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Martingales and Stochastic Integrals for Processes with a Multi-Dimensional Parameter

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1. Introduction

In this paper our interest is to develop a stochastic calculus of the Ito type for multi-parameter processes. The experience with stochastic integrals in one dimension makes it clear that the Ito calculus is a calculus of continuous-parameter martingales and local martingales [5, 7]. Thus, a useful generalization of the stochastic integral must necessarily involve a generalization of the martingale property to multidimensional parameter spaces. From this point of view, it is natural to consider martingales as random functions parameterized by subsets of R^n rather than points in R^n. Set inclusion provides a partial ordering in terms of which the martingale property can be defined in a natural way.

An important example of martingales with a partially ordered parameter is the following generalization of Brownian motion. Let $\bar{\mathscr{R}}^n$ denote the collection of all Borel sets in R^n having finite Lebesgue measure. Let $\{X_A, A\in\bar{\mathscr{R}}^n\}$ be a real Gaussian additive random set function with

$$EX_A=0, \qquad EX_A X_B=\mathscr{L}(A\cap B) \tag{1.1}$$

where $\mathscr{L}$ denotes the Lebesgue measure. Intuitively, X_A can be thought of as the integral over A of a Gaussian white noise. We note that for $n=1$ $X_{[0,t]}$ is just the ordinary Brownian motion. In the multidimensional case the process

$$W_{(z_1, z_2, \ldots, z_n)}=X_{[0,z_1]\times[0,z_2]\times\cdots\times[0,z_n]} \tag{1.2}$$

is a sample-continuous process provided a separable version is chosen, and the probability measure that it induces on $C([0,1]^n)$ generalizes the Wiener measure. The process defined by (1.2), which we shall call *Wiener process*, has been studied by a number of authors [6, 8, 9]. In particular, results of the Cameron-Martin type on absolutely continuous affine transformation of the Wiener measure have been obtained [8, 10].

Martingales with a partially ordered parameter are not new. Cairoli [1, 2], in particular, has considered martingales in a context very similar to ours, and has introduced a stochastic integral with respect to a two-parameter Wiener process, which will be referred to as integrals of the first type in this paper. We shall show that for Wiener processes with a two-dimensional parameter, stochastic integrals of a second type are necessary both for the purpose of representing

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Wiener functionals and martingales and for the development of a two-dimensional stochastic calculus. It should be noted that although Cairoli derived a differentiation formula for Wiener processes with a two-dimensional parameter, it involves differentials in one parameter at a time and hence is really a differentiation formula of one dimension.

In Section 2 we briefly introduce martingales with a partially ordered parameter and show that Gaussian white noise in general, and the Wiener process in particular, have a natural interpretation as martingales. In Section 3 we focus on the parameter space $T=[0,1]^n$ together with a partial ordering defined by componentwise ordering. By introducing increasing paths in T, we can study martingales and local martingales with the aid of the theory of one-parameter martingales. In Section 4, we consider functions $f(M_z, z)$ of a path independent martingale M_z which are themselves martingales. Using a differentiation formula for such functions on increasing paths we prove an important characterization of Wiener processes. We shall also prove that every square-integrable Wiener functional is expressible as $f(W_{\mathbf{1}}, \mathbf{1})$ $(\mathbf{1}=(1,1,\dots,1))$ for some f of this class. In Section 5, we introduce stochastic integrals of the first and second types and consider some of their properties. In Section 6, we shall develope a differentiation formula for two-parameter Wiener processes using the two types of stochastic integrals, and prove that every functional of a two-parameter Wiener process admits a representation in terms of integrals of these two types.

2. Martingales

Let $\mathscr{S}$ be a *directed set*. That is, $\mathscr{S}$ is a nonempty set partially ordered by a binary relation $\prec$ satisfying the condition that for every pair x, y in $\mathscr{S}$ there is a $z \in \mathscr{S}$ such that $x \prec z$ and $y \prec z$. Let $(\Omega, \mathscr{A}, \mathscr{P})$ be a probability space. A collection of *sub-σ-fields* $\{\mathscr{A}_s, s \in \mathscr{S}\}$ is said to be increasing if $s_1 \succ s_2 \Rightarrow \mathscr{A}_{s_1} \supseteq \mathscr{A}_{s_2}$. Given a family of random variables $\{X_s, s \in \mathscr{S}\}$ and an increasing collection $\{\mathscr{A}_s, s \in \mathscr{S}\}$, we shall say $\{X_s, \mathscr{A}_s, s \in \mathscr{S}\}$ is a *martingale* if $s \succ s_0$ implies

$$E^{\mathscr{A}_{s_0}} X_s = X_{s_0}, \qquad \text{almost surely}. \tag{2.1}$$

Let μ be a σ-finite Borel measure on R^n. Let $\overline{\mathscr{R}^n}$ denote the collection of all Borel sets of R^n which are μ-finite. Let $\{X_s, s \in \overline{\mathscr{R}^n}\}$ be a real Gaussian additive set function with $EX_s = 0$ and

$$EX_s X_{s'} = \mu(s \cap s'). \tag{2.2}$$

If we take $\mathscr{S}$ to be any subcollection of $\overline{\mathscr{R}^n}$ which is a directed set with respect to set inclusion, and take $\mathscr{A}_s$ to be the σ-field generated by $\{X_{s'}, s' \subseteq s\}$, then $\{X_s, \mathscr{A}_s, s \in \mathscr{S}\}$ is a martingale. More generally, we can take $\{\mathscr{A}_s, s \in \mathscr{S}\}$ to be any increasing collection such that X_{s_0} is $\mathscr{A}_s$-measurable if $s_0 \subseteq s$, and $\mathscr{A}_s$-indepedent if s_0 and s are disjoint. It is customary to refer to $\{X_s, s \in \overline{\mathscr{R}^n}\}$ as a Gaussian white noise. Thus, we see that a Gaussian white noise has a natural interpretation as a martingale. From (1.2) it is easy to see that the Wiener process $W_z, z \in R^n_+$, has a natural interpretation as a martingale with respect to the partial ordering defined by: $z \succ z' \Leftrightarrow z_i \geq z'_i$ for every i.

3. Martingales on Increasing Paths

Consider the parameter space $T=[0,1]^n$ together with the partial ordering

$$z \succ z' \Leftrightarrow z_i \geqq z'_i, \qquad i=1,2,\ldots,n.$$

We define a *path* in T as a continuous function θ: $[0,1] \to T$. We shall say a path is *increasing* if $\alpha > \beta \Rightarrow \theta(\alpha) \succ \theta(\beta)$. Let $\{M_z, \mathscr{F}_z, z \in T\}$ be a martingale and θ an increasing path. Clearly, $\{M_{\theta(t)}, \mathscr{F}_{\theta(t)}, t \in [0,1]\}$ is a one-parameter martingale. Therefore, a multi-parameter martingale defines a one-parameter martingale on every increasing path. Conversely, an n-parameter process which is a one-parameter martingale on every increasing path is a martingale. This is because if $z \succ z'$ then we can take the path $\theta(t)=z'+(z-z')t$ and find

$$E^{\mathscr{F}_{z'}} M_z = E^{\mathscr{F}_{\theta(0)}} M_{\theta(1)} = M_{\theta(0)} = M_{z'}.$$

The characterization of n-parameter martingales as one-parameter martingales on increasing paths allows one to make use of results in one-parameter martingale theory.

Let $\{M_z, \mathscr{F}_z, z \in T\}$ be a martingale such that almost all sample functions are continuous. Then for every increasing path θ, $\{M_{\theta(t)}, \mathscr{F}_{\theta(t)}, 0 \leqq t \leqq 1\}$ is a sample continuous martingale. As such, it is necessarily locally square integrable, and there exists a unique continuous increasing function A_t such that $M^2_{\theta(t)} - A_t$ is an $\mathscr{F}_{\theta(t)}$-local martingale [7]. We shall say $\{M_z, \mathscr{F}_z, z \in T\}$ is *path-independent* if A_1 is the same for all increasing paths θ having the same pair of endpoints $\theta(0)$ and $\theta(1)$. For a path-independent martingale M we can define a function $\langle M, M\rangle_z, z \in T$ as the increasing function A_1 for all paths connecting the points $\theta(0)=\mathbf{0}$ and $\theta(1)=z$. It will then follow that $\{M_z^2 - \langle M, M\rangle_z, \mathscr{F}_z, z \in T\}$ is a martingale if $EM_z^2 < \infty$ and otherwise a local martingale. Here, we define a multi-parameter *local martingale* as a process which is a local martingale on every increasing path. We can call $\langle M, M\rangle$ the *increasing process* of M, since $z \succ z' \Rightarrow \langle M, M\rangle_z \geqq \langle M, M\rangle_{z'}$. Conversely, a sample continuous martingale M is necessarily path-independent if we can find an increasing process $\langle M, M\rangle$ such that $M^2 - \langle M, M\rangle$ is a local martingale. It is easy to verify that a Wiener process is a path-independent martingale with

$$\langle W, W\rangle_z = \text{Volume}(\zeta \prec z). \tag{3.1}$$

Let $M_z = (M_{1z}, M_{2z}, \ldots, M_{mz})$ be a set of sample continuous local martingales so that each linear combination $\sum \alpha_i M_{iz}$ is a path-independent local martingale. Since both $M_{iz} + M_{jz}$ and $M_{iz} - M_{jz}$ are path-independent, we can define

$$\langle M_i, M_j\rangle_z = 1/4\left[\langle M_i + M_j, M_i + M_j\rangle_z - \langle M_i - M_j, M_i - M_j\rangle_z\right]. \tag{3.2}$$

For such an M a differentiation formula can be established almost immediately by using the differentiation rule for one-parameter martingales on increasing paths.

Let $f(u, z)$, $u \in R^m$, $z \in T$, be a real or complex valued function, having continuous mixed second partials with respect to the components of u and a continuous gradient with respect to z. We adopt the notation

$$f^i(u,z)=\frac{\partial f(u,z)}{\partial u_i},\quad f^{ij}(u,z)=\frac{\partial^2 f(u,z)}{\partial u_i\,\partial u_j},\quad \nabla f(u,z)=\operatorname{grad}_z f(u,z).$$

Let $\theta(t)$, $0\leqq t\leqq 1$, be an increasing path. Since $M_{i\theta(t)}$, $0\leqq t\leqq 1$, are one-parameter continuous local martingales the familiar differentiation formula of Ito and Kunita-Watanabe [7] yields

$$\begin{aligned} f(M_{\theta(t)},\theta(t))-f(M_{\theta(0)},\theta(0)) &= \sum_i \int_0^t f^i(M_{\theta(s)},\theta(s))\,dM_{i\theta(s)} \\ &\quad + \int_0^t \Big[\tfrac{1}{2}\sum_{i,j} f^{ij}(u,z)\,\nabla\langle M_i,M_j\rangle_z+\nabla f(u,z)\Big]_{\substack{z=\theta(s)\\ u=M_{\theta(s)}}}\cdot d\theta(s). \end{aligned} \tag{3.3}$$

Eq. (3.3) can be expressed in a simpler and more suggestive way as

$$\operatorname{grad} f(M_z,z)=\sum_i f^i(M_z,z)\,\nabla M_z+\tfrac{1}{2}\sum_{i,j} f^{ij}(M_z,z)\,\nabla\langle M_i,M_j\rangle_z+\nabla f(M_z,z). \tag{3.4}$$

4. Wiener Integral and Hermite Functionals

Let $\{W_z, z\in T=[0,1]^n\}$ be a Wiener process and let $\{\mathscr{F}_z, z\in t\}$ be an increasing family of σ-fields such that $\{W_z, \mathscr{F}_z, z\in T\}$ is a martingale. Let $L^2(T)$ denote the collection of all real valued functions ϕ on T such that $\int_T \phi^2(\zeta)\,d\zeta<\infty$. For $\phi\in L^2(T)$ the Wiener integral [6] $\int_T \phi(\zeta)\,W(d\zeta)$ is well defined by the following conditions:

(a) $\int_T \phi(\zeta)\,W(d\zeta)=W_z$ if ϕ is the indicator function of $\prod_{i=1}^n [0,z_i)$,

(b) $\int_T [a\phi(z)+b\psi(z)]\,W(d\zeta)=a\int_T \phi(\zeta)\,W(d\zeta)+b\int_T \psi(\zeta)\,W(d\zeta)$,

(c) $\int_T [\phi_n(\zeta)-\phi(\zeta)]^2\,d\zeta \xrightarrow[n\to\infty]{} 0 \Rightarrow \int_T \phi_n(\zeta)\,W(d\zeta)\xrightarrow[n\to\infty]{\text{q.m.}}\int_T \phi(\zeta)\,W(d\zeta)$.

The Wiener integral has the important properties:

$$E\Big[\int_T \phi(\zeta)\,W(d\zeta)\,|\,\mathscr{F}_z\Big]=\int_{\zeta\prec z}\phi(\zeta)\,W(d\zeta), \tag{4.1}$$

$$E\Big\{\Big[\int_T \phi(\zeta)\,W(d\zeta)\Big]^2\,|\,\mathscr{F}_z\Big\}=\int_{\zeta\prec z}\phi^2(\zeta)\,d\zeta. \tag{4.2}$$

If we define $M_z=\int_{\zeta\prec z}\phi(\zeta)\,W(d\zeta)$, then $\{M_z,\mathscr{F}_z, z\in T\}$ is a martingale. Furthermore, M is sample-continuous provided that a separable version is chosen. If we define

$$M_{iz}=\int_{\zeta\prec z}\phi_i(\zeta)\,W(d\zeta),\qquad i=1,2,\dots,m, \tag{4.3}$$

then

$$\langle M_i,M_j\rangle z=\int_{\zeta\prec z}\phi_i(\zeta)\,\phi_j(\zeta)\,d\zeta=V_{ij}(z) \tag{4.4}$$

so that any linear combination $\sum_{i=1}^{m} \alpha_i M_{iz}$ is a path-independent martingale. Eq. (3.3) now becomes

$$\begin{aligned} f(M_{\theta(t)}, \theta(t)) - f(M_{\theta(0)}, \theta(0)) = \sum_i \int_0^t f^i(M_{\theta(s)}, \theta(s))\, dM_{i\theta(s)} \\ + \int_0^t \left[\tfrac{1}{2} \sum_{i,j} f^{ij}(u, z)\, \nabla V_{ij}(z) + \nabla f(u,z)\right]_{\substack{z=\theta(s) \\ u=M_{\theta(s)}}} \cdot d\theta(s). \end{aligned} \tag{4.5}$$

It is clear that if $f(u, z)$ satisfies the equation

$$\tfrac{1}{2} \sum_{i,j} f^{ij}(u, z)\, \nabla V_{ij}(z) + \nabla f(u, z) = 0, \tag{4.6}$$

then $f(M_z, z)$ is necessarily a local martingale. Therefore, the class of f satisfying (4.6) is of some interest.

Let $f_\alpha(u, z)$ be defined by

$$f_\alpha(u, z) = \exp\left\{i \sum_j u_j \alpha_j + \tfrac{1}{2} \sum_{j,k} \alpha_j \alpha_k V_{jk}\right\}. \tag{4.7}$$

It is easy to verify that f_α satisfies (4.6) for every $\alpha \in R^m$. Therefore, any linear combination of elements of the family $\{f_\alpha, \alpha \in R^m\}$ satisfies (4.6) and so does the limit of a suitably convergent sequence of linear combinations. Specific examples include the polynomials

$$f_k(u, z) = (-i)^{k_1+k_2+\cdots+k_m} \left[\frac{\partial^{k_1+k_2+\cdots+k_m}}{\partial\alpha_1^{k_1}\, \partial\alpha_2^{k_2} \ldots \partial\alpha_m^{k_m}} f_\alpha(u, z)\right] \Bigg|\, \alpha_j = 0, \quad 1 \leqq j \leqq m, \tag{4.8}$$

and

$$f(u, z) = \int_{R^m} f_\alpha(u, z)\, d\mu(\alpha)$$

where μ is any Borel measure satisfying

$$\int_{R^m} \|\alpha\|^2 \, |f_\alpha(u, z)|\, d\mu(\zeta) < \infty, \qquad u \in R^m, \; z \in T.$$

A celebrated result of Cameron and Martin [3] states that every square-integrable functional of a Wiener process $\{W_z, z \in T\}$ can be represented in a series of Hermite functionals, a Hermite functional being a product of the form

$$\prod_{\nu=1}^{m} H_{p_\nu} \int_T \phi_\nu(\zeta)\, W(d\zeta)$$

where H_p are Hermite polynomials.

Theorem 4.1. *For every Hermite functional there exists a function $f(u, z)$, $u \in R^m$, $z \in T$, such that*

$$\begin{aligned} \prod_{\nu=1}^{n} H_{p_\nu}\left(\int_T \phi_\nu(\zeta)\, W(d\zeta)\right) &= f(M_{\mathbf{1}}, \mathbf{1}) \\ &= \sum_i \int_0^1 f^i(M_{\theta(s)}, \theta(s))\, dM_{i\theta(s)} + f(\mathbf{0}, \mathbf{0}) \end{aligned} \tag{4.10}$$

where θ is any increasing path connecting $\mathbf{0}$ *and* $\mathbf{1}$, *and*

$$M_{\nu z} = \int_{\zeta \prec z} \phi_\nu(\zeta)\, W(d\zeta).$$

Proof. For each $p=(p_1, p_2, \ldots, p_m)$, $\prod_{\nu=1}^{m} H_{p_\nu}(u_\nu)$ is a polynomial in $u=(u_1, u_2, \ldots, u_m)$ of degree p. Therefore, we can write

$$\prod_{\nu=1}^{n} H_{p_\nu}(u_\nu) = \sum_{k \leqq p} \beta_{pk} f_k(u, \mathbf{1})$$

where f_k satisfy (4.8). It follows that there is a function

$$f(u, z) = \sum_{k \leqq p} \beta_{pk} f_k(u, z)$$

satisfying (4.8) such that

$$\prod_{\nu=1}^{m} H_{p_\nu}(u_\nu) = f(u, \mathbf{1}).$$

Eq. (4.10) follows immediately from (4.5).

5. Stochastic Integrals

Let $T=[0,1]^n$ and let $\{W_z, \mathscr{F}_z, z \in T\}$ be a Wiener process. Integrals of the form

$$I_1(\phi) = \int_T \phi(z)\, W(dz) \tag{5.1}$$

where ϕ is a random function satisfying appropriate conditions can be defined in a way so as to generalize the Ito integral. Cairoli [2] has done this for $n=2$, and extension to arbitrary n poses no problem. We shall refer to integrals of the form (5.1) as stochastic integrals of the first type. The definition and some of the properties of these integrals are summarized below.

Let $\phi(\omega, z)$ satisfy the following conditions.

H_1: $\phi(\omega, z)$ is a bimeasurable function of (ω, z) with respect to $\mathscr{F} \otimes \mathscr{S}$ where $\mathscr{S}$ denotes the σ-algebra of Borel sets in T.

H_2: For each $z \in T$, ϕ_z is $\mathscr{F}_z$-measurable.

H_3: $\int_T E\phi_z^2 < \infty$.

Suppose ϕ is *simple*, i.e., $\phi(\omega, z) = \phi_\nu(\omega)$, $z \in \Delta_\nu$, $\nu = 1, 2, \ldots, k$, and $\phi = 0$ elsewhere, where Δ_ν are disjoint rectangles

$$\Delta_\nu = \prod_{i=1}^{n} [a_i^\nu, b_i^\nu).$$

Then we define

$$I_1(\phi) = \sum_\nu \phi_\nu \Delta_\nu W$$

where we have used ΔW to denote the white noise integral over a rectangle Δ. That is, if $\Delta = \prod_{i=1}^{n} [a_i, b_i)$, then

$$\Delta W = \sum_z (-1)^{\Pi(z)} W_z$$

where the sum is taken over the 2^n vertices $\{z\colon z_i=a_i, b_i\}$ and $\Pi(z)$ is the number of b_i's in z. The definition of I_1 is then extended to ϕ satisfying H_1-H_3 by a standard completion argument. Principal properties of I_1 include the following,

(a) linearity: $I_1(\alpha\phi+\beta\psi)=\alpha I_1(\phi)+\beta I_1(\psi)$,

(b) inner product: $EI_1(\phi)\cdot I_1(\psi)=\int\limits_T \phi_z\psi_z\,dz$,

(c) martingale: $E(I_1(\phi)|\mathscr{F}_z)=\int\limits_{\zeta\prec z}\phi(\zeta)\,W(d\zeta)$.

It is clear that I_1 is a straightforward generalization of the Ito integral. Henceforth, we shall adopt the rather obvious notation $\int\limits_0^z$ instead of $\int\limits_{\zeta\prec z}$.

Stochastic integrals of a second type, to be denoted by

$$I_2(\psi)=[\int\limits_T\int\limits_T]\,\psi(z_1,z_2)\,W(dz_1)\,W(dz_2)$$

will turn out to be necessary if a full-fledged multi-parameter stochastic calculus is to be developed. For motivation consider the following simple example. Let

$$X_z=W_z^2-\int\limits_0^z d\zeta=W_z^2-\prod_{i=1}^n z_i$$

which is easily seen to be a martingale. To obtain a stochastic integral representation for X, subdivide the rectangle $\prod\limits_{i=1}^n[0,z_i)$ by a rectangular subdivision $\{\Delta_\nu\}$. We shall use $\nu\succ\mu$ to denote $z\succ z'$ for all $z\in\Delta_\nu$ and $z'\in\Delta_\mu$. We use $\Delta_\nu W$ to denote the white noise integral over Δ_ν (i.e., $\Delta_\nu W=\int\limits_{\Delta_\nu}W(dz)$) so that $W_z=\sum\limits_\nu\Delta_\nu W$ and

$$X_z=(\sum_\nu\Delta_\nu W)^2-\prod_{i=1}^n z_i=\sum_\nu[(\Delta_\nu W)^2-\int\limits_{\Delta_\nu}d\zeta]+2\sum_\nu\sum_{\mu\prec\nu}\Delta_\mu W\Delta_\nu W$$
$$+\sum_{\nu,\mu\text{ unordered}}\Delta_\nu W\Delta_\mu W.$$

As $\int\limits_{\Delta_\nu}d\zeta\to 0$, the first sum goes to zero, the second sum converges to $2\int\limits_0^z W_\zeta\,W(d\zeta)$ which is an integral of the first type, and the last will converge to a stochastic integral of the second type $\left[\int\limits_0^z\int\limits_0^z\right]W(d\zeta)\,W(d\zeta')$. Hence, we will have a representation

$$X_z=2\int\limits_0^z W_\zeta\,W(d\zeta)+\left[\int\limits_0^z\int\limits_0^z\right]W(d\zeta)\,W(d\zeta').$$

We now turn to the definition of the integral of the second type.

For two points z and z' in T we use $z\vee z'$ to denote

$$(\max(z_1,z_1'),\max(z_2,z_2'),\dots,\max(z_n,z_n')).$$

Let G denote the subset of $T\times T$ consisting of pairs of points (z,z') which are unordered, and let h_G denote the indicator function of this set, so that

$$\begin{aligned} h_G(z,z')&=1 && \text{if } z,z' \text{ are unordered}\\ &=0 && \text{if } z,z' \text{ are ordered}.\end{aligned}$$

Let $\psi(\omega, z, z')$ be a random function defined on $\Omega \times T \times T$ satisfying:

H'_1: $\psi(\omega, z, z')$ is jointly measurable with respect to $\mathscr{F} \otimes \mathscr{S} \otimes \mathscr{S}$.

H'_2: for each pair z, z' the function $\psi(\omega, z, z')$ is measurable with respect to $\mathscr{F}_{z \vee z'}$.

H'_3: $E \int_T \int_T \psi^2(z, z')\, dz, dz' < \infty$.

Let ψ be a simple function, i.e., there exist rectangles Δ_1 and Δ_2 such that

$$\begin{aligned} \psi(\omega, z, z') &= \alpha(\omega), \qquad z \in \Delta_1,\ z' \in \Delta_2 \\ &= 0 \qquad\qquad \text{otherwise}. \end{aligned}$$

First, assume that $\Delta_1 \times \Delta_2 \subset G$. Then $I_2(\psi)$ is defined by

$$I_2(\psi) = \alpha \Delta_1 W \Delta_2 W.$$

Without the assumption $\Delta_1 \times \Delta_2 \subset G$, we define $I_2(\psi)$ as follows. Let there be an ε-lattice defined on T. For each $z \in T$ there is a largest lattice point $[z]^\varepsilon$ satisfying $[z]^\varepsilon \prec z$, because if a and b are lattice points then $a \vee b$ is again a lattice point. We now define an approximation to $I_2(\psi)$ by

$$I_2^\varepsilon(\psi) = \sum_{i^\varepsilon, j^\varepsilon} \psi(i^\varepsilon, j^\varepsilon)\, h_G(i^\varepsilon, j^\varepsilon)\, \Delta_i \varepsilon W \Delta_j \varepsilon W$$

where the summation is taken over all lattice points, or equivalently, because of h_G, over all unordered pairs of lattice points. If ε_2 is a subpartition of ε_1 then

$$E(I_2^{\varepsilon_1} - I_2^{\varepsilon_2})^2 = E\alpha^2 \int_T \int_T \left[h_G([z]^{\varepsilon_1}, [z']^{\varepsilon_1}) - h_G([z]^{\varepsilon_2}, [z']^{\varepsilon_2})\right]^2 dz\, dz'.$$

It follows from the result of the appendix that $E(I_2^{\varepsilon_1} - I_2^{\varepsilon_2})^2$ converges to zero as ε_1 and $\varepsilon_2 \to 0$. Therefore, I_2^ε converges in quadratic mean and we can define

$$I_2(\psi) = \lim_{\varepsilon \to 0} \text{ in q.m. } I_2^\varepsilon(\psi).$$

It is easy to verify that this definition is consistent with the definition for $\Delta_1 \times \Delta_2 \subset G$.

We can now extend the definition of $I_2(\psi)$ to all functions satisfying $H'_1 - H'_3$ by approximating ψ by linear combinations of simple functions in the usual way. It follows that $I_2(\psi)$ is defined for all ψ satisfying $H'_1 - H'_3$ and inherits from $I_2(\psi)$ the following properties:

(a) *linearity:* $\quad I_2(a\psi + b\psi') = a I_2(\psi) + b I_2(\psi')$,

(b) *off diagonal:* $\quad I_2(\psi) = I_2(h_G \psi)$,

(c) *inner product:* $\quad E I_2(\psi) I_2(\psi') = E \int_T \int_T h_G(z, z')\, \psi(z, z')\, \psi'(z, z')\, dz\, dz'$,

(d) *orthogonality:* $\quad E I_1(\phi) I_2(\psi) = 0$,

(e) *martingale:* $\quad E[I_2(\psi) | \mathscr{F}_z] = \left[\int_0^z \int_0^z\right] \psi(\zeta, \zeta')\, W(d\zeta)\, W(d\zeta')$.

These properties are easily verified for $I_2^\varepsilon(\psi)$ and extended to $I_2(\psi)$ by standard arguments.

Cairoli [1] has proved that if M_z, $z \in T=[0,1]^n$, is a martingale then there exist constants $A_{p,n}$ such that

$$E\big(\sup_{z\in T} |M_z|^p\big) \leqq A_{p,n} \sup_{z\in T} (E\,|M_z|^p).$$

Using this inequality for $p=2$, we can easily show that separable versions of the martingales

$$M_z = \int_0^z \phi_\zeta = W(d\zeta), \qquad X_z = \left[\int_0^z \int_0^z\right] \psi_{\zeta,\zeta'}\, W(d\zeta)\, W(d\zeta')$$

are sample continuous. Proofs follow exactly the same line as the one-dimensional case, viz., by showing that M and X are a.s. uniform limits of sample continuous martingales, and will not be repeated here.

Both I_1 and I_2 can be extended to integrands ϕ and ψ which do not satisfy H_3 and H_3' but instead satisfy the conditions

$$\int_T \phi_z^2\, dz < \infty \qquad \text{a.s.}$$

$$\int_T \int_T \psi^2(z,z')\, dz\, dz' \qquad \text{a.s.}$$

The extension is by approximating ϕ (resp. ψ) by a sequence of bounded functions $\phi_n(\psi_n)$ converging almost surely to ϕ (resp. ψ) at every point z (resp. every pair (z,z')). I_1 and I_2 can then be defined as

$$I_1(\phi) = \lim_{n\to\infty} \text{in prob. } I_1(\phi_n), \qquad I_2(\psi) = \lim_{n\to\infty} \text{in prob. } I_2(\psi_n).$$

So defined, I_1 and I_2 retain most of the properties, except that they need not be square integrable and need not have the martingale property.

We will show in the next section that for the *two-parameter* case stochastic integrals of the first and second types are *complete* in the sense that they suffice for the representation of every Wiener functional. For higher dimensional parameters this is not the case, which suggests that stochastic integrals of other types need to be defined for the general n-parameter case. As yet, it is not clear to us how this should be done.

6. Representation of Two-Parameter Wiener Functionals

We begin with an elementary differentiation formula. Let $\{W_z, \mathscr{F}_z, z\in[0,1]^2\}$ be a two-parameter Wiener process, and define a martingale

$$M_z = (M_{1z}, M_{2z}, \ldots, M_{mz})$$

by the Wiener integrals

$$M_{\nu z} = \int_0^z \phi_\nu(\zeta)\, W(d\zeta) \tag{6.1}$$

where ϕ_ν are non-random functions in $L^2([0,1]^2)$. We shall derive a differentiation formula for $f(M_z, z)$ where f is a function satisfying

$$\tfrac{1}{2} \sum_{i,j} f^{ij}(u,z)\, \nabla V_{ij}(z) + \nabla f(u,z) = 0 \tag{6.2}$$

where

$$V_{ij}(z)=\int_0^z \phi_i(\zeta)\,\phi_j(\zeta)\,d\zeta.$$

We note that (6.2) is identical to (4.6) so that $f(M_z, z)$ is a local martingale which has a representation on any increasing path θ given by

$$f(M_{\theta(t)},\theta(t))-f(M_{\theta(0)},\theta(0))=\sum_i\int_0^t f^i(M_{\theta(s)},\theta(s))\,dM_{i\theta(s)}. \tag{6.3}$$

Theorem 6.1. *Let f satisfy* (6.2) *and have continuous mixed partial derivatives with respect to the components of u through the third order. Then,*

$$\begin{aligned} &f[M(z_1,z_2),(z_1,z_2)]-f[M(z_1,0),(z_1,0)]-f[M(0,z_2),(0,z_2)]+f[M(0,0),(0,0)]\\ &\quad=\int_0^z\sum_i f^i(M_\zeta,\zeta)\,\phi_i(\zeta)\,W(d\zeta)\\ &\qquad+\tfrac{1}{2}\left[\int_0^z\int_0^z\right]\sum_{i,j} f^{ij}(M_{\zeta\vee\zeta'},\zeta\vee\zeta')\,\phi_i(\zeta)\,\phi_j(\zeta')\,W(d\zeta)\,W(d\zeta'). \end{aligned} \tag{6.4}$$

Proof. It is clear that we only need to prove (6.4) for the case $z=(1,1)$, since the general case follows from the martingale property of both sides. Now, let the unit square T be partitioned by a sequence of square subdivisions. It is convenient to take the squares to be of the same size (say δ_k) in each partition and we assume $\delta_k \xrightarrow[k\to\infty]{} 0$. We can order the lattice points of each partition in some arbitrary way and denote them by $z_{k\nu}=(x_{k\nu}, y_{k\nu})$. We can now write

$$\begin{aligned} &f(M(1,1),(1,1))-f(M(1,0),(1,0))-f(M(0,1),(0,1))+f(M(0,0),(0,0))\\ &\quad=\sum_\nu\{f(M(x_{k\nu}+\delta_k, y_{k\nu}+\delta_n),(x_{k\nu}+\delta_k, y_{k\nu}+\delta_k))\\ &\qquad-f(M(x_{k\nu}+\delta_k, y_{k\nu}),(x_{k\nu}+\delta_k, y_{k\nu}))-f(M(x_{k\nu}, y_{k\nu}+\delta_k),(x_{k\nu}, y_{k\nu}+\delta_k))\\ &\qquad+f(M(x_{k\nu}, y_{k\nu}),(x_{k\nu}, y_{k\nu}))\}. \end{aligned}$$

Since f satisfies (6.2), we can use (6.3) for the bracketed terms and write

$$\begin{aligned} &f(M(1,1),(1,1))-f(M(1,0),(1,0))-f(M(0,1),(0,1))+f(M(0,0),(0,0))\\ &\quad=\sum_\nu\sum_i\int_0^1\{f^i[M(x_{k\nu}+\delta_k, y_{k\nu}+s\delta_k),(x_{k\nu}+\delta_k, y_{k\nu}+s\delta_k)]\\ &\qquad\cdot M_i(x_{k\nu}+\delta_k, y_{k\nu}+\delta_k ds)\\ &\qquad-f^i[M(x_{k\nu}, y_{k\nu}+s\delta_k),(x_{k\nu}, y_{k\nu}+s\delta_k)]\cdot M_i(x_{k\nu}, y_{k\nu}+\delta_k ds)\}\\ &\quad=\sum_\nu\sum_i\int_{y_{k\nu}}^{y_{k\nu}+\delta_k}\{f^i[M(x_{k\nu}+\delta_k, y),(x_{k\nu}+\delta_k, y)]\,M_i(x_{k\nu}+\delta_\nu, dy)\\ &\qquad-f^i[M(x_{k\nu}, y),(x_{k\nu}, y)]\,M_i(x_{k\nu}, dy)\}. \end{aligned}$$

Because of the forward-difference nature of one-parameter stochastic integrals, we can write

$$f(M(1,1),(1,1))-f(M(1,0),(1,0))-f(M(0,1),(0,1))+f(M(0,0),(0,0))$$
$$=\lim_{k\to\infty}\text{in prob.}\sum_{v}\sum_{i}\{f^i[M(x_{kv}^+,y_{kv}),(x_{kv}^+,y_{kv})]\,[M_i(x_{kv}^+,y_{kv}^+)-M_i(x_{kv}^+,y_{kv})]$$
$$-f^i[M(x_{kv},y_{kv}),(x_{kv},y_{kv})]\,[M_i(x_{kv},y_{kv}^+)-M_i(x_{kv},y_{kv})]\}$$

where $x_{kv}^+=x_{kv}+\delta_k$ and $y_{kv}^+=y_{kv}+\delta_k$.

Rearranging terms and using (6.3) for the difference

$$f^i[M(x_{kv}^+,y_{kv}),(x_{kv}^+,y_{kv})]-f^i[M(x_{kv},y_{kv}),(x_{kv},y_{kv})]$$
$$=\int_{x_{kv}}^{x_{kv}+\delta}\sum_j f^{ij}[M(x,y_{kv}),(x,y_{kv})]\,M_j(dx,y_{kv})$$

we find

$$f[M(1,1),(1,1)]-f[M(1,0),(1,0)]-f[M(0,1),(0,1)]+f[M(0,0),(0,0)]$$
$$=\lim_{k\to\infty}\text{in prob.}\{\sum_{v}\sum_{i}f^i[M(x_{kv},y_{kv}),(x_{kv},y_{kv})]\,\Delta_{kv}M_i$$
$$+\sum_{v}\sum_{i,j}f^{ij}[M(x_{kv},y_{kv}),(x_{kv},y_{kv})]\cdot(\delta_{kv}^1M_i)(\delta_{kv}^2M_j)\}$$

where we have adopted the notations

$$\Delta_{kv}M_i=M_i(x_{kv}^+,y_{kv}^+)-M_i(x_{kv},y_{kv}^+)-M_i(x_{kv}^+,y_{kv})+M_i(x_{kv},y_{kv}),$$
$$\delta_{kv}^1M_i=M_i(x_{kv}^+,y_{kv})-M_i(x_{kv},y_{kv}),$$
$$\delta_{kv}^2M_i=M_i(x_{kv},y_{kv}^+)-M_i(x_{kv},y_{kv}).$$

From (6.1) we have $\Delta_{kv}M_i\sim\phi_i(x_{kv},y_{kv})\,\Delta_{kv}W$. Therefore,

$$\lim_{k\to\infty}\text{in prob.}\sum_{v}f^i[M(x_{kv},y_{kv}),(x_{kv},y_{kv})]\,\Delta_{kv}M_i=\int_T f^i(M_\zeta,\zeta)\,\phi_i(\zeta)\,W(d\zeta).$$

Now, we observe that for any function g

$$\sum_{v}\sum_{\substack{\mu\\ v\neq\mu}}g(x_{kv}\vee x_{k\mu},y_{kv}\vee y_{k\mu})\,\Delta_{kv}M_i\,\Delta_{k\mu}M_j=\sum_{v}g(x_{kv},y_{kv})[\delta_{kv}^1M_i\,\delta_{kv}^2M_j+\delta_k^2M_i\delta_{kv}^1M_j].$$

Since $f^{ij}=f^{ji}$, we have

$$\lim_{k\to\infty}\text{in prob.}\sum_{i,j}\sum_{v}f^{ij}[M(x_{kv},y_{kv}),(x_{kv},y_{kv})]\,(\delta_{kv}^1M_i)(\delta_{kv}^2M_j)$$
$$=\tfrac{1}{2}\lim_{k\to\infty}\text{in prob.}\sum_{i,j}\sum_{v\neq\mu}f^{ij}[M(z_{kv}\vee z_{k\mu}),z_{kv}\vee z_{k\mu}]\,\Delta_{kv}M_i\,\Delta_{k\mu}M_j$$
$$=\tfrac{1}{2}\Big[\int_{T\times T}\Big]f^{ij}(M_{\zeta\vee\zeta'},\zeta\vee\zeta')\,\phi_i(\zeta)\,\phi_j(\zeta')\,W(d\zeta)\,W(d\zeta').$$

The proof of Theorem 6.1 is now complete.

We now observe that as a corollary of Theorem 6.1 we have the following:

Corollary. *Let X be a square integrable functional of $\{W_z, z \in T\}$. Then X has a representation of the form*

$$X = \int_T \phi_\zeta W(d\zeta) + \Big[\int_{T\times T}\Big] \psi_{\zeta,\zeta'} W(d\zeta) W(d\zeta') + EX.$$

Proof. First, we observe that from (4.10) and (6.4) every Hermite functional has a representation

$$\prod_{\nu=1}^{n} H_{p_\nu}\Big(\int_T \psi_\nu(\zeta) W(d\zeta)\Big) = \text{constant} + \int_T \sum_i f^i(M_\zeta, \zeta) \phi_i(\zeta) W(d\zeta)$$
$$+ \tfrac{1}{2}\Big[\int_{T\times T}\Big] \sum_{i,j} f^{ij}(M_{\zeta\vee\zeta'}, \zeta\vee\zeta') \phi_i(\zeta) \phi_j(\zeta') W(d\zeta) W(d\zeta').$$

The assertion of the corollary now follows from completeness of Hermite functionals in the space of square-integrable Wiener functionals and from q.m. closure of stochastic integrals.

If we denote by $\mathscr{W}_z$ the σ-field generated by $\{W_\zeta, \zeta \prec z\}$, then it is obvious from the corollary that every square-integrable $\mathscr{W}_z$-martingale is of the form

$$M_z = M_0 + \int_0^z \phi_\zeta W(d\zeta) + \left[\int_0^z \int_0^z\right] \psi_{\zeta,\zeta'} W(d\zeta) W(d\zeta').$$

Results of this section generalize the well-known result in the one-dimensional case that every square integrable Wiener functional and martingale can be represented as an Ito stochastic integral [4, 6, 7]. They also provide an interesting connection with multiple Wiener integrals with a two-dimensional parameter. In [6] Ito proved the formula

$$\int_T \cdots \int_T \phi_1(z_1) \ldots \phi_n(z_{p_1+p_2+\cdots+p_n}) W(dz_1) \ldots W(dz_{p_1+p_2+\cdots+p_n})$$
$$= \prod_{\nu=1}^{n} \frac{H_{p_\nu}\left(\frac{1}{\sqrt{2}} \int_T \phi_\nu(z) W(dz)\right)}{(\sqrt{2})^{p_\nu}}$$

where the left hand side is a multiple Wiener integral, $\{\phi_1, \ldots, \phi_n\}$ is an orthonomial system and H_p are Hermite polynomials. It follows that if we denote $M_{iz} = \int_0^z \phi_i(\zeta) W(d\zeta)$ then there exists a polynomial $f(u, z)$ in u satisfying (6.2) such that

$$\int_T \cdots \int_T \phi_1(z_1) \ldots \phi_n(z_{p_1+p_2+\cdots+p_n}) W(dz_1) \ldots W(dz_{p_1+p_2+\cdots+p_n})$$
$$= \int_T \sum_i f^i(M_{\zeta,\zeta}) \phi_i(\zeta) W(d\zeta)$$
$$+ \tfrac{1}{2}\Big[\int_{T\times T}\Big] \sum_{i,j} f^{ij}(M_{\zeta\vee\zeta'}, \zeta\vee\zeta') \phi_i(\zeta) \phi_j(\zeta') W(d\zeta) W(d\zeta')$$
$$+ \text{constant}.$$

Appendix

In this appendix we state and prove a lemma which is referred to in Section 5.

Let $\Delta_1, \Delta_2, \ldots, \Delta_m$ be a set of m rectangles in $T=[0,1]^n$, and let h_Δ denote the indicator function of $\Delta_1 \times \Delta_2 \times \cdots \times \Delta_m$ so that $h_\Delta(\mathbf{t}), \mathbf{t} \in T^m$, is defined by

$$\begin{aligned} h_\Delta(\mathbf{t}) &= 1, \quad \mathbf{t} \in \prod_{\nu=1}^{m} \Delta_\nu \\ &= 0 \quad \text{elsewhere}. \end{aligned}$$

Now define sets F and G in T^m as follows:

$$\mathbf{t}=(t_1, t_2, \ldots, t_m) \in F \Leftrightarrow t_m \succ t_i, \qquad i=1,2,\ldots,m-1$$

$\mathbf{t} \in G \Leftrightarrow t_m$ and t_{m-1} are unordered and

$$t_m \vee t_{m-1} \succ t_i, \qquad i=1,2,\ldots,m-1.$$

Let h_F and h_G denote the indicator functions of F and G. Let T be partitioned by

$$T_N = \left\{ \left(\frac{k_1}{N}, \frac{k_2}{N}, \ldots, \frac{k_n}{N} \right), k_\nu = 0, 1, \ldots, N \right\}.$$

For each $z \in T$, $[z]^N$ denotes the largest lattice point dominated by z. For $\mathbf{t} \in T^m$ we denote $[\mathbf{t}]^N = ([t_1]^N, \ldots, [t_m]^N)$. Finally, let $d\mathbf{t}$ denote the m-fold product of the Lebesgue measure on T.

Lemma.
$$\int_{T^m} [h_\Delta(\mathbf{t})\, h_F(\mathbf{t}) - h_\Delta([\mathbf{t}]^N)\, h_F([\mathbf{t}]^N)]^2\, d\mathbf{t} \xrightarrow[N \to \infty]{} 0,$$
$$\int_{T^m} [h_\Delta(\mathbf{t})\, h_G(\mathbf{t}) - h_\Delta([\mathbf{t}]^N)\, h_G([\mathbf{t}]^N)]^2\, d\mathbf{t} \xrightarrow[N \to \infty]{} 0.$$

Proof. The proof for h_G and h_F are nearly identical, and we will use h to denote both of them. Adding and subtracting $h_\Delta([\mathbf{t}]^N)\, h(\mathbf{t})$, we get

$$\begin{aligned} &\int_{T^m} [h_\Delta(\mathbf{t})\, h(\mathbf{t}) - h_\Delta([\mathbf{t}]^N)\, h([\mathbf{t}]^N)]^2\, d\mathbf{t} \\ &\quad \leqq 2 \int_{T^m} h^2(\mathbf{t}) [h_\Delta(\mathbf{t}) - h_\Delta([\mathbf{t}]^N)]^2\, d\mathbf{t} + 2 \int_{T^m} h_\Delta^2([\mathbf{t}]^N) [h(\mathbf{t}) - h([\mathbf{t}]^N)]^2\, d\mathbf{t} \\ &\quad \leqq 2 \int_{T^m} [h_\Delta(\mathbf{t}) - h_\Delta([\mathbf{t}]^N)]^2\, d\mathbf{t} + 2 \int_{T^m} [h(\mathbf{t}) - h([\mathbf{t}]^N)]^2\, d\mathbf{t}. \end{aligned}$$

The first integral obviously goes to zero. The integrand in the second integral is either one or zero. Take the case $h = h_F$. In order that $|h_F(\mathbf{t}) - h_F([\mathbf{t}]^N)| = 1$, we must have for some $i \neq m$ $[t_m]^N \succ [t_i]^N$ and $t_m \not\succ t_i$. (The other case, $t_m \succ t_j$, $[t_m]^N \not\succ [t_j]^N$ for some $j \neq m$, is impossible.) Therefore, a necessary condition for $|h_F(\mathbf{t}) - h_F([\mathbf{t}]^N)| = 1$ is that at least one $t_i\,(i \neq n)$ should differ from t_m in one of its coordinates by no more than $1/N$. We shall now upper-bound the Lebesgue measure of this set in T^m. Assume that points are placed in T at random with a uniform distribution. A sequence of m independent samples gives us $t_1, t_2, \ldots, t_m$. The probability that t_1 and t_m differ in the first coordinate by $\frac{1}{N}$ or less is upper-bounded by $2/N$, similarly, for the other coordinates and t_j, $2 \leqq j \leqq m-1$. Therefore, the probability that at least one $t_i (i \neq m)$ and t_m differ in at least one coordinate by no more than $1/N$ is bounded by $\frac{2^n(m-1)}{N}$ which goes to zero as $N \to \infty$. The case of h_G follows very similar lines of argument and we omit the details.

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