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Joint Continuity of the Local Times of Markov Processes*

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Summary. Consider a Markov process on the real line with a specified transition density function. Certain conditions on the latter are shown to be sufficient for the almost sure existence of a local time of the sample function which is jointly continuous in the state and time variables.

1. Introduction and Summary

The purpose of this work is to present new sufficient conditions for the joint continuity of the local time of a homogeneous Markov process. These conditions are stated in terms of the transition density function.

The concept of the local time of a stochastic process was introduced by Levy [10] in the case of the Brownian motion process. His work was developed by Trotter [14], who proved, in the Brownian case, that the local time is almost surely jointly continuous. Since then the subject has developed in two different directions, namely, those of Markov processes and Gaussian processes, respectively. The methods in these two areas are generally different. (See the survey of German and Horowitz [7].) The author has been interested in the development of a general theory to cover a large class of stochastic processes which would include both Markovian and Gaussian classes. Recently, the author showed that one of the central concepts in the Gaussian case, local nondeterminism, could be usefully extended to a class of processes which includes a large class of Markov processes [3].

In the present work we again extend the methods of the Gaussian case to prove joint continuity of the local time for a class of processes which includes Markov processes with given transition densities. The most general known result for the joint continuity of the local time of a Markov process is that of Getoor and Kesten [9]. The hypothesis of their theorem is stated as a con-

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dition on the distribution of a first passage time. The latter distribution is, of course, determined by the transition distribution function. Our theorem is different from theirs in that the hypothesis is stated as explicit conditions on the transition density. The two sets of conditions are not strictly comparable because the relation between the transition density and the first passage time distribution is not known in a form sufficient to link the hypotheses.

We review the definition of local time. Let $X(t)$, $0 \leqq t \leqq 1$, be a real valued measurable function. (It is not necessarily the sample function of a stochastic process. Furthermore, the time domain $[0,1]$ may be replaced by an arbitrary measure space, and $X(t)$ may assume vector or more general values.) For each pair of linear Borel sets $A \subset (-\infty, \infty)$ and $I \subset [0,1]$, define $\nu(A, I)$ = Lebesgue measure $(s\colon X(s) \in A,\ s \in I)$. If, for fixed I, $\nu(\cdot, I)$ is absolutely continuous as a measure of sets A, then its Radon-Nikodym derivative, which we denote as $\alpha_I(x)$, is called the local time of X relative to I. It satisfies

$$\nu(A, I) = \int_A \alpha_I(x)\, dx. \tag{1.1}$$

If $X(t)$ is a stochastic process, then we say that the local time exists almost surely if it exists for almost all sample functions. In the particular case where $I = [0, t]$, for $0 \leqq t \leqq 1$, we put

$$\alpha(x, t) = \alpha_{[0,t]}(x). \tag{1.2}$$

The local time is said to be jointly continuous if $\alpha(x, t)$ is continuous in (x, t).

Our main result is Theorem 3.1 concerning the local time of a homogeneous Markov process with transition density $p(t; x, y)$. The following conditions on the latter function are sufficient for the almost sure joint continuity:

i) $p(t; x, y)$ is continuous in its variables on $t > 0$.

ii) For any real compact set K,

$$\lim_{\varepsilon \to 0} \sup_{x, y \in K} \int_0^\varepsilon p(t; x, y)\, dt = 0.$$

iii) For each x, there exists $\varepsilon > 0$ such that

$$\int_0^t p(s; x, x)\, ds = O(t^\varepsilon), \qquad \text{for } t \to 0.$$

iv) There exist positive real A and γ such that for every x and h and every $h' > 0$,

$$\int_0^{h'} |p(t; x, x+h) - p(t; x, x)|\, dt \leqq A |h\, h'|^\gamma,$$

$$\int_0^{h'} |p(t; x+h, x) - p(t; x, x)|\, dt \leqq A |h\, h'|^\gamma.$$

2. Preliminary Results and the Existence and Joint Continuity of the Local Time

Let $X(t)$, $0 \leqq t \leqq 1$, be a separable, measurable stochastic process. Assume that the finite-dimensional distributions are absolutely continuous with respect to

Lebesgue measure; and, for distinct $t_1, \dots, t_k$ in $[0,1]$, and for arbitrary real $x_1, \dots, x_k$, let $p(x_1, \dots, x_k; t_1, \dots, t_k)$ be the joint density of $X(t_1), \dots, X(t_k)$ at $(x_1, \dots, x_k)$. Define the function,

$$q(x_1, \dots, x_k) = \int_0^1 \dots \int_0^1 p(x_1, \dots, x_k; t_1, \dots, t_k)\, dt_1 \dots dt_k. \tag{2.1}$$

It is clear that p is a symmetric function of the pairs (x_i, t_i), so that q is a symmetric function of $x_1, \dots, x_k$. By an application of Fubini's theorem, we obtain

$$\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} q(x_1, \dots, x_k)\, dx_1 \dots dx_k = 1,$$

so that q is finite almost everywhere.

If the time domain $[0,1]$ in (2.1) is replaced by an arbitrary Borel subset I of $[0,1]$, then the corresponding function q is also a function of I:

$$q_I(x_1, \dots, x_k) = \int_I \dots \int_I p(x_1, \dots, x_k; t_1, \dots, t_k)\, dt_1 \dots dt_k. \tag{2.2}$$

We now state an extension of a result in [7] which seems to be known but has not been published. (I am indebted to N.R. Shieh for first bringing it to my attention.)

Lemma 2.1. *If, for some $k \geqq 2$, the function q_I in* (2.1) *is continuous on R^k, then the local time α_I exists almost surely, and*

$$E \prod_{i=1}^{k} \alpha_I(x_i) = q_I(x_1, \dots, x_k). \tag{2.3}$$

Proof. To simplify the typography, we put $I = [0,1]$. For every $\varepsilon > 0$, the definition (2.1) implies

$$\begin{aligned}(2\varepsilon)^{-1} &\int_0^1 \int_0^1 P(|X(s) - X(t)| \leqq \varepsilon)\, ds\, dt \\ &= (2\varepsilon)^{-1} \int_{R^{k-2}} \dots \int \Big\{ \iint_{|x_1 - x_2| \leqq \varepsilon} q(x_1, x_2, x_3, \dots, x_k)\, dx_1\, dx_2 \Big\}\, dx_3 \dots dx_k,\end{aligned}$$

which, by the continuity and integrability of q, has a finite limit for $\varepsilon \to 0$. Hence, by the result of Geman and Horowitz [7], Theorem 21.15, α exists almost surely.

The proof of (2.3) now follows by a familiar argument. Let $\chi(y)$ be the indicator function of $[-1,1]$, and define

$$\alpha_n(x) = \int_0^1 \frac{n}{2} \chi(n[X(t) - x])\, dt. \tag{2.4}$$

For arbitrary $x_1, \dots, x_k$ and positive integers $n_1, \dots, n_k$, we have

$$E \prod_{i=1}^{k} \alpha_{n_i}(x_i) = \int_{R^k} q(y_1, \dots, y_k) \prod_{i=1}^{k} \frac{n_i}{2} \chi(n_i(y_i - x_i)) \prod_{i=1}^{k} dy_i. \tag{2.5}$$

If x_i in (2.5) is restricted to a compact subset of the real line, then the integrand in (2.5) vanishes outside a compact subset of R^k, and q, by the hypothesis of continuity, is uniformly continuous on the subset. Hence the arguments of Geman and Horowitz [7], Sect. 25, imply that $\alpha_0(x) = \lim_{n\to\infty} \alpha_n(x)$ exists almost surely for almost all x in a compact subset; that $\alpha_0(x)$ is actually a version of $\alpha(x)$; and that the convergence of the right hand member of (2.5) to the right hand member of (2.3) for $n_1, \ldots, n_k \to \infty$ implies that the relation (2.3) holds for $\alpha_0(x)$ as well as for $\alpha(x)$.

In [1] we employed the classical Kolmogorov criterion to find sufficient conditions on the distributions of the process for the almost sure continuity of the local time as a function of the spatial variable. This was done in the context of Gaussian processes but the discussion could also have been extended to more general processes. Let $f(x_1, \ldots, x_k)$ be an arbitrary real valued function on R^k, and define the difference operator $\theta_{j,h}$ as

$$\theta_{j,h} f(x_1, \ldots, x_k) = f(x_1, \ldots, x_{j-1}, x_j + h, x_{j+1}, \ldots, x_k) - f(x_1, \ldots, x_j, \ldots, x_k).$$

Let $\alpha_I(x)$ be the local time; then, by Lemma 2.1,

$$E(\alpha_I(x+h) - \alpha_I(x))^k = \prod_{j=1}^{k} \theta_{j,h} q_I(x, \ldots, x). \tag{2.6}$$

This was used in [1] in applying the Kolmogorov criterion to α_I: The latter is almost surely continuous in x for fixed I if there exist $b > 0$ and $\varepsilon > 0$ such that

$$\left| \prod_{j=1}^{k} \theta_{j,h} q_I(x, \ldots, x) \right| \leqq b|h|^{1+\varepsilon}, \qquad \text{for } h > 0. \tag{2.7}$$

This criterion was extended in [2] in the Gaussian case to a condition for the joint continuity of the local time in the space and time variables, that is, the continuity of the function $\alpha(x, t)$ in (1.2): The sufficient conditions are, in addition to (2.7),

$$\left| \prod_{j=1}^{k} \theta_{j,h} q_I(x, \ldots, x) \right| \leqq b|h|^{1+\varepsilon} (\text{mes } I)^{1+\varepsilon} \tag{2.8}$$

for all x, I and h; and, for each x and I,

$$|q_I(x, \ldots, x)| \leqq b(\text{mes } I)^{1+\varepsilon} \tag{2.9}$$

where b and ε may depend on x. Pitt [12] and Geman and Horowitz [7] have used another criterion not strictly comparable to the one above. More recent work in this area is due to German, Horowitz and Rosen [8], Cuzick and Du Preez [4], and Ehm [6].

All previous work on the computation of the integral in the continuity criterion has been done in terms of the modulus of the joint characteristic function of the process. Such computations are possible for Gaussian processes and processes with stationary independent increments where the characteristic function is of a given explicit form. However, they are not possible for other important classes of processes such as Markov processes where the probability distributions are characterized by the transition density function rather than the characteristic function.

Note that (2.8) is stronger than (2.7). In the following theorem we give the precise conditions for a general process under which (2.8) and (2.9) form a set of sufficient conditions for the joint continuity of the local time.

Theorem 2.1. *If* $q_I(x_1, \ldots, x_k)$ *is continuous for some even* $k \geqq 2$ *and every subinterval* I, *then* (2.8) *and* (2.9) *form a set of sufficient conditions for the joint continuity of the local time.*

Proof. The assumptions permit the use of the formula (2.6). The proof is now exactly the same as in the Gaussian case considered in [2].

Theorem 2.2. *In verifying the conditions* (2.8) *and* (2.9), *it suffices to consider the function* q *in* (2.1) *as equal to* $k!$ *times the function*

$$\bar{q}(x_1, \ldots, x_k) = \int_{0 \leqq t_1 < \ldots < t_k \leqq 1} \cdots \int p(x_1, \ldots, x_k; t_1, \ldots, t_k)\, dt_1 \ldots dt_k. \tag{2.10}$$

Proof. By the remark following (2.1), the latter integral may be written as

$$\int_{0 \leqq t_1 < \ldots < t_k \leqq 1} \cdots \int \sum_\sigma p(x_{\sigma 1}, \ldots, x_{\sigma k}; t_1, \ldots, t_k)\, dt_1, \ldots, dt_k, \tag{2.11}$$

where the sum is over all permutations σ of $(1, \ldots, k)$. This implies that $q(x, \ldots, x) = k!\, \bar{q}(x, \ldots, x)$, so that $\bar{q}$ may be used in place of q in (2.9). Since the operators $\theta_{j,h}$ in (2.9) are linear and $\prod_{j=1}^{k} \theta_{j,h}$ is invariant under permutations, (2.11) implies that

$$\begin{aligned}
&\prod_{j=1}^{k} \theta_{j,h}\, q(x, \ldots, x) \\
&\quad = \int_{0 \leqq t_1 < \ldots < t_k \leqq 1} \cdots \int \sum_\sigma \sum_{j=1}^{k} p(x, \ldots, x; t_1, \ldots, t_k)\, dt_1 \ldots dt_k \\
&\quad = k! \prod_{j=1}^{k} \theta_{j,h}\, \bar{q}(x, \ldots, x).
\end{aligned}$$

3. Application to a Markov Process

Let $X(t)$, $0 \leqq t \leqq 1$, be a real valued homogeneous Markov process having a transition density function $p(t; x, y)$ representing the conditional density of $X(t)$ at the point y given $X(0) = x$. Let x_0 be an arbitrary real number representing the starting point of the process, and let the function p in formula (2.1) be the joint density of $X(t_1), \ldots, X(t_k)$. If $t_1 < \ldots < t_k$, then, by the Markov property, we have

$$p(x_1, \ldots, x_k; t_1, \ldots, t_k) = p(t_1; x_0, x_1)\, p(t_2 - t_1; x_1, x_2) \ldots p(t_k - t_{k-1}; x_{k-1}, x_k). \tag{3.1}$$

Therefore, the function $\bar{q}$ in (2.10) takes the form

$$\bar{q}_I(x_1, \ldots, x_k) = \int_{t_1<\ldots<t_k,\, t_1,\ldots,t_k \in I} \cdots \int p(t_1; x_0, x_1)\, p(t_2 - t_1; x_1, x_2) \ldots p(t_k - t_{k-1}; x_{k-1}, x_k)\, dt_1 \ldots dt_k. \tag{3.2}$$

Theorem 3.1. *Let $X(t)$, $0 \leqq t \leqq 1$, be a homogeneous Markov process on the real line with the transition density $p(t; x, y)$. Assume the following conditions:*

i) *The function $p(t; x, y)$ is jointly continuous for $t > 0$.*

ii) *For each compact real set K,*

$$\lim_{\varepsilon \to 0} \sup_{x, y \in K} \int_0^\varepsilon p(t; x, y)\, dt = 0. \tag{3.3}$$

iii) *For every x, there exists $\varepsilon > 0$ such that*

$$\int_0^t p(s; x, x)\, ds = O(t^\varepsilon), \quad \text{for } t \to 0. \tag{3.4}$$

iv) *There exist $A > 0$ and $\gamma > 0$ such that for every x and h, and every $h' > 0$,*

$$\int_0^{h'} |p(t; x, x+h) - p(t; x, x)|\, dt \leqq A |h\, h'|^\gamma$$

$$\int_0^{h'} |p(t; x+h, x) - p(t; x, x)\, dt \leqq A |h\, h'|^\gamma. \tag{3.5}$$

Then the local time exists and is jointly continuous, almost surely.

Proof. The continuity of the function q in (2.1) is implied by that of $\bar{q}$ in (2.10); this follows from (2.11). In the case of the function $\bar{q}_I$ in (3.2), the integrand is, by assumption i), continuous in all its variables over the domain of integration. The assumption ii) implies uniform integrability with respect to $t_1, \ldots, t_k$ near the boundary of the domain, where $|t_i - t_j|$ is small for some i and j. Therefore, the integral $\bar{q}_I$ in (3.2) is continuous in $(x_1, \ldots, x_k)$.

Next we verify condition (2.8). By Theorem 2.2 it suffices to consider the function $\bar{q}_I$ in (3.2). Let $k > 2$ be an arbitrary even positive integer, and write $\bar{q}_I$ as

$$q_I = \int_{\substack{t_1<\ldots<t_k \\ t_1,\ldots,t_k \in I}} \cdots \int p(t_1; x_0, x_1)\, p(t_k - t_{k-1}; x_{k-1}, x_k) \cdot \prod_{j=1}^{\frac{k}{2}-1} p(t_{2j} - t_{2j-1}; x_{2j-1}, x_{2j})\, p(t_{2j+1} - t_{2j}; x_{2j}, x_{2j+1})\, dt_1 \ldots dt_k.$$

Since the difference operator is linear, we may express $\prod_{j=1}^{k} \theta_{j,h} \bar{q}_I$ as

$$\left(\prod_{j=0}^{\frac{k}{2}-1} \theta_{2j+1,h} \cdot \theta_{k,h} \cdot \prod_{j=1}^{\frac{k}{2}-1} \theta_{2j,h} \right) \bar{q}_I$$

$$= \prod_{j=0}^{\frac{k}{2}-1} \theta_{2j+1,h} \cdot \theta_{k,h} \int \cdots \int p(t_1; x_0, x_1)\, p(t_k - t_{k-1}; x_{k-1}, x_k) \cdot \prod_{j=1}^{\frac{k}{2}-1} \theta_{2j,h}\, p(t_{2j} - t_{2j-1}; x_{2j-1}, x_{2j})\, p(t_{2j+1} - t_{2j}; x_{2j}, x_{2j+1})\, dt_1 \ldots dt_k. \tag{3.6}$$

Apply the difference operator $\theta_{2j,h}$ in the integrand in (3.6); then the product becomes

$$\prod_{j=1}^{\frac{k}{2}-1} \{p(t_{2j}-t_{2j-1}; x_{2j-1}, x+h)\, p(t_{2j+1}-t_{2j}; x+h, x_{2j+1}) \\ -p(t_{2j}-t_{2j-1}; x_{2j-1}, x)\, p(t_{2j+1}-t_{2j}; x, x_{2j+1})\}.$$

When we apply the operators θ for the variables of the remaining indices, we obtain for (3.6) a sum of $2^{k/2+1}$ terms of the form

$$\pm \underset{\substack{t_1<\ldots<t_k \\ t_1,\ldots,t_k\in I}}{\int\cdots\int} p(t_1; x_0, x_1)\, p(t_k-t_{k-1}; x_{k-1}, x_k) \\ \cdot \prod_{j=1}^{\frac{k}{2}-1} \{p(t_{2j}-t_{2j-1}; x_{2j-1}, x+h)\, p(t_{2j+1}-t_{2j}; x+h, x_{2j+1}) \\ -p(t_{2j}-t_{2j-1}; x_{2j-1}, x)\, p(t_{2j+1}-t_{2j}; x, x_{2j+1})\}\, dt_1 \ldots dt_k, \tag{3.7}$$

where the variables x_{2j+1}, $j=0, 1, \ldots, k/2$ assume the values x and $x+h$. We estimate the typical term (3.7). Its absolute value is at most equal to

$$\int_I p(s; x_0, x_1)\, ds \cdot \int_0^{h'} p(s; x_{k-1}, x_k)\, ds \\ \cdot \prod_{j=1}^{\frac{k}{2}-1} \int_0^{h'}\int_0^{h'} |p(s; x_{2j-1}, x+h)\, p(t; x+h, x_{2j+1}) - p(s; x_{2j-1}, x)\, p(t; x, x_{2j+1})|\, ds\, dt, \tag{3.8}$$

where h' = length of I, and $x_{2j+1}=x$ or $x+h$, $j=0, 1, \ldots, k/2-1$, $x_k=x$ or $x+h$.

To complete the proof it suffices to estimate the product of double integrals in (3.8). Put $i=2j$; then, by the triangle inequality, the ith factor in (3.8) is at most equal to the sum of the terms

$$\int_0^{h'}\int_0^{h'} p(s; x_{i-1}, x+h)\, |p(t; x+h, x_{i+1}) - p(t; x, x_{i+1})|\, ds\, dt \tag{3.9}$$

and

$$\int_0^{h'}\int_0^{h'} p(t; x, x_{i+1})\, |p(s; x_{i-1}, x+h) - p(s; x_{i-1}, x)|\, ds\, dt. \tag{3.10}$$

The double integral (3.9) is equal to

$$\int_0^{h'} p(s; x_{i-1}, x+h)\, ds \cdot \int_0^{h'} |p(t; x+h, x_{i+1}) - p(t; x, x_{i+1})|\, dt,$$

which, by assumptions i), ii) and iv), and the fact that $x_i=x$ or $x+h$, is at most equal to a constant times $|hh'|^{\gamma}$. The integral (3.10) has a similar estimate. From this we infer that the product in (3.8) is at most equal to a constant multiple of

$$|hh'|^{\gamma(k-3)}.$$

Since k may be taken arbitrarily large, it follows that condition (2.8) is satisfied if k is chosen so that $\gamma(k-3)>1$.

Finally, we verify condition (2.9). As before, we put $I=[a,b]$, and then $h'=b-a$. According to (3.2), the function $\bar{q}_I(x,\ldots,x)$ is at most equal to

$$\int_a^b p(t;x_0,x_1)\,dt\cdot\left(\int_0^{h'} p(t;x,x)\,dt\right)^{k-1}.$$

By assumptions i), ii) and iii), the expression above is of the order

$$|h'|^{\varepsilon(k-1)},$$

so that (2.9) is satisfied for k chosen so that $\varepsilon(k-1)>1$. The proof is now complete.

4. Examples and Comparisons with the Theorem of Getoor and Kesten

The hypothesis of Theorem 3.1 requires a specified smoothness of the transition density function in its variables (t,x,y) for $t\to 0$ and for $|x-y|\to 0$. The corresponding theorem of Getoor and Kesten in [9] requires a specified smoothness of the function

$$q(x,y)=1-E^x(e^{-T_y})\,E^y(e^{-T_x}) \tag{4.1}$$

for $|x-y|\to 0$, where T_z is the first passage time to z, and $E^z(\cdot)$ is expectation given $X(0)=z$.

Example 4.1. Let $X(t)$, $t\geqq 0$, be a process with real stationary independent increments. Let

$$E e^{iu(X(t)-X(s))}=e^{-(t-s)\Psi(u)},\qquad 0<s<t,$$

be the Lévy representation of the characteristic function. If

$$\int_{-\infty}^{\infty}|e^{-t\Psi(u)}|\,du<\infty$$

for $t>0$, then, by the inversion formula, we have

$$p(t;x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iu(y-x)-t\Psi(u)}\,du. \tag{4.2}$$

Standard calculations show that conditions i), ii) and iii) of Theorem 3.1 are valid if

$$\int_{-\infty}^{\infty}\frac{1-e^{-t\operatorname{Re}\Psi(u)}}{\operatorname{Re}\Psi(u)}\,du=O(t^{\varepsilon}),\qquad \text{for } t\to 0. \tag{4.3}$$

Assumption iv) is valid if

$$\int_{-\infty}^{\infty}|\sin uh|\,\frac{1-e^{-h'\operatorname{Re}\Psi(u)}}{\operatorname{Re}\Psi(u)}\,du=O(|hh'|^{\gamma}), \tag{4.4}$$

because by (4.2),

$$|p(t;x,x+h)-p(t;x,x)|$$
$$\leqq (2\pi)^{-1}\int_{-\infty}^{\infty}|e^{-iuh}-1|\,|e^{-t\Psi(u)}|\,du$$
$$=(2\pi)^{-1}\int_{-\infty}^{\infty}2|\sin\tfrac{1}{2}uh|\,|e^{-t\Psi(u)}|\,du.$$

It can be shown that the conditions hold for the symmetric stable process of index α, $1<\alpha\leqq 2$.

The conditions (4.3) and (4.4) are not comparable to the sufficient condition for the corresponding result of Getoor and Kesten in [9]. Put

$$\delta(u)=\sup_{|x|\leqq u}\frac{1}{\pi}\int_{-\infty}^{\infty}(1-\cos x\lambda)\,\mathrm{Re}\frac{1}{1+\Psi(\lambda)}\,d\lambda.$$

Their condition

$$\sum_{n=1}^{\infty}\{\delta(2^{-n}\}^{1/2}<\infty \tag{4.5}$$

is sufficient for joint continuity. The latter is a condition on the smoothness of the cosine transform of $\mathrm{Re}\,1/1+\Psi(u)$ at the origin. This is not strictly comparable to the conditions (4.3) and (4.4).

Example 4.2. Define

$$\hat{p}(s;x,y)=\int_0^{\infty}e^{-st}p(t;x,y)\,dt,\qquad s>0;$$

then, for a large class of Markov processes, the Laplace transform $E^x(e^{-sT_y})$ appearing in (4.1) can be expressed in terms of functions $\hat{p}$. For example, if $X(t)$ is a general diffusion process satisfying a backward equation, then $E^x(e^{-sT_y})$ and $\hat{p}(s;x,y)$ satisfy the same equation as functions of x. (See, for example, [13].) Thus the condition of Getoor and Kesten is stated in terms of $\hat{p}$ while our conditions in Theorem 3.1 are given in terms of p itself. On the one hand, the smoothness of the function $q(x,y)$ in (4.1) for $|x-y|\to 0$ requires an estimate of $|\hat{p}(1;x,y)-\hat{p}(1;y,y)|$. On the other hand, the corresponding hypothesis (iv) requires an estimate of

$$\int_0^{h'}|p(t;x,y)-p(t;y,y)|\,dt$$

for $h'\to 0$ and $|x-y|\to 0$. In general, one estimate cannot be derived from the other.

As an illustration of the difference of the computations needed to verify the respective hypotheses, let us consider a well known example to which both the Getoor-Kesten theorem and our Theorem 3.1 may be applied. Let $X(t)$ be the Ornstein-Uhlenbeck process, where

$$p(t;x,y)=[2\pi(1-e^{-2t})]^{-1/2}\exp\left[-\frac{(y-xe^{-t})^2}{2(1-e^{-2t})}\right]. \tag{4.6}$$

The conditions of Theorem 3.1 are verified by a direct computation with (4.6). In order to verify the hypothesis of [9], one first obtains $E^x(e^{-sT_y})$ in terms of solutions to the Laplace transform version of the backward equation. From this it follows that $E^x(e^{-sT_y})$ is a ratio of functions of the form $e^{x^2/4}D_{-s}(-x)$, where $D_v(z)$ is a parabolic cylinder function (see [13, 11], p. 323). The verification of the condition of [9] is then done in terms of the latter function.

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Note Added in Proof. The arguments in the proof of Theorem 3.1 can also be used to establish Hölder conditions for the local time in x and t. This was indicated to me by N.R. Shieh.