Z. Wahrscheinlichkeitstheorie verw. Gebiete
69, 23-35 (1985)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Continuity of Local Times for Lévy Processes

M.T. Barlow*

Statistical Laboratory, University of Cambridge, Cambridge, England CB2 1SB

1. Introduction

In this paper we shall obtain a sufficient condition for the continuity of the local time of a 1-dimensional Lévy process, which improves that of Getoor and Kesten [7].

Let X_t be a 1-dimensional Lévy process: that is a process with stationary independent increments. We denote by P^x the law of X starting at $x \in \mathbb{R}$, and by E^x expectation with respect to P^x. We write P, E for P^0, E^0. The characteristic function of X is given by

$$E e^{i\lambda X_t} = E e^{i\lambda(X_{t+s}-X_s)} = e^{-t\psi(\lambda)}, \tag{1.1}$$

where

$$\psi(\lambda) = -ia\lambda + \tfrac{1}{2}\sigma^2\lambda^2 - \int_{-\infty}^{\infty} \left(e^{i\lambda y} - 1 - \frac{i\lambda y}{1+y^2}\right) \nu(dy). \tag{1.2}$$

Here ν is a measure on $\mathbb{R}$ satisfying $\int (1 \wedge |y|^2)\,\nu(dy) < \infty$, $\nu(\{0\}) = 0$. We may, and shall, take a version of X_t which is right-continuous with left limits: X is then strongly Markovian.

We shall restrict ourselves to Lévy processes for which

$$0 \text{ is regular for } \{0\} \tag{1.3}$$

and for which

$$P^x(X_t = y \text{ for some } t \geqq 0) > 0 \quad \text{for all } x, y. \tag{1.4}$$

The first of these conditions ensures that, for each x, a local time L_t^x exists, while the second excludes the (in this context) uninteresting case when X is compound Poisson.

---

* This research was conducted while the author was a visiting scientist at the University of British Columbia

These conditions may be expressed in terms of a, σ^2 and ν: a necessary and sufficient condition for (1.3) and (1.4) to hold is that

$$\int_0^\infty \operatorname{Re} \frac{1}{1+\psi(\lambda)} d\lambda < \infty, \tag{1.5}$$

and

$$\text{either } \sigma^2 > 0, \quad \text{or } \int (|x| \wedge 1)\, \nu(dx) = \infty. \tag{1.6}$$

(See Bretagnolle [5], Kesten [9].)

If X satisfies (1.3) and (1.4) then a version of $L_t^x(\omega)$ exists which is jointly measurable in (x, t, ω) - see [2] or [3, V.3.41]. The normalisation of L is somewhat arbitrary; here we shall normalise L so that it is the occupation density of X; we then have

$$\int_0^t f(X_s)\, ds = \int_{-\infty}^{\infty} f(a)\, L_t^a\, da \qquad \text{for all bounded Borel } f. \tag{1.7}$$

Conditions which ensure that $(x, t) \to L_t^x$ can be chosen to be jointly continuous in (x, t) have been given by Trotter [16] (for Brownian motion), Boylan [4] (for a class of Markov processes), and by Getoor and Kesten [7], who also found conditions on X under which $(x, t) \to L_t^x$ could not be continuous. Millar and Tran [13] improved the latter result, showing that, under the same conditions, $L_t^{\cdot}$ is unbounded in every interval around X_0. There is a gap between the necessary and sufficient conditions for the continuity of L, which is illustrated by some examples in [7].

The key result of this paper is given in Sect. 2, where we obtain an estimate on the size of $|L^a - L^b|$, when a and b are close together, which improves the estimate in [3, V.3.28]. Feeding this estimate into Garsia's lemma (very much as in [7]) we obtain a sufficient condition for the joint continuity of L, and a modulus of continuity in x for L_t^x. If X is Brownian motion this modulus of continuity is a constant multiple of the correct modulus of continuity (see McKean [12], Ray [15], Knight [11]).

Theorem 1.1. *Let X_t be a right-continuous 1-dimensional Lévy process with a characteristic function given by* (1.1) *and* (1.2), *and satisfying* (1.3) *and* (1.4). *Let*

$$\delta_0(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} (1 - \cos \lambda x) \operatorname{Re} \frac{1}{1+\psi(\lambda)} d\lambda, \tag{1.8}$$

and

$$\delta(x) = \sup_{|u| \leq x} \delta_0(u). \tag{1.9}$$

Then if

$$\sum_{n=1}^{\infty} \delta(2^{-n})^{\frac{1}{2}} n^{-\frac{1}{2}} < \infty, \tag{1.10}$$

there exists a version of L_t^x which is jointly continuous in (x, t). Further, if

$$\rho(y) = \int_0^y (\log(1+u^{-2}))^{\frac{1}{2}}\, d\delta^{\frac{1}{2}}(u), \tag{1.11}$$

then ρ is finite, and for all $t \geqq 0$ there exists $\varepsilon_t(\omega)$ with $P^x(\varepsilon_t > 0)$ such that

$$|L_s^a - L_s^b| \leqq c(\sup_x L_t^x)^{\frac{1}{2}} \rho(|b-a|) \quad \text{for } 0 \leqq s \leqq t,\ |a-b| < \varepsilon_t(\omega), \text{ a.s.} \tag{1.12}$$

Remarks. 1. We may compare (1.10) with the condition of [7], which is $\sum_n \delta(2^{-n})^{\frac{1}{2}} < \infty$, and also with Boylan's condition $\sum_n n\delta(2^{-n})^{\frac{1}{2}} < \infty$.

2. In [13] Millar and Tran proved that, if

$$\limsup_{\alpha \to \infty} (\log \alpha) \int \operatorname{Re} \frac{1}{\alpha + \psi(\lambda)} d\lambda > 0, \tag{1.13}$$

then $P^0(\sup_{x \in \mathbb{Q}} L_t^x = +\infty) = 1$ for every $t > 0$. There is a small gap between (1.10) and (1.13), which will be illustrated by some examples in Sect. 4.

In the remainder of this section we introduce the notation which will be used in Sects. 2 and 3. X will always be a Lévy process satisfying (1.3) and (1.4). We set

$$T_x = \inf\{t \geqq 0 \colon X_t = x\} \quad \text{for } x \in \mathbb{R},$$

and

$$\psi^\alpha(x) = E^0 e^{-\alpha T_x} = E^y e^{-\alpha T_{x+y}} \quad \text{for } \alpha > 0.$$

It is well known (see Bretagnolle [5]) that, under (1.3) and (1.4), the potential kernel has a continuous density $u^\alpha(x)$, so that, for every bounded Borel f,

$$E^x \int_0^\infty e^{-\alpha s} f(X_s)\, ds = \int_{-\infty}^\infty u^\alpha(y-x) f(y)\, dy.$$

Using the density of occupation formula (1.7) we deduce that

$$u^\alpha(x) = E^0 \int_0^\infty e^{-\alpha s} dL_s^x, \tag{1.14}$$

and, applying the strong Markov property of X at time T_x, we have

$$u^\alpha(x) = \psi^\alpha(x) u^\alpha(0). \tag{1.15}$$

It is shown in [7] that

$$u^\alpha(x) + u^\alpha(-x) = \frac{1}{\pi} \int_{-\infty}^\infty \cos \lambda x \operatorname{Re} \frac{1}{\alpha + \psi(\lambda)} d\lambda, \tag{1.16}$$

from which it is immediate that

$$\delta_0(x) = 2u^1(0) - u^1(x) - u^1(-x). \tag{1.17}$$

The layout of the paper is as follows. In Sect. 2 we perform some preliminary calculations, and then go on to obtain the fundamental estimate on $|L^a - L^b|$, (Proposition 1.6, Lemma 1.7). The calculations with Garsia's lemma are made in Sect. 3, and in Sect. 4 we give applications of Theorem 1.1 to the range of a Lévy process (see [1]) and [10]), and to the modulus of continuity of the local time of a stable process. Finally, we describe a family of asymmetric processes, which includes the asymmetric Cauchy process, and which illustrate the gap between (1.10) and (1.13).

2. Preliminary Estimates

For a, b in $\mathbb{R}$ let

$$h(a,b)=h(b-a)=E^a L^a_{T_b}. \tag{2.1}$$

Lemma 2.1. (i) *For each* $\lambda>0$

$$E^a \int_0^{T_b} e^{-\lambda s}\, dL^a_s = u^\lambda(0)(1-\psi^\lambda(b-a)\,\psi^\lambda(a-b)). \tag{2.2}$$

(ii) $h(x) \geqq u^1(0)(1-\psi^1(x))\,\psi^1(-x)$ *for all* x.

Proof. From (1.14) and the strong Markov property,

$$\begin{aligned} u^\lambda(0) &= E^a \int_0^{T_b} e^{-\lambda s}\, dL^a_s + E^a e^{-\lambda T_b} E^b \int_0^\infty e^{-\lambda s}\, dL^a_s \\ &= E^a \int_0^{T_b} e^{-\lambda s}\, dL^a_s + \psi^\lambda(b-a)\,\psi^\lambda(a-b)\, u^\lambda(0), \end{aligned}$$

proving (i). Since $h(x)=E^0 \int_0^{T_x} dL^0_s \geqq E^0 \int_0^{T_x} e^{-s}\, dL^0_s$, (ii) is an immediate consequence of (i).

Corollary 2.2. $h(x)=h(-x)$.

Proof. From (2.2) we see that, for all $\lambda>0$,

$$E^a \int_0^{T_b} e^{-\lambda s}\, dL^a_s = E^b \int_0^{T_a} e^{-\lambda s}\, dL^b_s: \quad \text{the result follows on letting } \lambda \to 0.$$

We recall that $\psi^\lambda(x) \to 1$ as $x \to 0$ (see [5]).

Lemma 2.3. *Suppose that* $|x|$ *is small enough so that* $\psi^1(x) \geqq 3/4$. *Then*

$$h(x) \leqq 4u^1(0)(1-\psi^1(x)\,\psi^1(-x)). \tag{2.3}$$

Proof. Let $t>0$: then

$$\begin{aligned} \psi^1(x) &= E^0 e^{-T_x} 1_{(T_x \leqq t)} + E^0 e^{-T_x} 1_{(T_x > t)} \\ &\leqq P^0(T_x \leqq t) + e^{-t} P^0(T_x > t). \end{aligned}$$

Rearranging, we have that $P^0(T_x \leqq t) \geqq (1-e^{-t})^{-1}(\psi^1(x)-e^{-t})$, and, in particular, $P^0(T_x \leqq \log 2) \geqq \frac{1}{2}$.

Now $h(x)=E^0 L^0_{T_x \wedge t} + E^0(L^0_{T_x} - L^0_{T_x \wedge t})$, and

$$E^0(L^0_{T_x} - L^0_{T_x \wedge t}) = E^0 1_{(T_x > t)}(L^0_{T_x} - L^0_t) = E 1_{(T_x > t)} E^{X_t} L^0_{T_x}.$$

However, $E^y L^0_{T_x} \leqq E^0 L^0_{T_x}$ for any y, and so $E^0(L^0_{T_x} - L^0_{T_x \wedge t)}) \leqq P^0(T_x > t)\, h(x)$.

Thus

$$\begin{aligned} P^0(T_x \leqq t)\, h(x) &\leqq E^0 L^0_{T_x \wedge t} \\ &\leqq E^0 e^t \int_0^{T_x} e^{-s}\, dL^0_s \\ &= e^t u^1(0)(1 - \psi^1(x)\, \psi^1(-x)), \end{aligned}$$

by (2.2), and taking $t = \log 2$, we obtain (2.3).

Lemma 2.4. *Suppose that $|x|$ is sufficiently small so that $\psi^1(x) \geqq 3/4$. Then*

$$3/4\, \delta_0(x) \leqq h(x) \leqq 4\, \delta_0(x). \tag{2.4}$$

Proof. It is shown in Getoor and Kesten [7, p. 297] that

$$\delta_0(x) = u^1(0)(1 - \psi^1(x) + 1 - \psi^1(-x)).$$

Thus

$$\delta_0(x) = u^1(0)(1 - \psi^1(x)\, \psi^1(-x)) + u^1(0)(1 - \psi^1(x))(1 - \psi(-x)), \tag{2.5}$$

and combining this with (2.3) we obtain the right hand side of (2.4). Adding $(1 - \psi^1(x))$ to the term inside the final bracket of (2.5) we obtain

$$\delta_0(x) \leqq u^1(0)(1 - \psi^1(x)\, \psi^1(-x)) + (1 - \psi^1(x))\, \delta_0(x),$$

and from this and Lemma 2.1(ii) the left hand inequality of (2.4) is immediate.

Let a, b be fixed points in $\mathbb{R}$; we now proceed to obtain an estimate on the tail of the distribution of $L^a_{\cdot} - L^b_{\cdot}$. Set $h = h(a,b) = h(b,a)$.

Let Y_t be a symmetric Markov chain, with state space $\{a, b\}$, $Y_0 = a$, and Q-matrix given by $q_{ab} = q_{ba} = h$, $q_{aa} = q_{bb} = -h$. (If X is time-changed by the inverse of $L^a + L^b$, we shall see that the resulting process has the same law as Y killed at the time $L^a_\infty + L^b_\infty$.) For $x = a$, $x = b$, let $H^x_t = \int_0^t 1_{(Y_s = x)}\, ds$, and let τ_t be the right-continuous inverse of H^a: $\tau_t = \inf\{s : H^a_s > t\}$. Set $M_t = H^a_{\tau_t} - H^b_{\tau_t} = t - H^b_{\tau_t}$: it is easily seen that M is a martingale.

Let N be the number of excursions Y makes to b before time τ_t, and let Z_i be the length of the i^{th} excursion to b. Then $H^b_{\tau_t} = \sum_{i=1}^N Z_i$, N is Poisson with mean $h^{-1} t$, the Z_i are exponential with mean h, and $N, Z_1, Z_2, \ldots$ are independent (see, for example, Ito [17]). Therefore $E M_t^2 = E \sum_{i=1}^N Z_i^2 - t^2 = 2ht$. We require an estimate on the tail of the distribution of M_t: this could be done by direct calculation, but it will be simpler to use a Bernstein's inequality for martingales.

Lemma 2.5 (Theorem 1.6 and Proposition 3.3 [8]). *Let $\xi_1, \ldots, \xi_n$ be a sequence of martingale differences relative to a filtration $(\mathcal{G}_r)$. Suppose that there exist constants c, $\sigma^2 > 0$ such that*

(i) $\xi_i \leqq c$,

(ii) $\sum_{i=1}^{n} E(\xi_i^2 | \mathscr{G}_{i-1}) \leqq \sigma^2$.

Then, for all $x > 0$,

$$P\left(\max_{1 \leqq r \leqq n} \sum_{i=1}^{r} \xi_i > x\right) \leqq \exp\left(-\frac{x^2}{2\sigma^2} \psi\left(\frac{cx}{\sigma^2}\right)\right),$$

where $\psi(\lambda) = 2\lambda^{-2} \int_0^{\lambda} \log(1+t)\, dt$.

Lemma 2.6. For any $x > 0$, $t > 0$,

$$P(\sup_{s \leqq t} M_s > x) \leqq \exp\left(-\frac{x^2}{4ht}\right).$$

Proof. Let $n \geqq 1$, $\xi_i = M_{it/n} - M_{(i-1)t/n}$, and $\mathscr{G}_i = \sigma(M_s, s \leqq it/n)$. By the strong Markov property of Y, the ξ_i are actually independent, and so $\sum_{i=1}^{n} E(\xi_i^2 | \mathscr{G}_{i-1}) = EM_t^2 = 2ht$. As M_t makes only downward jumps $\xi_i \leqq t/n$, and, applying Lemma 2.5, we have

$$P\left(\max_{1 \leqq r \leqq n} \sum_{i=1}^{r} \xi_i > x\right) \leqq \exp\left(-\frac{x^2}{4ht} \psi\left(\frac{x}{2hn}\right)\right).$$

Let $n \to \infty$: then $\psi(x/2hn) \to 1$, and we deduce the desired result.

Now let $\tau_t = \tau_t^a$ be the right-continuous inverse of L_t^a.

Proposition 2.7. *For all* $x > 0$, $t > 0$, $y \in \mathbb{R}$,

$$P^y(\sup_{s \leqq \tau_t} (L_s^a - L_s^b) > x) \leqq \exp(-x^2/4th(a,b)). \tag{2.6}$$

Proof. Using the Strong Markov property of X at time T_a, it is clearly sufficient to prove (2.6) in the case $y = a$. Under P^a, $L_{T_b}^a$ is exponential with mean h - see Kesten [10]. It follows from this and the Strong Markov property of X that, if $\sigma_t = \inf\{s: L_s^a + L_s^b > t\}$, then X_{σ_t} is a Markov Chain killed at time $\zeta = L_\infty^a + L_\infty^b$ with state space $\{a, b\}$, and Q-matrix

$$Q' = \begin{pmatrix} -h(a,b) & h(a,b)\,p(a,b) \\ h(b,a)\,p(b,a) & -h(b,a) \end{pmatrix},$$

where $p(x,y) = P^x(T_y < \infty)$. Let Y', Y'' be independent copies of Y, but with $Y_0' = a$, $Y_0'' = b$, and let

$$Z_t = \begin{cases} X_{\sigma_t} & 0 \leqq t < \zeta \\ Y'_{t-\zeta} & \text{if } Z_{\zeta-} = b \\ Y''_{t-\zeta} & \text{if } Z_{\zeta-} = a. \end{cases}$$

Then Z has the same law as Y. Let N_t be the martingale obtained from Z in the same way that M was from Y: we have, for $t < L_\infty^a$, $L_{\tau_t}^a - L_{\tau_t}^b = N_t$. Then, by

Lemma 2.6,

$$P^a(\sup_{s\leqq\tau_t}(L^a_s-L^b_s)>x)=P^a(\sup_{u\leqq t}(L^a_{\tau_u}-L^b_{\tau_u})>x)$$
$$\leqq P^a(\sup_{u\leqq t} N_u>x)$$
$$\leqq \exp(-x^2/4th(a,b)).$$

The final form of our estimate on $L^a_\cdot - L^b_\cdot$ is the following.

Lemma 2.8. *For each $\lambda>0$, $x>0$, $y\in\mathbb{R}$*

$$P^y(\sup_{s\geqq 0}|\lambda\wedge L^a_s-\lambda\wedge L^b_s|>x)\leqq 2\exp\left(-\frac{x^2}{4\lambda h(a,b)}\right). \tag{2.7}$$

Proof. Suppose $\omega\in\{\sup_s(\lambda\wedge L^a_s-\lambda\wedge L^b_s)>x\}$. Then, for some s, $\lambda\wedge L^a_s(\omega)-\lambda\wedge L^b_s(\omega)>x$. As $\lambda\wedge L^a_s(\omega)\leqq\lambda$, we must have $L^b_s(\omega)<\lambda$, and so $\lambda\wedge L^a_s(\omega)>x+L^b_s(\omega)$. Hence, if $u=s\wedge\tau_\lambda(\omega)$, $L^a_u(\omega)=\lambda\wedge L^a_s(\omega)>x+L^b_s(\omega)\geqq x+L^b_u(\omega)$, and therefore $\omega\in\{\sup_{s\leqq\tau_\lambda}(L^a_s-L^b_s)>x\}$. (2.7) is now immediate from (2.6).

3. A Sufficient Condition for Continuity

We are now ready to obtain a modulus of continuity for L from the estimate (2.7) by using Garsia's lemma, which we use in the following form.

Lemma 3.1 (see Garsia, [6]). *Let p and Ψ be strictly increasing functions on $[0,\infty)$, such that $p(0)=\Psi(0)=0$, $\lim_{t\to\infty}\Psi(t)=\infty$, and Ψ is convex. Let $N>0$, and let f be a measurable function on $[-N,N]$ such that*

$$\int_{-N}^{N}\int_{-N}^{N}\Psi\left(\frac{|f(x)-f(y)|}{p(|x-y|)}\right)dx\,dy\leqq\Gamma<\infty.$$

Then for (Lebesgue) almost all x,y in $[-N,N]\times[-N,N]$,

$$|f(x)-f(y)|\leqq 8\int_0^{|x-y|}\Psi^{-1}(\Gamma u^{-2})\,dp(u).$$

Proof of Theorem 1.1. We begin by remarking that (1.10) implies that the function ρ defined in (1.11) is finite. For

$$\rho(1)=\sum_{n=1}^{\infty}\int_{2^{-n}}^{2\cdot 2^{-n}}(\log(1+u^{-2}))^{\frac{1}{2}}\,d\delta^{\frac{1}{2}}(u)$$
$$\leqq c\sum_n n^{\frac{1}{2}}(\delta^{\frac{1}{2}}(2\cdot 2^{-n})-\delta^{\frac{1}{2}}(2^{-n}))$$
$$\leqq c\sum_n((n+1)^{\frac{1}{2}}-n^{\frac{1}{2}})\,\delta^{\frac{1}{2}}(2^{-n})\leqq c\sum_n n^{-\frac{1}{2}}\,\delta^{\frac{1}{2}}(2^{-n})<\infty.$$

(In this calculation, and elsewhere, c denotes a universal constant, the value of which may change from line to line.)

Let $\Psi(x)=e^{x^2}-1$, so that $\Psi^{-1}(y)=(\log(1+y))^{\frac{1}{2}}$, and let

$$g(x)=\sup_{|u|<x} h(u), \qquad p(\lambda,x)=(8\lambda g(x))^{\frac{1}{2}}, \tag{3.1}$$

$$\phi(y)=\int_0^y \left(\log\left(1+\frac{1}{u^2}\right)\right)^{\frac{1}{2}} dg^{\frac{1}{2}}(u). \tag{3.2}$$

By Lemma 2.4, if $|x|$ is sufficiently small, $3/4\,\rho(x)\leqq\phi(x)\leqq 4\rho(x)$; and in particular ϕ is finite.

Set

$$Y(\lambda,a,b)=\sup_{s\geqq 0}|\lambda\wedge L_s^a-\lambda\wedge L_s^b|.$$

By (2.7)

$$\begin{aligned}P^x\left(\Psi\left(\frac{Y(\lambda,a,b)}{p(\lambda,|a-b|)}\right)>\alpha\right)&=P^x(Y(\lambda,a,b)\geqq p(\lambda,|a-b|)(\log(1+\alpha))^{\frac{1}{2}})\\&\leqq 2\exp\left(-\frac{\log(1+\alpha)\cdot 8\lambda g(|a-b|)}{4\lambda h(a,b)}\right)\\&\leqq 2\exp(-2\log(1+\alpha))=2(1+\alpha)^{-2}.\end{aligned}$$

Hence, integrating, we have

$$E^x\Psi\left(\frac{Y(\lambda,a,b)}{p(\lambda,|a-b|)}\right)\leqq 2. \tag{3.3}$$

Now let $N>1$ be fixed, let $I=[-N,N]$, and for $\lambda>0$ let

$$\Gamma_\lambda=\int_I\int_I \Psi\left(\frac{Y(\lambda,a,b)}{p(\lambda,|a-b|)}\right)da\,db;$$

then, by (3.3) and Fubini,

$$E^x\Gamma_\lambda\leqq 2N^2. \tag{3.4}$$

So, for each λ, $P^x(\Gamma_\lambda<\infty)=1$, and hence, throwing out a P^x-null set, $\Gamma_\lambda(\omega)<\infty$ for all $\lambda\in\mathbb{Q}_+$, $\omega\in\Omega$. Now for any $t\geqq 0$, $\lambda\in\mathbb{Q}_+$,

$$\int_I\int_I \Psi\left(\frac{|\lambda\wedge L_t^a-\lambda\wedge L_t^b|}{p(\lambda,|a-b|)}\right)da\,db\leqq\Gamma_\lambda<\infty,$$

and therefore, using Garsia's lemma, with $p(\lambda,\cdot)$ for $p(\cdot)$, there exists a set $A(t,\lambda,\omega)\subset I\times I$ such that $|A(t,\lambda,\omega)|=0$ and, for $(a,b)\in(I\times I)\cap A^c(t,\lambda,\omega)$,

$$|\lambda\wedge L_t^a-\lambda\wedge L_t^b|\leqq 8\int_0^{|a-b|}\Psi^{-1}(u^{-2}\Gamma_\lambda)\,d_u p(\lambda,u). \tag{3.5}$$

The right-hand side of (3.5) is

$$\begin{aligned}&8\int_0^{|a-b|}\left(\log\left(\frac{u^2+\Gamma_\lambda}{u^2+1}\cdot\frac{u^2+1}{u^2}\right)\right)^{\frac{1}{2}}(8\lambda)^{\frac{1}{2}}dg^{\frac{1}{2}}(u)\\&\quad\leqq 8^{\frac{3}{2}}\lambda^{\frac{1}{2}}[(\log(1\vee\Gamma_\lambda))^{\frac{1}{2}}g^{\frac{1}{2}}(|a-b|)+\phi(|a-b|)]<\infty.\end{aligned}$$

Following Getoor-Kesten [7], we define, for all (x,t,ω),

$$\bar{L}_t^x = \limsup_{m\to\infty} \frac{m}{2} \int_{x-1/m}^{x+1/m} L_t^y\, dy. \tag{3.6}$$

Let $I_0 = [-N+1, N-1]$: using (3.5) we have

$$|\lambda \wedge \bar{L}_t^a - \lambda \wedge \bar{L}_t^b| \leqq c\lambda^{\frac{1}{2}}(\phi(|b-a|) + (g(|b-a|)\log(1 \vee \Gamma_\lambda))^{\frac{1}{2}})$$
$$\text{for all } t \geqq 0,\ (a,b) \in I_0 \times I_0,\ \lambda \in \mathbb{Q}_+. \tag{3.7}$$

Now (see [7]), $P^z(\bar{L}_t^x = L_t^x) = 1$ for all (x,t), so that $\bar{L}^x$ is a version of L^x.

Let

$$F = \{\omega\colon \bar{L}_s^a(\omega) = L_s^a(\omega) \text{ for all } s \in \mathbb{Q}_+,\ a \in \mathbb{Q}, \text{ and } s \to L_s^a(\omega) \text{ is continuous for all } a \in \mathbb{Q}\}.$$

Then $P^z(F) = 1$, and it follows from (3.7) that, on F, $(x,t) \to \lambda \wedge \bar{L}_t^x$ is continuous on $I_0 \times \mathbb{R}_+$ for $\lambda \in \mathbb{Q}_+$. To conclude that $\bar{L}_t^x$ is continuous, we need to show that $\sup_{x\in I_0} \bar{L}_t^x < \infty$.

From (3.7), setting $b = 0$, we have

$$\lambda \wedge \bar{L}_t^a \leqq \lambda \wedge \bar{L}_t^0 + \lambda^{\frac{1}{2}}[c\phi(N) + cg^{\frac{1}{2}}(N)(\log(1 \vee \Gamma_\lambda))^{\frac{1}{2}}] \quad \text{for all } a \in I_0.$$

Hence, on $\{\sup_{x\in I_0} \bar{L}_t^x > \lambda\}$,

$$\lambda \leqq \lambda \wedge \bar{L}_t^0 + \lambda^{\frac{1}{2}}(c\phi(N) + cg(N)^{\frac{1}{2}}(\log(1 \vee \Gamma_\lambda))^{\frac{1}{2}}).$$

Now if $y^2 \leqq A + By$, then $y^2 \leqq 2A + B^2$, and so, on $\{\sup_{x\in I_0} \bar{L}_t^x > \lambda\}$,

$$2\bar{L}_t^0 + c(\phi(N) + g(N)^{\frac{1}{2}}(\log(1 \vee \Gamma_\lambda))^{\frac{1}{2}})^2 \geqq \lambda.$$

Thus

$$P(\sup_{x\in I_0} \bar{L}_t^x > \lambda) \leqq P(\bar{L}_t^0 > \lambda/4) + P\left(\Gamma_\lambda \geqq \exp\left(\frac{\lambda}{4c\,g(N)} - \frac{\phi(N)^2}{g(N)}\right)\right)$$
$$\leqq P(\bar{L}_t^0 > \lambda/4) + (1 + 2N^2)\, e^{\phi(N)^2/g(N)}\, e^{-\lambda/4cg(N)}.$$

Integrating, we have

$$E \sup_{x\in I_0} \bar{L}_t^x \leqq 4E\bar{L}_t^0 + cN^2\, h(N)^{-1}\, e^{\phi(N)^2/g(N)}, \tag{3.8}$$

so that, in particular, $\sup_{x\in I_0} \bar{L}_t^x < \infty$ P^z-a.s. Taking λ greater than this supremum, it follows that $(x,t) \to \bar{L}_t^x$ is jointly continuous on $[-N+1, N-1] \times \mathbb{R}_+$ for any $N \geqq 1$, and hence on $\mathbb{R} \times \mathbb{R}_+$.

To conclude the proof of Theorem 1.1 it remains to establish the modulus of continuity (1.12). We may choose $\varepsilon > 0$ small enough so that $\phi(u) \leqq 4\rho(u)$ for $0 < u < \varepsilon$. Now let $t > 0$, let $N = N(\omega)$ be large enough so that $|X_s| \leqq N$ for $0 \leqq s \leqq t$, let $\lambda \in \mathbb{Q}_+$ be such that $\sup_x L_t^x \leqq \lambda \leqq 2\sup_x L_t^x$; then if $\varepsilon_t(\omega)$ is chosen small enough so that $(g(\varepsilon_t(\omega))\log(1 \vee \Gamma_\lambda(\omega)))^{\frac{1}{2}} \leqq \phi(\varepsilon_t(\omega))$, (1.12) follows immediately from (3.7).

4. Applications and Examples

1. Range of a Lévy Process

Let $R_t(\omega)=\{X_0(\omega), X_{t-}(\omega)\}\cup\{X_s(\omega), X_{s-}(\omega), 0<s<t\}$. For $t>0$ R_t is a.s. a closed perfect set of positive Lebesgue measure, and in [1] it was shown that either, with probability 1, R_t is nowhere dense, or else, with probability 1, $R_t=cl(\text{int}(R_t))$ and R_t contains an open interval around X_0. In [10] Kesten gave examples of symmetric Lévy processes for which R_t is nowhere dense, and proved that if for some $\varepsilon>0$ $\delta(2^{-n})<n^{-1-\varepsilon}$, then R_t contains an open interval around X_0 a.s. Since (as is shown in [10]) if $L^{\cdot}$ is continuous then R_t must contain an open interval around X_0, Theorem 1.1 gives immediately a slight improvement on Kesten's result.

Corollary 4.1. *If* $\sum_n \delta(2^{-n})^{\frac{1}{2}} n^{-\frac{1}{2}}<\infty$, *then* R_t *contains an interval around* X_0, *a.s.*

Remark. In Pruitt and Taylor [14] it is proved that the asymmetric, but not completely asymmetric, Cauchy process has a nowhere dense range.

2. Modulus of Continuity in Space of the Local Time of a Stable Process

The symmetric stable process of index α, where $0<\alpha\leqq 2$, is a Lévy process for which $Ee^{i\lambda X_t}=e^{-t|\lambda|^\alpha}$. For $0<\alpha<2$ this process arises by taking $\nu(dx)=c_\alpha|x|^{-1-\alpha}dx$, for $\alpha=2$ the process is a multiple of Brownian motion. For $1<\alpha\leqq 2$ these processes have local times, which Boylan [4] proved are jointly continuous in (x,t).

Using the notation of Theorem 1.1, it is not hard to see that

$$\delta_0(x)\sim c_\alpha|x|^{\alpha-1} \quad \text{as } x\to 0,$$

while $\rho(y)\leqq c_\alpha\left(y^{\alpha-1}\log\frac{1}{y}\right)^{\frac{1}{2}}$ for sufficiently small y. Hence, by (1.12), for all $t\geqq 0$ there exists $\varepsilon_t(\omega)$ with $P(\varepsilon_t>0)=1$ such that

$$|L_s^a-L_s^b|\leqq c_\alpha(\sup_x L_t^x)^{\frac{1}{2}}\left(|b-a|^{\alpha-1}\log\frac{1}{|b-a|}\right)^{\frac{1}{2}}$$
$$\text{for } 0\leqq s\leqq t,\quad |b-a|<\varepsilon_t(\omega). \tag{4.1}$$

Here c_α is a constant depending only on α.

Remarks. 1. The modulus of continuity which follows from the results in [7] is $c|b-a|^{(\alpha-1)/2}\log|b-a|^{-1}$.

2. In the case of Brownian motion the exact modulus of continuity of $L_t^{\cdot}$ is known (see [12] and [15]); (4.1) is only worse than this exact modulus by a constant factor.

3. A Family of Asymmetric Lévy Processes, Including Some Critical Cases

Let $a=\sigma^2=0$, and

$$\nu(dx)=x^{-2}g(1/x)(p\,1_{(x>0)}+q\,1_{(x<0)}), \tag{4.2}$$

where $p,\ q>0,\ p+q=1$, and where

$$g(y)=(\log|y|)^{\alpha}(\log\log|y|)^{\beta}\,1_{(|y|>e)}, \qquad \alpha,\beta\in\mathbb{R}. \tag{4.3}$$

This family of processes includes the asymmetric Cauchy ($\alpha=\beta=0$, $p\neq q$), and contains both processes with continuous local time, and processes with unbounded local time. We shall examine the continuity of $L^{\cdot}_{\cdot}$ for these processes, but as some of the calculations of $\psi(\lambda)$ and $\delta_0(x)$ are rather tedious, we shall just present an outline. We shall denote unimportant constants whose values depend only on α,β by c, c', and use the notation $a(x)\approx b(x)$ as $x\to\infty$ to mean that $a(x)/b(x)$ is bounded away from 0 and ∞ as $x\to\infty$.

We begin by checking (1.6):

$$\begin{aligned}\int_{-1}^{1}|x|\,\nu(dx)&=\infty \qquad \text{if } \alpha>-1, \text{ or } \alpha=-1 \text{ and } \beta\geqq -1\\ &<\infty \qquad \text{otherwise.}\end{aligned} \tag{4.4}$$

We restrict our attention to the first case: in the second the paths of X have finite variation, and X does not have a local time.

From (1.2) we have

$$\psi(\lambda)\sim c|\lambda|\,g(\lambda)+ic'(p-q)\,\lambda(\log|\lambda|)\,g(\lambda) \qquad \text{as } |\lambda|\to\infty. \tag{4.5}$$

Note that $\operatorname{Im}\psi(\lambda)\gg\operatorname{Re}\psi(\lambda)$ if $p\neq q$: the asymmetry plays an important role. It follows immediately that as $|\lambda|\to\infty$

$$\begin{aligned}\operatorname{Re}\frac{1}{1+\psi(\lambda)}&\approx[|\lambda|(\log|\lambda|)^2 g(\lambda)]^{-1} \qquad \text{if } p\neq q\\ &\approx[|\lambda|\,g(\lambda)]^{-1} \qquad \text{if } p=q\end{aligned} \tag{4.6}$$

Many aspects of the detailed sample-path behaviour of X are governed by the behaviour of $\operatorname{Re}(1+\psi(\lambda))^{-1}$ as $\lambda\to\infty$: we see that the symmetric process with parameter α behaves in a similar fashion to the asymmetric process of index $\alpha-2$. From now on we take $p>q$. Checking (1.5),

$$\begin{aligned}\int\operatorname{Re}\frac{1}{1+\psi(\lambda)}\,d\lambda&<\infty \qquad \text{if } \alpha>-1, \text{ or } \alpha=-1,\ \beta>1\\ &=\infty \qquad \text{otherwise.}\end{aligned} \tag{4.7}$$

From (4.4), (1.5) and (1.6), we see that, in the first case, X has a local time: we restrict ourselves to this case.

We now consider the condition (1.13):

$$\int \mathrm{Re}\frac{1}{y+\psi(\lambda)}\,d\lambda \geqq c\int_y^\infty \frac{d\lambda}{\lambda(\log\lambda)^2\, g(\lambda)}$$
$$\approx c(\log y)^{-1-\alpha}(\log\log y)^{-\beta}.$$

Hence

$$\liminf_{y\to\infty} \log y \int \mathrm{Re}\frac{1}{y+\psi(\lambda)}\,d\lambda > 0 \qquad \text{if } \alpha<0, \text{ or } \alpha=0,\ \beta\leqq 0,$$

and in these cases, by [13], $L_t^{\cdot}$ is unbounded.

We now check (1.10): with a little work we find that

$$\delta_0(x) \approx \frac{1}{(\log 1/x)\, g(1/x)} \qquad \text{as } x\to 0,$$

so that $\delta(2^{-n}) \approx n^{-\alpha-1}(\log n)^{-\beta}$, and

$$\sum_{n=1}^{\infty} \delta(2^{-n})^{\frac{1}{2}} n^{-\frac{1}{2}} < \infty \qquad \text{if } \alpha>0, \text{ or } \alpha=0,\ \beta>2$$
$$= \infty \qquad \text{otherwise.}$$

In the first case $L_{\cdot}^{\cdot}$ is jointly continuous by Theorem 1.1. The cases $\alpha=0$, $0<\beta\leqq 2$ are not covered by either (1.10) or (1.13), so that the behaviour of $L_t^{\cdot}$ is unknown. We conjecture that in these cases L is unbounded.

We may summarise the properties of L for these processes as follows: recall that we are taking $p>q$.

$\alpha<-1$, $\alpha=-1$, $\beta<-1$	paths of finite variation, no local time;
$\alpha=-1$, $-1\leqq\beta\leqq 1$	paths of infinite variation, no local time;
$\alpha=-1$, $\beta>1$; $-1<\alpha<0$; $\alpha=0$, $\beta\leqq 0$	local time exists, unbounded in every interval by (1.13);
$\alpha=0$, $0<\beta\leqq 2$	local time exists, other properties unknown;
$\alpha=0$, $\beta>2$; $\alpha>0$	local time continuous by (1.10).

Acknowledgement. I wish to thank E. Perkins and S.J. Taylor for a number of conversations on this topic, and Charles Goldie for telling me about the martingale version of Bernstein's inequality. Especially, I wish to thank R. Bass for his interest in this work, and for a number of very helpful comments.

References

1. Barlow, M.T.: Zero-one laws for the excursions and range of a Lévy process. Z. für Wahrscheinlichkeitstheorie verw. Gebiete **55**, 149–163 (1981)
2. Blumenthal, R.M., Getoor, R.K.: Local times for Markov processes. Z. für Wahrscheinlichkeitstheorie verw. Gebiete **3**, 50–74 (1964)

3. Blumenthal, R.M., Getoor, R.K.: Markov processes and potential theory. New York: Academic Press 1968
4. Boylan, E.S.: Local times for a class of Markov processes. Ill. J. Math. **8**, 19–39 (1964)
5. Bretagnolle, J.: Resultats de Kesten sur les processus à accroisements indépendants. Sem. de Probability V, 21–36, Lect. Notes in Math. **191**. Berlin, Heidelberg, New York: Springer 1971
6. Garsia, A.M.: Continuity properties of multi-dimensional Gaussian processes. 6th Berkeley Symposium on Math. Stat. Probability, Vol. 2, 369–374, Berkeley (1970)
7. Getoor, R.K., Kesten, H.: Continuity of local times for Markov processes. Comp. Math. **24**, 277–303 (1972)
8. Freedman, D.A.: On tail probabilities for martingales. Ann. Probability **3**, 100–118 (1975)
9. Kesten, H.: Hitting probabilities of single points for processes with stationary independent increments. Memoir no. 93, Amer. Math. Soc. (1969)
10. Kesten, H.: Lévy processes with a nowhere dense range. Indiana Univ. Math. J. **25**, 45–64 (1976)
11. Knight, F.: Random walks and the sojourn density process of Brownian motion. TAMS **107**, 56–86 (1963)
12. McKean, H.P., jr.: A Hölder condition for Brownian local time. J. Math. Kyoto, **1-2**, 195–201 (1962)
13. Millar, P.W., Tran, L.T.: Unbounded local times. Z. für Wahrscheinlichkeitstheorie verw. Gebiete **30**, 87–92 (1974)
14. Pruitt, W.E., Taylor, S.J.: The local structure of the sample paths of asymmetric Cauchy processes. To appear
15. Ray, D.B.: Sojourn times of a diffusion process. Ill. J. Math., **1**, 615–630 (1963)
16. Trotter, H.F.: A property of Brownian motion paths. Ill. J. Math. **2**, 425–433 (1958)
17. Itô, K.: Poisson point processes attached to Markov processes. 6th Berkeley Symp. on Math. Stat. Probability, Vol. **3**, 225–239 (1971)

Received February 10, 1984