,n} < x\}. \tag{3.6}$$

Thus, putting

$$U_n(\alpha; x) = P_n(\xi_n(x) < \alpha), \tag{3.7}$$

we have from Corollary 2.1 the following result.

Theorem 3.1. *Let $\{a_n\}$ and $\{b_n\}$ be characteristic sequences of the extremes. Then for each t, as $n \to +\infty$,*

$$\lim P_n(X^*_{n-t,n} < a_n x + b_n) = F_t(x) \tag{3.8}$$

exists if, and only if, as $n \to +\infty$,

$$\lim U_n\left(\frac{a}{n}; a_n x + b_n\right) = U(a; x) \tag{3.9}$$

exists and is a proper distribution function. $F_t(x)$ is necessarily of the form

$$F_t(x) = \sum_{k=0}^{t} \frac{1}{k!} \int_0^{+\infty} a^k e^{-a} \, dU(a; x). \tag{3.10}$$

In order to apply Corollary 2.1 to obtaining Theorem 3.1 we have just to observe that, in view of (3.6), the existence of the limits in (3.8) is equivalent to $v_n(a_n x + b_n)$ having a limit law $\{p_k(x)\}$. The requirement that $\{p_k(x)\}$ be a proper distribution is equivalent to (3.4), which, by the conclusion of Lemma 3.1, is equivalent to the sequences $\{a_n\}$ and $\{b_n\}$ being characteristic sequences of the extremes. The theorem thus follows.

Two special cases have been investigated for $t=0$ by Berman [2] and [3]. In [2], he restricted himself to the choices of $\{a_n\}$ and $\{b_n\}$ when, roughly speaking, they can come up as normalizing sequences for the maximum of independent, identically distributed random variables. More precisely, let $\{a_n\}$ and $\{b_n\}$ be such that there is a distribution function $F(x)$ such that, as $n \to +\infty$,

$$\lim F^n(a_n x + b_n) = G(x), \tag{3.11}$$

a proper distribution function. From classical theory it is well known (and easily follows from (3.11)) that (3.11) is equivalent to

$$n[1 - F(a_n x + b_n)] \to -\log G(x).$$

Thus the condition (3.9) can be written in the form that if $\{a_n\}$ and $\{b_n\}$ are such that (3.11) is satisfied with some distribution function $F(x)$, then a criterion for

(3.8) is the existence of the limit

$$\lim P_n(\xi_n(a_n x+b_n)/[1-F(a_n x+b_n)]<a),$$

as $n\to+\infty$.

The result of [3] is a special case of the following form which may also serve as an example to Theorem 3.1. Let $G(x, y)$ be a distribution function in x for given y, and let $F_n(y)$ be another distribution function. Let the joint distribution function of $(X_{i_1,n}, X_{i_2,n}, \ldots, X_{i_k,n})$ be given by

$$\int_{-\infty}^{+\infty} G(x_1, y)\, G(x_2, y) \ldots G(x_k, y)\, dF_n(y).$$

Then

$$\xi_n(x)=1-G(x, y)$$

and thus (y signifying the random point)

$$\begin{aligned} U_n(\alpha; x)&=P_n(\xi_n(x)<\alpha)=P_n(G(x, y)>1-\alpha)\\ &=\int_{J(\alpha;\, x)} dF_n(y)\end{aligned}$$

where $J(\alpha; x)$ is the y-set determined by

$$G(x, y)>1-\alpha.$$

The criterion (3.9) therefore reduces to the existence of the limit

$$\lim \int_{J(a/n;\, a_n x+b_n)} dF_n(y), \tag{3.12}$$

as $n\to+\infty$. Berman [3] deals with the case when $G(x, y)=H(x-y)$ with a distribution function $H(z)$ and $F_n(y)=F(y)$ for all n. With these choices, $J(\alpha; x)$ reduces to the interval $(-\infty, x-H^{-1}(1-\alpha))$ and (3.12) to the existence of the limit

$$\lim F\left(a_n x+b_n-H^{-1}\left(1-\frac{a}{n}\right)\right) \tag{3.13}$$

as $n\to+\infty$. If $F=H$ is the normal $(0, \zeta)$ distribution, (3.13) has limit $F(x)$ itself with $a_n=1$, $b_n=(2\zeta \log n)^{1/2}$, independently of a. Hence we get a degenerate case $F_t(x)=F(x)$ for each t, which is a non-characteristic case for the extremes. We intentionally excluded such a case, though evidently the whole theory of Section 2, and thus of Section 3, remains unchanged if we drop our requirement of the limit $\{p_k\}$ in Theorem 2.1 being a proper distribution. In this case, in our criterions, the limiting distribution $U(a)$ is not necessarily a proper distribution function and the normalizing constants $\{a_n\}$ and $\{b_n\}$ are not necessarily characteristic sequences of the extremes.

Another direct consequence of Theorem 2.1, in view of Corollary 2.3, is a necessary and sufficient condition for the existence of the limit of $F_{n,t}(x)$ in (3.1) for each fixed t, when the $X_{j,n}$ are independent and N is a random variable, independently distributed of the X's. Though the theorem below extends the best known one, due to Thomas [11], the emphasis is rather on the fact that our Theorem 2.1 unifies this theory with those of exchangeable variables.

Theorem 3.2. *Let $X_{j,n}$, $1 \leqq j \leqq N(n)$, be independent random variables with common distribution function $G_n(x)$. Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers such that, as $n \to +\infty$, $G_n(a_n x + b_n) \to 1$. If $N(n)$ is a sequence of positive integer valued random variables, independently distributed of the X's, then*

$$P_n(X^*_{N(n)-t,n} < a_n x + b_n) = F_{n,t}(x)$$

has a limit $F_t(x)$ for each t if, and only if, as $n \to +\infty$,

$$\lim P_n(N(n)/[1 - G_n(a_n x + b_n)] < a) = U(a; x)$$

exists. When it exists,

$$F_t(x) = \sum_{k=0}^{t} \frac{1}{k!} \int_0^{+\infty} a^k e^{-a} \, dU(a; x).$$

If $\{a_n\}$ and $\{b_n\}$ are characteristic sequences of the extremes, then $U(a; x)$ is a proper distribution function in a.

The theorem follows from Corollary 2.3 by observing that, with the notations of (3.5) and (3.6),

$$P_n(v_{N(n)}(x) = k) = \int_0^{+\infty} \binom{\lambda}{k} [1 - G_n(x)]^k G_n(x)^{\lambda - k} \, dP_n(N(n) < \lambda).$$

As pointed our after Corollary 2.3, it is interesting to note that the distributions $F_t(x)$ in Theorems 3.1 and 3.2 are formally the same. Hence, results on exchangeable variables can immediately be transformed into theorems on extremes of independent variables with random sample size.

Thomas [11] obtained the conclusion of Theorem 3.2 for $t = 0$ with the restriction that $\{a_n\}$ and $\{b_n\}$ should satisfy (3.11) for some $F(x)$.

We conclude this section with a remark. If the sequence $X_{j,n}$, $1 \leqq j \leqq N(n)$, is an arbitrary sequence of random variables, we can appeal to Corollary 2.4 for obtaining limit laws for $X^*_{N-t,n}$, after normalization. This is made possible by Kendall's representation theorem and by my comparison method obtained in [5] (see the last paragraph of Section 2). While such an application is mainly of theoretical value at the present stage, it may be of interest to remark that the form of the limit law will be exactly the same as for the class of exchangeable variables. It follows from the form of p_k in Corollaries 2.1 and 2.4.

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