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On Strassen-Type Laws of the Iterated Logarithm for Gaussian Elements in Abstract Spaces

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1. Introduction

Since Strassen's fundamental result on the limit-set form of the law of the iterated logarithm (LIL) for the Brownian Motion process [21], several authors have investigated its extension to more general Gaussian processes ([11, 13, 15, 17]). Starting with an idea developed in [13], an attempt is made here to give a unified approach to this problem by setting some general conditions under which the LIL for Gaussian processes with (almost all) sample paths in a separable Fréchet space F can be characterized in terms of the unit ball of a certain Hilbert space contained in F.

2. Preliminaries

A number of definitions and facts used repeatedly, will be reviewed in this section. At the same time we shall establish a consistent notation.

A linear topological space will be denoted by L, and by L^* we shall mean its topological dual, i.e. the set of all continuous linear functionals on L. If A is a subset of L, $\bar{A}$ is its closure in L and $\mathrm{sp}(A)$ represents the linear subspace generated by all finite linear combinations of elements of A. If $x^* \in L^*$, $\langle x^*, x\rangle$ is the evaluation of x^* at x. We shall always be concerned with linear spaces over the field of real numbers. At times it will be more convenient to replace the symbol $\langle x^*, x\rangle$ by $x^*(x)$. For more special cases, we shall use the letters F, B and H instead. A Fréchet space, or F-space, is a locally convex, linear topological space which is metrizable and complete and we shall denote it with the letter F. Likewise, the letters B and H will be used to denote a general Banach and Hilbert space respectively. The real line will always be denoted by $\mathbb{R}$, while $\mathbb{R}^n$ will represent the n-fold cartesian product of $\mathbb{R}$ with itselt. The σ-algebra generated by the open sets of L will be denoted by $\mathscr{B}(L)$ or simply by $\mathscr{B}$ when the underlying space is clearly understood. Completeness of measure spaces will always be assumed.

Given a probability space $(\Omega, \mathscr{F}, P)$ let X be a map from Ω into L. If for each $x^* \in L^*$, $x^*(X)$ is a Gaussian random variable, then X is called a Gaussian random element on L. From now on the term random variable will be used only for mappings into $\mathbb{R}$. Although this definition qualifies X as a weak measurable map from $(\Omega, \mathscr{F})$ into $(L, \mathscr{B}(L))$, it is well known (cf. [1], p. 100) that if L is metrizable and separable then the inverse map of $\mathscr{B}(L)$, under X, is contained in $\mathscr{F}$. Thus, for the case where F is a separable Fréchet space, the set function $\mu = PX^{-1}$ defines a (Gaussian) measure on $\mathscr{B}(F)$.

A sequence $(X_n, n \geqq 1)$ of Gaussian elements on L is called a Gaussian sequence if for all positive integers $k, n_1, \ldots, n_k$ in (n) and elements $x_1^*, \ldots, x_k^*$ in L^*: $(\langle x_1^*, X_{n_1}\rangle, \ldots, \langle x_k^*, X_{n_k}\rangle)$ is a Gaussian random vector.

Let us suppose that L is a space of real functions on some set S. For $s \in S$ let δ_s: $L \to \mathbb{R}$ be the evaluation map at s, i.e. $\delta_s(f) = f(s)$ for $f \in L$. If δ_s is continuous and X is a random element inducing a Gaussian measure μ on $\mathscr{B}(L)$, the set $(X(s), s \in S)$, where $X(s) = \delta_s(X)$ is a Gaussian process. (Relations between the classical definition of Gaussian processes and Gaussian measures on linear topological spaces have been studied in [19].) Suppose $EX(s) = 0$ for all $s \in S$. Then the bilinear map $R: S \times S \to \mathbb{R}$ defined by $R(s,t) = EX(s)\,X(t) = \int_L x(s)\,x(t)\,\mu(dx)$ is a symmetric positive-definite kernel. It will be referred to, in the sequel, as the covariance function of μ, or X. The Hilbert space $H(R) \equiv \overline{\mathrm{sp}}\,(R(s,\cdot): s \in S)$, the completion of $\mathrm{sp}(R(s,\cdot): s \in S)$, is called the reproducing kernel Hilbert space (RKHS) generated by R. It is characterized by the following property: $h(s) = \langle h, R(s,\cdot)\rangle_H$, $h \in H(R)$, where $\langle\cdot,\cdot\rangle_H$ is the inner product on $H(R)$ (cf. Aronszajn [2]).

Let F be a separable Fréchet space with topology defined by a countable family of semi-norms $(p_i, i \geqq 1)$ and X be a Gaussian element with values in F. Suppose $Ex^*(X) = 0$ for all x^* in F^* and let $R(x^*, y^*) = Ex^*(X)\,y^*(X)$. Denote by θ^* the isometric isomorphism or congruence from $H' \equiv \overline{\mathrm{sp}}\,(R(x^*,\cdot): x^* \in F^*)$ onto $\overline{\mathrm{sp}}\,(x^*(X): x^* \in F^*)$ in $L_2(\Omega, \mathscr{F}, P)$, the collection of all (classes of) random variables on $(\Omega, \mathscr{F}, P)$ with finite second moment. (For each x^* set $\theta^*(R(x^*,\cdot)) = x^*(X)$.) Then $H' \subset F^{**}$ so that if τ is the natural imbedding of F into its second topological dual we have, by letting $H = \tau^{-1}(H') \subset F$ and $\langle x, y\rangle_H = \langle \tau x, \tau y\rangle_{H'}$ that $\theta = \theta^* \tau$ is a congruence from H onto $\overline{\mathrm{sp}}\,(x^*(X): x^* \in F^*)$. (A detailed discussion of this point will appear elsewhere.) The Hilbert space H, which is separable and whose norm will be denoted by $\|\cdot\|_H$ has the following properties (cf. [6, 8, 14, 18]):

i) if $x \in H$, then $p_i(x) \leqq A_i \|x\|_H$ where A_i is a constant independent of x. Thus the injection: $H \to F$ is continuous and $F^* \subset H^*$.

ii) if μ is the Gaussian measure induced by X on F, then $\mu|_H$ is the canonical normal distribution on H and $p_i(\cdot)$ is a measurable semi-norm on H;

iii) the closure of $\bar{H}$ of H in F is the support of μ.

The space H is also referred to, in the literature, as the generator of μ. Let $\mathscr{C}(T)$ be the Fréchet space of all real-valued continuous functions on a σ-compact metric space T. If X is a random element inducing a Gaussian measure μ on $\mathscr{C}(T)$ then the generator of μ coincides with the RKHS $H(R) \equiv \overline{\mathrm{sp}}\,(R(s,\cdot): s \in T)$ where $R(t,s) = E\delta_t(X)\,\delta_s(X)$ (cf. [8, 18]).

Let now $(e_i, i \geqq 1)$ be a complete orthonormal system (CONS) in H. Let $\xi_i = \theta(e_i)$, where θ has been defined above. Then $(\xi_i, i \geqq 1)$ is a sequence of independent $N(0,1)$ random variables and for almost all $\omega \in \Omega$:

$$X(\omega) = \sum_{i=1}^{\infty} \xi_i(\omega)\, e_i \tag{1'}$$

where the series is convergent in the F-topology. ([7, 10, 14, 18]).

3. Preparatory Lemmas

If E is a Hausdorff topological space, a subset A of E is said to be sequentially compact if every sequence in A has at least a subsequence which converges to some point of E. The set of limit points of A in E will be denoted by $\mathscr{L}_E(A)$ or simply by $\mathscr{L}(A)$ if E is understood. When no confusion arises it will be also called the limit-set of A.

The following result, that we state here as a lemma for easy reference, will be useful in our discussion.

Lemma 3.1 (cf. [12, 16]). *Let $(Y_n, n \geqq 1)$ be a Gaussian sequence of $N(0, \sigma^2)$ random variables. Suppose:*

$$\lim_{r\to\infty} \max_{|m-n|>r} E Y_n Y_m \leqq 0.$$

Then:

$$\limsup_{n\to\infty} (2 \lg n)^{-\frac{1}{2}} Y_n = \sigma \qquad \text{a.e.}$$

Lemma 3.2. *Let $(V_\infty^{(n)}, n \geqq 1)$ be a sequence of random elements with values in $\mathbb{R}^\infty$. Let the components $\xi_i^{(n)}$ of $V_\infty^{(n)}$ be independently distributed $N(0,1)$ random variables. For all $N, N \geqq 1$, the sequence $(V_N^{(n)}, n \geqq 1)$ of vectors: $V_N^{(n)} = (\xi_1^{(n)}, \ldots, \xi_N^{(n)})$ is assumed to be a Gaussian sequence. If, for all i and j:*

$$\lim_{r\to\infty} \max_{|m-n|>r} |E \xi_i^{(n)} \xi_j^{(m)}| = 0, \tag{1}$$

then for all $N, N \geqq 1$, the sequence: $((2 \lg n)^{-\frac{1}{2}} V_N^{(n)}, n \geqq 2)$ is almost surely, a sequentially compact subset of $\mathbb{R}^N$ and

$$\mathscr{L}((2 \lg n)^{-\frac{1}{2}} V_N^{(n)}) = B_N \qquad a.e. \tag{2}$$

where $B_N = \left(x \in \mathbb{R}^N : \sum_{i=1}^{N} x_i^2 \leqq 1\right)$ and $\mathscr{L}(\cdot)$ denotes the limit-set in $\mathbb{R}^N$.

Proof. To prove this lemma we shall follow an idea due to Finkelstein ([5], p. 609, Lemma 2). For an arbitrary $N, N \geqq 1$, let $(V_N^{(n)}, n \geqq 1)$ be a sequence of random vectors in $\mathbb{R}^N$ defined as in the statement of the lemma. Let $T \in (\mathbb{R}^N)^*$ be a non-null functional over $\mathbb{R}^N$. Set $Y_n = T V_N^{(n)} = \sum_{i=1}^{N} a_i \xi_i^{(n)}$, $a_i \in \mathbb{R}$, and let $\|\cdot\|$ denote the Euclidean norm on the space $\mathbb{R}^N$. Then $(Y_n, n \geqq 1)$ is a Gaussian sequence of

$N(0, \|T\|^2)$ random variables. Since:

$$\max_{|m-n|>r} EY_n Y_m \leqq \max_{|m-n|>r} \left(\sum_{i=1}^{N} \sum_{j=1}^{N} |a_i a_j| \, |E\xi_i^{(n)} \xi_j^{(m)}|\right) \xrightarrow[r\to\infty]{} 0$$

by (1), we get from Lemma 3.1:

$$\limsup_{n\to\infty} (2\lg n)^{-\frac{1}{2}} Y_n = \|T\| \qquad \text{a.e.} \tag{3}$$

It then easily follows that:

$$\mathscr{L}((2\lg n)^{-\frac{1}{2}} V_N^{(n)}) \subset B_N \qquad \text{a.e.} \tag{4}$$

On the other hand, let $x \in S_N = (x \in \mathbb{R}^N: \|x\| = 1)$. Then for all $n \geqq 2$:

$$\|(2\lg n)^{-\frac{1}{2}} V_N^{(n)} - x\|^2 = \|(2\lg n)^{-\frac{1}{2}} V_N^{(n)}\|^2 + \|x\|^2 - 2\langle (2\lg n)^{-\frac{1}{2}} V_N^{(n)}, x\rangle. \tag{5}$$

From (3) we must have:

$$\limsup_{n\to\infty} \langle (2\lg n)^{-\frac{1}{2}} V_N^{(n)}, x\rangle = \|x\| = 1 \qquad \text{a.e.}$$

Therefore, for almost all ω we can choose a subsequence (n_k) such that:

$$\lim_{k\to\infty} \langle (2\lg n_k)^{-\frac{1}{2}} V_N^{(n_k)}, x\rangle = 1.$$

Thus, using (5), for any $\varepsilon > 0$, and all large k:

$$\|(2\lg n_k)^{-\frac{1}{2}} V_N^{(n_k)} - x\|^2 \leqq 1 + (1+\varepsilon)^2 - 2(1-\varepsilon) \leqq 5\varepsilon.$$

Hence:

$$S_N \subset \mathscr{L}((2\lg n)^{-\frac{1}{2}} V_N^{(n)}) \qquad \text{a.e.} \tag{6}$$

Since (6) holds for all N we then also have:

$$S_{N+1} \subset \mathscr{L}((2\lg n)^{-\frac{1}{2}} V_{N+1}^{(n)}) \qquad \text{a.e.} \tag{7}$$

Let P_N denote the projection: $\mathbb{R}^{N+1} \to \mathbb{R}^N$. Since $B_N = P_N(S_{N+1})$, relation (7) implies:

$$B_N \subset \mathscr{L}((2\lg n)^{-\frac{1}{2}} V_N^{(n)}) \qquad \text{a.e.}$$

Lemma 3.3. *Let $(V_\infty^{(n)}, n \geqq 1)$ be a sequence of random elements defined as in the previous lemma and having components satisfying condition (1). If $(e_i, i \geqq 1)$ is an orthonormal system (OS) in some Hilbert space H, then for all N, $N \geqq 1$:*

$$\left((2\lg n)^{-\frac{1}{2}} \sum_{i=1}^{N} e_i \xi_i^{(n)}, n \geqq 2\right)$$

is sequentially compact in H and

$$\mathscr{L}_H\left((2\lg n)^{-\frac{1}{2}} \sum_{i=1}^{N} e_i \xi_i^{(n)}\right) = K_N \qquad \text{a.e.}$$

where $K_N = (x \in H: x \in \operatorname{sp}(e_1, \ldots, e_N), \|x\|_H \leqq 1)$.

Proof. Let θ_N denote the congruence between $\mathbb{R}^N$ and $\mathrm{sp}(e_1, \ldots, e_N)$ defined in terms of the bases. It then follows from Lemma 3.2 that:

$$\begin{aligned}\mathscr{L}_H\left((2\lg n)^{-\frac{1}{2}}\sum_{i=1}^{N} e_i \xi_i^{(n)}\right) &= \mathscr{L}_H(\theta_N[(2\lg n)^{-\frac{1}{2}} V_N^{(n)}]) \\ &= \theta_N(B_N) \\ &= K_N.\end{aligned}$$

We have used here the fact that, under a continuous map, sequential compactness is preserved and the image of a limit-set is the limit-set of the image points.

Let $\mathscr{C}(T)$ denote the space of all continuous functions from T, a σ-compact metric space, into $\mathbb{R}$. We shall endow $\mathscr{C}(T)$ with the topology of uniform convergence on compacta. With this topology $\mathscr{C}(T)$ is a separable F-space (Edwards [3], p. 205). Since $T = \bigcup(C_i, i \geqq 1)$, where C_i are compact, we may assume $C_i \subset C_{i+1}$ for all i. The mapping p_i: $\mathscr{C}(T) \to \mathbb{R}$ defined as: $p_i(x) = \sup(|x(t)|, t \in C_i)$ where $x \in \mathscr{C}(T)$ is a semi-norm and the metric d, constructed in the following way:

$$d(x, y) = \sum_{j=1}^{\infty} 2^{-j}[p_j(x-y)/(1+p_j(x-y))]$$

is compatible with the $\mathscr{C}(T)$-topology.

If x, x_n are elements of $\mathscr{C}(T)$ we set:

$$x^{(i)} = x|_{C_i}; \qquad x_n^{(i)} = x_n|_{C_i}.$$

If R is a continuous positive definite kernel on $T \times T$ and $H = H(R)$ is the reproducing kernel Hilbert space generated by R then $H(R) \subset \mathscr{C}(T)$, set theoretically. We also set: $R_i = R|_{C_i \times C_i}$, $H_i = H(R_i)$. Finally, let $\|\cdot\|_i = \|\cdot\|_{H_i}$, $K_i = (x \in H_i\colon \|x\|_i \leqq 1)$, $K = (x \in H\colon \|x\|_H \leqq 1)$ and $\mathscr{C}(C_i)$ be the space of all continuous functions: $C_i \to \mathbb{R}$ endowed with the sup-norm topology.

Lemma 3.4. *K is closed in $\mathscr{C}(T)$.*

Proof. Suppose $(x_n, n \geqq 1) \subset K$ and $x_n \to x$ in $\mathscr{C}(T)$ as $n \to \infty$. We are to show that $x \in K$. By assumption: $x_n^{(i)} \to x^{(i)}$, as $n \to \infty$ uniformly on C_i. From a theorem of Aronszajn ([2], p. 351) we know that for each i, H_i is the class of the restrictions to C_i of the functions of H and if h_i is any such a restriction, $\|h_i\|_i$ is the minimum of $\|h\|_H$ taken over the class of all h in H such that: $h|_{C_i} = h_i$. Hence $\|x_n^{(i)}\|_i \leqq \|x_n\|_H \leqq 1$. Since K_i is compact in $\mathscr{C}(C_i)$ ([11], p. 256) we have $x^{(i)} \in K_i$. Thus the element x, $\mathscr{C}(T)$-limit of $(x_n, n \geqq 1)$ has the following properties:

$$x|_{C_i} = x^{(i)} \in H_i; \qquad \|x^{(i)}\|_i \leqq 1.$$

We have assumed that the sequence $(C_i, i \geqq 1)$ of compact subsets of T is increasing. If $h_i \in H_i$ we shall denote by h_{ij}, $j \leqq i$, the restriction of h_i to $C_j \subset C_i$. Hence the sequence $(H_i, i \geqq 1)$ of RKHS's H_i satisfies the following conditions:

(a) for every $h_i \in H_i$ and every $j \leqq i$: $h_{ij} \in H_j$;

(b) for every $h_i \in H_i$ and every $j \leqq i$: $\|h_{ij}\|_j \leqq \|h_i\|_i$;

(c) H_i has a reproducing kernel R_i.

Since $R=\lim_{i\to\infty} R_i$, we are able to conclude from another theorem of Aronszajn ([2], Thm. I p. 362) that $x\in H$. But then $\|x\|_H=\lim_{i\to\infty}\|x^{(i)}\|_i\leqq 1$. Hence $x\in K$.

In the following lemma, let $\mathscr{L}_{(i)}(\cdot)$ denote the limit-set in $\mathscr{C}(C_i)$.

Lemma 3.5. *Suppose there exists a closed subspace F of $\mathscr{C}(T)$ such that $H(R)\subset F$. Let $(x_n, n\geqq 1)$ be a sequence in F. If for each i, $(x_n^{(i)}, n\geqq 1)$ is sequentially compact in $\mathscr{C}(C_i)$ and:*

$$\mathscr{L}_{(i)}(x_n^{(i)})=K_i, \tag{9}$$

then $(x_n, n\geqq 1)$ is a sequentially compact subset of F and:

$$\mathscr{L}_F(x_n)=K.$$

Proof. Let (m) be any subsequence in (n). We shall show first that $(x_m, m\geqq 1)$ has a further subsequence converging to a point of K in the $\mathscr{C}(T)$-topology. Let $x_m^{(i)}=x_m|_{C_i}$. For $i=1$, it follows from (9) that there exists a subsequence $(x_{m_{1,k}}^{(1)}, k\geqq 1)$ such that, for some $x^{(1)}\in K_1$: $x_{m_{1,k}}^{(1)}\to x^{(1)}$, as $k\to\infty$ uniformly on C_1. Similarly $(x_{m_{1,k}}^{(2)}, k\geqq 1)$ contains a subsequence $(x_{m_{2,k}}^{(2)}, k\geqq 1)$ such that: $x_{m_{2,k}}^{(2)}\to x^{(2)}$, as $k\to\infty$, uniformly on C_2, for some $x^{(2)}$ in K_2. But then: $x_{m_{2,k}}^{(1)}\to x^{(1)}$, as $k\to\infty$, uniformly on C_1. And so on. By diagonalization we can find a subsequence $(m')\subset(m)$ such that: $x_{m'}^{(i)}\to x^{(i)}$, as $m'\to\infty$ uniformly on C_i, for all i. Hence $(x_{m'})$ is a Cauchy sequence in F. Let $y=\lim_{m'\to\infty} x_{m'}$. Then for each i: $y|_{C_i}=x^{(i)}\in K_i$ and by the same argument used in the proof of the previous lemma we conclude that $y\in K$. Let now x be any element in K. Given any positive ε, choose i_0 so that: $\sum_{i_0+1}^{\infty} 2^{-j}\leqq\varepsilon/2$. Then $y_0=x|_{C_{i_0}}\in K_{i_0}$. Thus by (9) there exists a subsequence $(n(j), j\geqq 1)$ of (n) for which:

$$y_{n(j)}\to y_0, \quad \text{as } j\to\infty, \tag{10}$$

uniformly on C_{i_0}. In (10) we have set: $y_{n(j)}=x_{n(j)}|_{C_{i_0}}$. Since: $p_1\leqq\cdots\leqq p_{i_0}$, from the definition of the metric d on $\mathscr{C}(T)$ we have:

$$d(x, x_{n(j)})\leqq[p_{i_0}(x-x_{n(j)})/(1+p_{i_0}(x-x_{n(j)}))]+\varepsilon/2\leqq\varepsilon,$$

for all large j. This shows that x is in $\mathscr{L}_F(x_n)$ and completes the proof.

Remark. One may notice that the set K defined above is actually compact in $\mathscr{C}(T)$.

4. Sequential Compactness of Certain Sequences of F-Valued Gaussian Elements

By using the notions of Section 2 and the additional results of Section 3 we are now able to prove the following result.

Theorem 4.1. *Let F denote a separable Fréchet space and $(X_n, n\geqq 1)$ be a Gaussian sequence on $(\Omega, \mathscr{F}, P)$ with values in F. Suppose that $\mu=PX_n^{-1}$ is independent of n and $Ex^*(X_n)=0$ for x^* in F^*. For $n\geqq 2$ set $Y_n=(2\lg n)^{-\frac{1}{2}}X_n$. If for all x^* and y^**

in F^*:

$$\lim_{r\to\infty} \max_{|m-n|>r} |E x^*(X_n) y^*(X_m)| = 0 \tag{11}$$

then, with probability one, $(Y_n, n \geqq 2)$ *is a sequentially compact subset of* F *and:*

$$\mathscr{L}_F(Y_n) = K$$

where K *is the unit ball in* H, *the generator of* μ.

Proof. We shall give the proof in three main steps. In step 1 the claim of the theorem will be established for B spaces and in step 2 the same conclusion will be reached for the $\mathscr{C}(T)$ case. The complete proof will then easily follow by regarding F as a closed subspace of $\mathscr{C}(\mathbb{R})$.

Step 1. Suppose X_n take values in a separable B-space. Let $(e_i, i \geqq 1)$ be a CONS in H and θ_n be the congruence between H and $\overline{\text{sp}}(x^*(X_n)\colon x^* \in B^*)$. If $\xi_i^{(n)} = \theta_n(e_i)$, then according to Equation (1′) of Section 2 we can write for all n:

$$X_n = \sum_{i=1}^{\infty} e_i \xi_i^{(n)}, \qquad \text{a.e.} \tag{12}$$

where the series is convergent in the norm of B. Let this norm be denoted by $\|\cdot\|_B$. Applying Fernique's theorem ([4]) to the Gaussian element $X_1 - \sum_{i=1}^{N} e_i \xi_i^{(1)}$ we obtain:

$$E \exp\left(\alpha^2 \left\| X_1 - \sum_{i=1}^{N} e_i \xi_i^{(1)} \right\|_B^2\right) < \infty \tag{13}$$

if $0 < \alpha^2 < (1/24 s^2) \lg(h_N/1 - h_N)$, where $h_N = P\left(\left\| X_1 - \sum_{i=1}^{N} e_i \xi_i^{(1)} \right\|_B \leqq s\right) > \frac{1}{2}$. Since $h_N \to 1$, as $N \to \infty$ by Equation (12), if ε is any positive number, α can chosen equal to $1/\varepsilon$ provided N is sufficiently large. Taking into account that X_n induces on B a measure independent of n, we can then write:

$$P\left(\left\| X_n - \sum_{i=1}^{N} e_i \xi_i^{(n)} \right\|_B \geqq (2 \lg n)^{\frac{1}{2}} \varepsilon\right) = P\left(\alpha \left\| X_1 - \sum_{i=1}^{N} e_i \xi_i^{(1)} \right\|_B \geqq (2 \lg n)^{\frac{1}{2}} \alpha \varepsilon\right)$$
$$\leqq A_N n^{-2}$$

where $N = N(\varepsilon)$ has been chosen so that (13) is satisfied. Thus, by the Borel-Cantelli lemma:

$$P(\|X_n - Y_n^N\|_B < \varepsilon, \text{ all large } n) = 1 \tag{14}$$

where $Y_n^N = (2 \lg n)^{-\frac{1}{2}} \sum_{i=1}^{N} e_i \xi_i^{(n)}$. Note that the random variables $\xi_i^{(n)}$ appearing in the series expansion (12) may be assumed to be of the form $x_i^*(X_n)$ for some $x_i^* \in B^*$ independent of n (cf. [14]). Hence by (11):

$$\lim_{r\to\infty} \max_{|m-n|>r} |E \xi_i^{(m)} \xi_j^{(n)}| = 0 \qquad \text{for all } i \text{ and } j;$$

consequently it follows from Lemma 3.3 that $\mathscr{L}_H(Y_n^N) \subset K_N \subset K$ a.e. We recall, cf. Section 2, that on H the norm $\|\cdot\|_B$ is weaker than $\|\cdot\|_H$ is the sense that there exists a positive constant A such that:

$$\|x\|_B \leqq A\|x\|_H \tag{15}$$

for all x in H. Using (14) it is now easy to see that, with probability one, $(Y_n, n \geqq 2)$ is a sequentially compact subset of B and $\mathscr{L}_B(Y_n)$ is contained is an ε-neighborhood of K. Since ε is arbitrary, $\mathscr{L}_B(Y_n)$ is contained in K a.e. To prove $K \subset \mathscr{L}_B(Y_n)$ it is enough to show that for any x in a countable dense subset of K, the event $(\|x - Y_n\| < \varepsilon)$ occurs i.o. with probability one, for any prefixed $\varepsilon > 0$. Thus given such an x, let N be chosen so that the following two conditions are simultaneously satisfied:

$$\left\| x - \sum_{i=1}^{N} e_i x_i \right\|_H < \varepsilon/3A \qquad (x_i = \langle x, e_i \rangle_H),$$

$$P(\|Y_n - Y_n^N\|_B < \varepsilon/3, \text{ all large } n) = 1.$$

(The constant A is the same as in (15)). Then, with probability one:

$$\begin{aligned} \|x - Y_n\|_B &\leqq \left\| x - \sum_{i=1}^{N} e_i x_i \right\|_B + \left\| \sum_{i=1}^{N} e_i x_i - Y_n^N \right\|_B + \|Y_n^N - Y_n\|_B \\ &\leqq (2\varepsilon/3) + A \left\| \sum_{i=1}^{N} e_i x_i - Y_n^N \right\|_H \end{aligned}$$

for all large n. Hence $x \in \mathscr{L}_B(Y_n)$ by Lemma 3.3.

Step 2. Suppose that X_n take values in a closed subspace F of $\mathscr{C}(T)$. Letting $R(t, s) = E\delta_t(X_n)\delta_s(X_n)$ then R is continuous on $T \times T$ and $H(R) \subset F$. Furthermore $H(R)$ is the generator of $\mu = PX_n^{-1}$ (cf. [18] p. 292). Write $T = \bigcup(C_i, i \geqq 1)$ where C_i are compact and $C_i \subset C_{i+1}$. For each i let $X_n^{(i)} = \pi_i(X_n)$ where $\pi_i\colon F \to \mathscr{C}(C_i)$ is defined by $\pi_i(x) = x|_{C_i}$, $x \in F$. Since for each $x^* \in \mathscr{C}^*(C_i)$, $x^*\pi_i \in F^*$, $\mu_i = \mu\pi_i^{-1}$ is the Gaussian measure on $\mathscr{C}(C_i)$ induced by $X_n^{(i)}$ and $Ex^*(X_n^{(i)}) = 0$. Moreover, $(X_n^{(i)}, n \geqq 1)$ is a Gaussian sequence and by condition (11):

$$\lim_{r \to \infty} \max_{|m-n| > r} |Ex^*(X_n^{(i)})\, y^*(X_m^{(i)})| = 0$$

for all x^* and y^* in $\mathscr{C}^*(C_i)$. By step 1, $((2 \lg n)^{-\frac{1}{2}} X_n^{(i)}, n \geqq 2)$ is, with probability one, a sequentially compact subset of $\mathscr{C}(C_i)$ and its set of limit points equals K_i a.e. K_i is the unit ball in the RKHS H_i generated by the covariance function of μ_i. Let $R_i = R|_{C_i \times C_i}$. Then one can easily check that:

$$R_i(t, s) = EX_n^{(i)}(t)\, X_n^{(i)}(s).$$

Hence the assertion follows from Lemma 3.5.

Step 3. Let X_n be F-valued, where F is a separable Fréchet space. Then there exists a topological isomorphism ϕ from F onto a closed subspace F' of $\mathscr{C}(\mathbb{R})$ (cf. [9], p. 218). Let $X_n' = \phi(X_n)$. Then $(X_n', n \geqq 1)$ is an F'-valued Gaussian sequence that satisfies the conditions of the theorem. Consequently by step 2 $((2 \lg n)^{-\frac{1}{2}} X_n', n \geqq 2)$ is, with probability one, sequentially compact in F' with limit-set equal to

K', the unit ball in the RKHS H' generated by the covariance function of X'_n. Now the Hilbert space $H=\phi^{-1}(H')$ with inner product $\langle x, y\rangle_H = \langle \phi x, \phi y\rangle_{H'}$ is the generator of $\mu = PX_n^{-1}$ ([18], p. 298). Since ϕ^{-1} is continuous, the conclusion follows by applying the fact mentioned in the proof of Lemma 3.3. This completes the proof of the theorem.

5. Some Applications of Theorem 4.1

In this section we shall derive a number of theorems related to the study, initiated by Strassen [21], of the law of the iterated logarithm in its functional form.

(a) We start by establishing the main result of Lai that is closely connected, in its formulation, to our Theorem 4.1.

Theorem 5.1 ([13], p. 9). *Suppose $X(t)$, $t\in[0,1]$, is a separable real valued Gaussian process with mean zero and continuous covariance function $R(t,s)$ satisfying:*

A) $E[X(t)-X(s)]^2 \leqq g^2(|t-s|)$

where g is a continuous nondecreasing function on $[0,1]$ such that $\int_1^\infty g(e^{-u^2})\,du<\infty$. Let $(X_n(t), t\in[0,1], n\geqq 1]$ be a sequence of Gaussian processes defined on the same probability space and having the same distribution as the process $X(t)$, and set $Y_n(t)=(2\lg n)^{-\frac{1}{2}} X_n(t)$.

Then, with probability one, the sequence $(Y_n(\cdot), n\geqq 2)$ is sequentially compact in $\mathscr{C}[0,1]$ and its set of limit points in $\mathscr{C}[0,1]$ is contained in the unit ball K of the reproducing kernel Hilbert space $H(R)$ corresponding to the process $X(t)$. Letting $\mathscr{G}_n=(X_j(t), t\in[0,1], 1\leqq j\leqq n)$ and $\mathscr{G}=\bigcup(\mathscr{G}_n, n\geqq 1)$, suppose furthermore that:

B) *$\mathscr{G}$ is a Gaussian family of random variables (i.e. any finite number of elements in $\mathscr{G}$ is a normal random vector) such that:*

$$\lim_{\substack{n\to\infty\\(m-n)\to\infty}} E[E(X_m(t)|\mathscr{G}_n)]^2=0, \qquad t\in[0,1].$$

Then, with probability one, the set of limit points of $(Y_n(\cdot), n\geqq 2)$ in $\mathscr{C}[0,1]$ coincides with the set K.

Proof. Let $(\Omega, \mathscr{F}, P)$ be the probability space on which the processes $(X_n(t), t\in[0,1])$ are defined. Because of condition A), each process $(X_n(t), t\in[0,1])$ can be viewed as a Gaussian element $X_n\colon \Omega\to\mathscr{C}[0,1]$. As a consequence of B) $(X_n, n\geqq 1)$ is a Gaussian sequence such that

$$\lim_{r\to\infty}\max_{|m-n|>r} |EX_m(t)\,X_n(s)|=0 \tag{16}$$

for each t and s in $[0,1]$. Since:

$$|EX_n(t)\,X_m(s)|\leqq E^{\frac{1}{2}}[X_n(t)]^2\,E^{\frac{1}{2}}[X_m(s)]^2 = R^{\frac{1}{2}}(t,t)\,R^{\frac{1}{2}}(s,s)$$

and $R(t,s)=EX_n(t)\,X_n(s)$ is continuous on $[0,1]\times[0,1]$, if v_1 and v_2 denote the measures on $[0,1]$ corresponding to arbitrary elements x^* and y^* of $\mathscr{C}^*[0,1]$, it follows from (16) and Lebesgue dominated convergence theorem that

$$\max_{|n-m|>r} |E x^*(X_n)\, y^*(X_m)| \leqq \int_{[0,1]\times[0,1]} \max_{|n-m|>r} |EX_n(t)\, X_m(s)\, \|v_1(dt)\|\, v_2(ds)| \to 0 \qquad \text{as } r\to\infty.$$

By Theorem 4.1 the assertion is established.

(b) Let B denote a real, separable Banach space and let $(Z_n, n\geqq 1)$ be a sequence of independent, identically distributed Gaussian random elements on B such that $Ex^*(Z_n)=0$ for all x^* in B^*. For any two elements x^* and y^* in B^*, our assumptions on the sequence $(Z_n, n\geqq 1)$ imply:

$$E x^*(Z_n)\, y^*(Z_m)=0, \qquad \text{if } n\neq m,$$
$$E x^*(Z_n)\, y^*(Z_n)=C(x^*, y^*)$$

where $C(x^*, y^*)=\int_B x^*(x)\, y^*(x)\, \mu(dx)$, $\mu=PZ_n^{-1}$. Thus:

$$E x^*\left(\sum_{i=1}^{n} Z_i\right) y^*\left(\sum_{j=1}^{m} Z_j\right)=(n\wedge m)\, C(x^*, y^*). \tag{17}$$

Let now: $X_n=n^{-\frac{1}{2}}\sum_{i=1}^{n} Z_i$. Then $(X_n, n\geqq 1)$ is a Gaussian sequence in B such that for all n: $PX_n^{-1}=\mu$ and $Ex^*(X_n)=0$ for all x^* in B^*. Along the subsequence: $n_k=[c^k]$, $k\geqq 1$, $c>1$, we have from (17):

$$E x^*(X_{n_k})\, y^*(X_{n_h})=(\text{Const.})(c^{|k-h|})^{-\frac{1}{2}}.$$

Set $\zeta_n=(2n\,\lg\lg n)^{-\frac{1}{2}}\sum_{i=1}^{n} Z_i=(2\,\lg\lg n)^{-\frac{1}{2}}\, X_n$ and $Y_k=(2\lg k)^{-\frac{1}{2}}\, X_{n_k}$. From the results of the previous section we then have that $(Y_k, k\geqq 2)$ is a sequentially compact subset of B and:

$$\mathscr{L}_B(Y_k)=K \qquad \text{a.e.} \tag{18}$$

where K is the unit ball in H, the generator of μ. Since $\lg\lg n_k \sim \lg k$, as $k\to\infty$, Equation (18) implies that $K\subset \mathscr{L}_B(\zeta_n)$ a.e. On the other hand, given any $\varepsilon>0$, there exists a number $c=c(\varepsilon)$ sufficiently close to one, such that:

$$P\Big(\sup_{n_k\leqq n< n_{k+1}} \|\zeta_{n_k}-\zeta_n\|_B<\varepsilon, \text{ all large } k\Big)=1$$

([15], p. 106). In other words, $\mathscr{L}_B(\zeta_n)\subset K_\varepsilon$ a.e., where K_ε is an ε-neighborhood of K. Since ε is arbitrary this yields $\mathscr{L}_B(\zeta_n)\subset K$ a.e., i.e. the conclusion of the the following theorem has been established.

Theorem 5.2 ([15]). *Let B denote a real separable Banach space and $(Z_n, n\geqq 1)$ be a sequence of independent Gaussian random variables on $(\Omega, \mathscr{F}, P)$, having mean zero and taking values in B. Suppose $\mu=PZ_n^{-1}$ does not depend on n and set for $n\geqq 3$: $\zeta_n=(2n\,\lg\lg n)^{-\frac{1}{2}}(Z_1+\cdots+Z_n)$.*

If K is the unit ball in the Hilbert subspace of B which generates μ, then the sequence $(\zeta_n, n \geqq 3)$ a.e. converges to K and clusters at every point of K in the sense of B-norm.

(c) Let H be a real, separable Hilbert space. For $t>0$, let m_t denote the canonical normal distribution on H. Let $\|\cdot\|_B$ be a measurable norm on H and B denote the closure of H with respect to $\|\cdot\|_B$. Then B is a separable Banach space (cf. Gross [6], Kallianpur [8]). Let μ_t be the extension of m_t to the Borel subsets of B. We call μ_t the Wiener measure on B, generated by H with variance parameter t. Let now Ω_B denote the space of continuous functions ω: $[0, \infty) \to B$ such that $\omega(0)=0$ and $\mathscr{F}$ be the σ-field of Ω_B generated by the functions $\omega \to \omega(t)$. Then there exists a probability measure P on $\mathscr{F}$ such that if $0=t_0<t_1<\cdots<t_n$, then $\omega(t_j)-\omega(t_{j-1})$ $(j=1, \ldots, n)$ are independent increments and $\omega(t_j)-\omega(t_{j-1})$ has distribution $\mu_{t_j-t_{j-1}}$. Thus the stochastic process W_t defined on $(\Omega_B, \mathscr{F}, P)$ by $W_t(\omega)=\omega(t)$ has stationary independent increments and it is called Brownian Motion on B.

Let now $\mathscr{C}_B$ denote the space of continuous functions from $[0, 1]$ into B which vanish at zero. If $f \in \mathscr{C}_B$ let: $\|f\|_{C_B}=\sup(\|f(t)\|_B, t\in[0,1])$. Then $\mathscr{C}_B$ is a separable Banach space and the Brownian Motion on B induces a probability measure P on $\mathscr{C}_B$ which is a zero-mean Gaussian measure ([11], p. 254). Let $m>n$ and set $W_m=W(m\cdot)$, $W_n=W(n\cdot)$, $\mathscr{F}_n=\sigma(W(t), t\in[0,n])$. We define: $V_{m,n}=E(W_m \mid \mathscr{F}_n)$ and observe that $V_{m,n}$ is well defined as conditional expectation (cf. [20], p. 353). If f^* and g^* are any two elements in $\mathscr{C}_B^*$ we obtain by interchanging E and f^*:

$$E g^*(W_n) f^*(W_m)=E[g^*(W_n)\{E(f^*(W_m) \mid \mathscr{F}_n)\}]=E g^*(W_n) f^*(V_{m,n}).$$

Hence:

$$|E g^*(W_n) f^*(W_m)| \leqq \|g^*\| \, \|f^*\| \, E(\|W_n\|_{C_B} \|V_{m,n}\|_{C_B}).$$

Let δ_s denote the evaluation map: $\mathscr{C}_B \to B$ at $s\in[0,1]$. Then by using independence of increments:

$$\delta_s(V_{m,n})=E(W_m(s) \mid \mathscr{F}_n)=E(W(ms) \mid \mathscr{F}_n)=W(ms \wedge n).$$

Since by assumption $m>n$ we also have:

$$\begin{aligned}|E g^*(W_n) f^*(W_m)| &\leqq (\text{Const.})\, E\big(\sup_{0\leqq t\leqq 1} \|W(nt)\|_B \sup_{0\leqq t\leqq 1} \|W(mt\wedge n)\|_B\big)\\ &=(\text{Const.})\, E\big(\sup_{0\leqq t\leqq 1} \|W(nt)\|_B\big)^2.\end{aligned}$$

Now $X_n(\cdot)=n^{-\frac{1}{2}} W(n\cdot)$ induces the same measure P on $\mathscr{C}_B$ as W. Consequently:

$$E\big(\sup_{0\leqq t\leqq 1} \|W(nt)\|_B\big)^2=nE\big(\sup_{0\leqq t\leqq 1} \|W(t)\|_B\big)^2=nE(\|W\|_{C_B})^2$$

where $E\|W\|_{C_B}^2<\infty$ by Fernique's theorem ([4]). In other words:

$$E g^*(W_n) f^*(W_m)=(\text{Const.})(m\wedge n).$$

Again, we let $n_k=[c^k]$, $k\geqq 1$, $c>1$ and define:

$$\zeta_n(\cdot)=(2n \lg\lg n)^{-\frac{1}{2}} W(n\cdot), \qquad Y_k=(2\lg k)^{-\frac{1}{2}} X_{n_k}(\cdot).$$

Then ([11], p. 260) given any $\varepsilon>0$, it is possible to find a number c, sufficiently close to one such that:

$$P\Big(\sup_{n_k\leqq n<n_{k+1}} \|\zeta_{n_k}(\cdot)-\zeta_n(\cdot)\|_{C_B}<\varepsilon, \text{ all large } k\Big)=1.$$

The same argument used under b) shows that the following theorem holds:

Theorem 5.3 ([11]). *Let B denote a separable Banach space with norm $\|\cdot\|_B$ and $\mathscr{C}_B$ the Banach space of all continuous functions from $[0,1]$ to B. $\mathscr{C}_B$ is endowed with the norm of the uniform convergence on $[0,1]$. If $(W(t), t\geqq 0)$ is the Brownian Motion on B and for each $t\in[0,1]$ and $n\geqq 3$: $\zeta_n(t)=(2n \operatorname{lglg} n)^{-\frac{1}{2}} W(nt)$ then, with probability one, the sequence $(\zeta_n(\cdot), n\geqq 3)$ converges in $\mathscr{C}_B$ to a compact set K and clusters at every point of K. The set K is the unit ball in the Hilbert subspace of $\mathscr{C}_B$ which generates the Gaussian measure induced by W on $\mathscr{C}_B$.*

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