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Markov Additive Processes. II*

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1. Introduction

Markov additive processes were introduced in [2] in a general setting in the non-terminating case. Below we shall define time-homogeneous Markov additive processes in a modern setting. Throughout we use the notation and terminology of Blumenthal and Getoor [1]; in referring to it we write BG IV.3.5 to mean the expression or statement (3.5) in Chapter IV of [1].

We recall, in particular, that if $(G, \mathscr{G})$ and $(H, \mathscr{H})$ are measurable spaces and if $f: G \to H$ is measurable with respect to $\mathscr{G}$ and $\mathscr{H}$ then we write $f \in \mathscr{G}/\mathscr{H}$. If $H = \bar{R} = [-\infty, +\infty]$ and $\mathscr{H} = \bar{\mathscr{R}}$-the Borel subsets of $\bar{\mathbb{R}}$, then we write $f \in \mathscr{G}$ instead of $f \in \mathscr{G}/\bar{\mathscr{R}}$. Further, $b\mathscr{G} = \{f \in \mathscr{G}: f \text{ is bounded}\}$, $\mathscr{G}_+ = \{f \in \mathscr{G}: f \geqq 0\}$, $b\mathscr{G}_+ = \{f \in b\mathscr{G}: f \geqq 0\}$.

We will in addition introduce the following notation. If $\{\mathscr{G}_t; t \geqq 0\}$ is an increasing family of sub-σ-algebras of a σ-algebra $\mathscr{M}$ on a set Ω and if the function $T: \Omega \to [0, +\infty]$ is a stopping time with respect to the family $\{\mathscr{G}_t\}$ (that is, if $\{T \leqq t\} \in \mathscr{G}_t$ for every $t \geqq 0$), then we write $T \in s(\mathscr{G}_t)$.

Let $X = (\Omega, \mathscr{M}, \mathscr{M}_t, X_t, \theta_t, P^x)$ be a Markov process with state space $(E, \mathscr{E})$ augmented by Δ (cf. BG I.3.1 for definition). We let

$$\zeta = \inf\{t: X_t = \Delta\};$$

$\mathscr{K}$ will denote the completion of $\mathscr{K}^0 = \sigma(X_t;\ t \geqq 0)$ with respect to the family of measures $\{P^\mu: \mu \text{ a finite measure on } \mathscr{E}_\Delta\}$; $\mathscr{K}_t$ will be the completion of $\mathscr{K}_t^0 = \sigma(X_s: 0 \leqq s \leqq t)$ in $\mathscr{K}$ with respect to the same family $\{P^\mu\}$.

Let G be a topological group with the group operation denoted by "+", and let $\mathscr{G}$ be the Borel subsets of G; ($\mathscr{G}$ is the σ-algebra generated by the open sets in G). A family $A = \{A_t;\ t \geqq 0\}$ of functions defined on $(\Omega, \mathscr{K})$ and taking values in $(G, \mathscr{G})$ is called an *additive functional* of the Markov process X provided that

a) almost surely the mapping $t \to A_t$ is right continuous, has left-hand limits, and $A_0 = 0$, $A_{\zeta-} = A_\zeta$, $A_t = A_\zeta$ for $t \geqq \zeta$;

b) $A_t \in \mathscr{K}_t/\mathscr{G}$ for each $t \geqq 0$;

c) $A_{t+s} = A_t + A_s \circ \theta_t$ almost surely for each pair (t, s); $t, s \geqq 0$.

A family $M = \{M_t;\ t \geqq 0\}$ of real or complex valued functions defined on $(\Omega, \mathscr{K})$ is called a *multiplicative functional* of X provided that

a) almost surely $t \to M_t$ is right continuous;

b) $M_t \in \mathscr{K}_t$ for every $t \geqq 0$;

c) $M_{t+s} = M_t (M_s \circ \theta_t)$ almost surely for each pair (t, s); $t, s \geqq 0$.

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The following two definitions introduce our subject matter.

(1.1) *Definition.* Let $(E, \mathscr{E})$ be a measurable space, and let $F=\overline{\mathbb{R}}^m$, $\mathscr{F}=\overline{\mathscr{R}}^m$ for some fixed integer $m \geqq 1$. A family $\{Q_t;\ t \geqq 0\}$ of functions from $E \times (\mathscr{E} \times \mathscr{F})$ into $[0, 1]$ is said to be a semi-Markov transition function on $(E, \mathscr{E}, \mathscr{F})$ provided

a) for each $t \geqq 0$ and $x \in E$, $\Gamma \to Q_t(x, \Gamma)$ is a measure on $\mathscr{E} \times \mathscr{F}$;

b) for each $t, s \geqq 0$, $x \in E$, $A \in \mathscr{E}$, $B \in \mathscr{F}$,

$$Q_{t+s}(x, A \times B) = \int_{E \times F} Q_t(x, dy \times dz)\, Q_s(y, A \times (B-z))$$

where $B-z=\{u-z\colon u \in B\}$.

(1.2) *Definition.* Let $X=(\Omega, \mathscr{M}, \mathscr{M}_t, X_t, \theta_t, P^x)$ be a Markov process with state space $(E, \mathscr{E})$, write $(F, \mathscr{F})=(\overline{\mathbb{R}}^m, \overline{\mathscr{R}}^m)$, and let $Y=\{Y_t;\ t \geqq 0\}$ be a family of functions from $(\Omega, \mathscr{M})$ into $(F, \mathscr{F})$. Then $(X, Y)=(\Omega, \mathscr{M}, \mathscr{M}_t, X_t, Y_t, \theta_t, P^x)$ is called a *Markov additive process* (MAP) provided the following hold:

a) almost surely the mapping $t \to Y_t$ is right continuous, has left-hand limits, and satisfies $Y_0=0$, $Y_t=Y_\zeta$ for $t \geqq \zeta$;

b) for each $t \in [0, \infty)$, $Y_t \in \mathscr{M}_t/\mathscr{F}$;

c) for each $t \in [0, \infty)$, $A \in \mathscr{E}$, $B \in \mathscr{F}$, the mapping $x \to P^x\{X_t \in A,\ Y_t \in B\}$ of E into $[0, 1]$ is in $\mathscr{E}$;

d) for each $t, s \in [0, \infty)$, $Y_{t+s}=Y_t+Y_s \circ \theta_t$ almost surely;

e) for all $t, s \in [0, \infty)$, $x \in E_\Delta$, $A \in \mathscr{E}_\Delta$, $B \in \mathscr{F}$

$$P^x\{X_s \circ \theta_t \in A,\ Y_s \circ \theta_t \in B \mid \mathscr{M}_t\} = P^{X(t)}\{X_s \in A,\quad Y_s \in B\}.$$

By redefining Y on a set $\Gamma \in \mathscr{M}$ with $P^x(\Gamma)=0$ for all $x \in E_\Delta$ if necessary, we may assume that the regularity properties in (1.2a) hold for all $\omega \in \Omega$. We put $Y_\infty = \lim_{t \to \infty} Y_t$; clearly $Y_\infty = Y_\zeta$. Further, if Y is not continuous at ζ, we may redefine it at ζ so that $Y_{\zeta-}=Y_\zeta$; this will not alter the validity of other conditions. These alterations we will assume done without special mention.

It is easy to check that, if (1.2a, b, c) hold, then (1.2c) remains valid when E, $\mathscr{E}$, $[0, +\infty)$ are replaced by E_Δ, $\mathscr{E}_\Delta$, $[0, +\infty]$. Thus, by the monotone class theorem, $x \to P^x\{(X_t, Y_t) \in \Gamma\}$ is in $\mathscr{E}_\Delta$ for any $t \in [0, +\infty]$, $\Gamma \in \mathscr{E}_\Delta \times \mathscr{F}$.

The exceptional set in (1.2d) depends on both s and t generally. In view of the right continuity of Y assumed in (1.2a) however, for each fixed $t \geqq 0$, $Y_{t+s}=Y_t+Y_s \circ \theta_t$ for all $s \geqq 0$ except possibly on some set Γ_t with $P^x(\Gamma_t)=0$ for all x. In this connection we introduce the following

(1.3) *Definition.* A MAP (X, Y) is said to be *perfect* provided there exists a set $\Gamma \in \mathscr{M}$ with $P^x(\Gamma)=1$ for all x such that $Y_{t+s}(\omega)=Y_t(\omega)+Y_s(\theta_t\,\omega)$ for all $s, t \geqq 0$ whenever $\omega \in \Gamma$.

(1.4) *Definition.* A MAP (X, Y) is said to be strong Markov if X is strong Markov and

$$P^x\{X_t \circ \theta_T \in A,\ Y_t \circ \theta_T \in B \mid \mathscr{M}_T\} = P^{X(T)}\{X_t \in A,\ Y_t \in B\}$$

for all $x \in E_\Delta$, $A \in \mathscr{E}_\Delta$, $B \in \mathscr{F}$, $t \geqq 0$, $T \in s(\mathscr{M}_t)$.

We shall show in Section 3 that, if X is standard, then every MAP (X, Y) with Y non-decreasing real-valued is strong Markov.

(1.5) *Definition.* Let (X, Y) be a MAP. Y is said to be *continuous* if almost surely $t \to Y_t$ is continuous; Y is called a *pure jump* process if $t \to Y_t$ is a step function almost surely. When E is a topological space, Y is said to be *natural* if almost surely $t \to X_t$ and $t \to Y_t$ have no common discontinuities. Y is called *quasi-left-continuous* if $\lim_n Y(T_n) = Y(T)$ almost surely for every increasing sequence $\{T_n\} \subset s(\mathcal{M}_t)$ with $\lim_n T_n = T$.

If in Definition (1.2) the condition $Y_t \in \mathcal{M}_t/\mathcal{F}$ is replaced by $Y_t \in \mathcal{K}_t/\mathcal{F}$, then Y becomes an additive functional of X (in this case the conditions (1.2c) and (1.2e) are superfluous). If, further, Y is numerically valued and $t \to Y_t$ almost surely non-decreasing, then Y is an additive functional of X in the sense of BG IV.1.1. We remark that in this case our definitions of the terms perfect, strong Markov, continuous, etc. coincide with those given in BG IV.1.3, BG IV.1.11, BG IV.1.15.

Throughout the following X will be a standard Markov process: the state space E is locally compact with a countable base, $\mathcal{E}$ is the set of all Borel subsets of E, Δ is the point at infinity if E is noncompact and is an isolated point if E is compact; $\{\mathcal{M}_t\}$ is right continuous and "complete"; $t \to X_t$ is right continuous on $[0, \infty)$ almost surely; X is normal, strong Markov with respect to $\{\mathcal{M}_t\}$; X is quasi-left-continuous on $[0, \zeta)$. We will sometimes assume X to be a Hunt process (a standard process which is quasi-left-continuous on $[0, \infty)$); but this will always be explicitly stated.

In addition to the canonical σ-algebras $\{\mathcal{K}_t\}$ introduced already we define $\mathcal{L}$ to be the completion of $\mathcal{L}^0 = \sigma\{X_t; Y_t; t \geqq 0\}$ with respect to $\{P^\mu: \mu$ a finite measure on $\mathcal{E}_\Delta\}$ and $\mathcal{L}_t$ to be the completion in $\mathcal{L}$ of $\mathcal{L}_t^0 = \sigma(X_s, Y_s; 0 \leqq s \leqq t)$ with respect to the same family. We write $\mathcal{E}^*, \mathcal{F}^*, \mathcal{R}^*$ for the universally measurable subsets of $E, F, [0, \infty]$ respectively.

In the next section various preliminary results will be given; most of these will be without proofs. In Section 3 we examine the strong Markov property.

The process Y can be decomposed as

$$Y = A + Y^f + Y^c + Y^d$$

in a manner analogous to Lévy's decomposition of additive processes; here A is an additive functional of X, Y^f is a pure jump process whose jump times are "fixed" once the path of X is known, Y^c is continuous, Y^d increases or decreases by jumps but these times are *not* fixed by X. In Section 4, Y^f is further decomposed as

$$Y^f = Y^q + Y^n$$

where Y^q is quasi-left-continuous and not natural, whereas Y^n is natural and not quasi-left-continuous. In both cases their structures are completely characterized by using the known results for additive functionals. In Section 5 the component Y^d is examined; roughly speaking, it is of the form

$$Y_t^d = \int Z_t \, dM$$

where M is a Poisson random measure independent of X and Z_t is an additive functional of X.

2. Dependence of Y on X

Let $(X, Y)=(\Omega, \mathcal{M}, \mathcal{M}_t, X_t, Y_t, \theta_t, P^x)$ be a MAP with X taking values in $(E, \mathscr{E})$ and Y in $(F, \mathscr{F})=(\mathbb{R}^m, \mathscr{R}^m)$. X is standard; and $\{\mathscr{K}_t\}$, $\{\mathscr{L}_t\}$ are as defined before.

(2.1) **Proposition.** *For $x\in E$, $\Gamma\in\mathscr{E}\times\mathscr{F}$ define*

$$Q_t(x, \Gamma)=P^x\{(X_t, Y_t)\in\Gamma\}, \quad t\geqq 0.$$

Then $\{Q_t; t\geqq 0\}$ is a semi-Markov transition function.

Proof is omitted. Clearly, if $\{Q_t; t\geqq 0\}$ is a semi-Markov transition function and if

$$(2.2) \qquad P_t(x, A)=Q_t(x, A\times F), \quad x\in E, \; A\in\mathscr{E}$$

for all $t\geqq 0$, then $\{P_t; t\geqq 0\}$ is a Markov transition function. It was shown in Çinlar [2] that a MAP (X, Y) with a given semi-Markov transition function $\{Q_t\}$ exists if and only if there exists a Markov process X with transition function $\{P_t\}$ defined by (2.2). A MAP(X, Y) is not a Markov process in the sense of BG I.3.1, the main difference being in the way the shift operators θ_t work. However we have

(2.3) **Proposition.** *For each $x\in E_\Delta$, the stochastic process $\{(X_t, Y_t); t\geqq 0\}$ defined over $(\Omega, \mathcal{M}, P^x)$ is a Markov process adapted to $\{\mathcal{M}_t; t\geqq 0\}$ with state space $(E_\Delta\times F, \mathscr{E}_\Delta\times\mathscr{F})$ (in the sense of* BG I.1.1).

The transition function for this Markov process is translation invariant in the $(F, \mathscr{F})$ variables. For this reason Ezhov and Skorokhod [4] refer to it as a "Markov process with homogeneous second component." Of course, it is easy to construct a Markov process in the sense of BG I.3.1 from a given MAP. However, that is not of interest to us.

(2.4) **Proposition.** *For each $t\geqq 0$, $\theta_t\in\mathscr{L}_{s+t}/\mathscr{L}_s$ for any $s\geqq 0$; and therefore, $\theta_t\in\mathscr{L}/\mathscr{L}$. For any $Z\in b\mathscr{L}$, the mapping $x\to E^x[Z]$ is in $\mathscr{E}_\Delta^*$.*

Proof follows easily from the Definition (1.2). Following is a consequence of (1.2e) and (2.4).

(2.5) **Proposition.** *For each $t\geqq 0$ and $Z\in b\mathscr{L}$,*

$$E^x[Z\circ\theta_t|\mathcal{M}_t]=E^{X(t)}[Z].$$

The following propositions examine the conditional behavior of Y given X.

(2.6) **Lemma.** *For any $x\in E_\Delta$, $t\geqq 0$, $Z\in b\mathscr{L}_t^0$,*

$$E^x[Z|\mathscr{K}^0]=E^x[Z|\mathscr{K}_t^0].$$

Proof. The right-hand side has the required measurability property. Let $H\in\mathscr{K}_t^0$, $G\in\mathscr{K}^0$. Then, from the Markov property for X we have, since $Z\in\mathcal{M}_t$,

$$\begin{aligned} E^x[ZH(G\circ\theta_t)] &= E^x[ZHE^{X(t)}[G]] \\ &= E^x[HE^{X(t)}[G]\,E^x[Z|\mathscr{K}_t^0]] \\ &= E^x[E^x[HE^x[Z|\mathscr{K}_t^0]\,G\circ\theta_t|\mathscr{K}_t^0]] \\ &= E^x[H(G\circ\theta_t)\,E^x[Z|\mathscr{K}_t^0]]. \end{aligned}$$

Since functions of the form $H(G\circ\theta_t)$ with $H\in\mathscr{K}_t^0$ and $G\in\mathscr{K}^0$ generate $\mathscr{K}^0$ this completes the proof.

(2.7) **Proposition.** *For any* $x\in E_\Delta$, $t\geqq 0$, $Z\in b\mathscr{L}_t$,

$$E_x[Z|\mathscr{K}]=E^x[Z|\mathscr{K}_t].$$

Proof. The right-hand side has the proper measurability property. Since any $\Lambda\in\mathscr{K}$ differs from a set $\Lambda_0\in\mathscr{K}^0$ by a P^x-negligible set, it is sufficient to show that

$$E^x[Z;\Lambda]=E^x[E^x[Z|\mathscr{K}_t];\Lambda], \tag{2.8}$$

for $\Lambda\in\mathscr{K}^0$. By the definition of $\mathscr{L}_t$, there exist $Z_x\in b\mathscr{L}_t^0$ and $\Gamma\in\mathscr{L}^0$ such that $\{Z\neq Z_x\}\subset\Gamma$ and $P^x(\Gamma)=0$. Then

$$E^x[Z;\Lambda]=E^x[Z_x;\Lambda]=E^x[E^x[Z_x|\mathscr{K}_t^0];\Lambda] \tag{2.9}$$

for any $\Lambda\in\mathscr{K}^0$ by Lemma (2.6). On the other hand, (2.9) holds for $\Lambda\in\mathscr{K}_t^0\subset\mathscr{K}^0$ also, which implies that

$$E^x[Z|\mathscr{K}_t]=E^x[Z_x|\mathscr{K}_t^0] \tag{2.10}$$

since for any $\Gamma\in\mathscr{K}_t$ there is $\Lambda\in\mathscr{K}_t^0$ and $M,N\in\mathscr{K}^0$ such that $\Lambda-N\subset\Gamma\subset\Lambda\cup M$ and $P^x(N)=P^x(M)=0$. Now (2.8), (2.9), (2.10) together yield the desired result.

In the preceding proposition the dependence of conditional expectations on x can be dispensed with by choosing the proper versions:

(2.11) **Proposition.** *For any* $Z\in b\mathscr{L}$ *there is a random variable* $W\in b\mathscr{K}$ *satisfying*

$$E^x[Z|\mathscr{K}]=W$$

for all $x\in E_\Delta$.

Proof. a) Fix $t\geqq 0$; let $\{J_n\}$ be an increasing sequence of finite subsets of $[0,t]$, each containing 0 and t, and such that $J=\bigcup_n J_n$ is dense in $[0,t]$. Let $\mathscr{K}^n$ be the completion in $\mathscr{K}$ of $\sigma(X_s;\, s\in J_n)$ with respect to $\{P^\mu$: μ a finite measure on $\mathscr{E}_\Delta\}$. Since X is standard, E is locally compact with a countable base and X is right continuous. Therefore (cf. BG III.2.2 and the remark following it), we have

$$\bigcup_n \mathscr{K}^n=\mathscr{K}_t. \tag{2.12}$$

b) There is no loss of generality in assuming $Z\in b\mathscr{L}_t$. Since $\mathscr{E}$ has a countable base and J_n is finite, $\mathscr{E}^{J_n}$ has a countable base and therefore $\sigma(X_t;\, t\in J_n)$ has a countable base. Thus, we can apply a result in Doob [3, p. 344] to obtain a version of

$$W_n^x=E^x[Z|\mathscr{K}^n] \tag{2.13}$$

so that the mapping $(x,\omega)\to W_n^x(\omega)$ is in $\mathscr{E}_\Delta^*\times\mathscr{K}^n$. Define

$$W_n=W_n^{X(0)} \tag{2.14}$$

for each n; then $W_n\in\mathscr{K}^n$ and for any $\Lambda\in\mathscr{K}^n$

$$E^x[W_{n+1};\Lambda]=E^x[W_{n+1};\Lambda\cap\{X_0=x\}]=E^x[W_{n+1}^x;\Lambda\cap\{X_0=x\}]$$

by the normality of X and (2.14). Now using (2.13), the fact that $\mathscr{K}^n \subset \mathscr{K}^{n+1}$, (2.14) and the normality of X again, we get

$$E^x[W_{n+1};\ \Lambda] = E^x[W_n;\ \Lambda].$$

Hence $\{W_n, \mathscr{K}^n\}$ is a martingale with respect to the measure P^x for any x. Since $W_n \in \mathscr{K}^n \subset \mathscr{K}_t$ for each n and since the W_n are uniformly bounded (because Z is bounded), $W = \limsup W_n \in b\mathscr{K}_t$. By the martingale convergence theorem, for each x, $\lim W_n$ exists and $P^x\{\lim W_n = W\} = 1$. By the normality of X, (2.13) and (2.14) we have

$$E^x[W|\mathscr{K}^n] = W_n = E^x[Z|\mathscr{K}^n];$$

thus, letting $n \to \infty$, we have

$$W = E^x[Z|\bigcup_n \mathscr{K}^n] = E^x[Z|\mathscr{K}_t] \tag{2.15}$$

by (2.12) for any $x \in E_\Delta$. Since $Z \in b\mathscr{L}_t$, by Proposition (2.7), the last term in (2.15) is $E^x[Z|\mathscr{K}]$.

(2.16) *Notation.* Making use of the preceding proposition we will simply write EZ for the version of $E^x[Z|\mathscr{K}]$ constructed in the proof above for any $Z \in b\mathscr{L}$. Note that if $Z \in b\mathscr{L}_t$ then $EZ \in b\mathscr{K}_t$.

(2.17) **Proposition.** *For any* $t \geqq 0$, $x \in E_\Delta$, *and* $Z \in b\mathscr{L}$,

$$E^x[Z \circ \theta_t | \mathscr{M}_t \vee \mathscr{K}] = (EZ) \circ \theta_t.$$

Proof. It is sufficient to show that

$$E^x[H(G \circ \theta_t)(Z \circ \theta_t)] = E^x[H \cdot (G \circ \theta_t)(EZ \circ \theta_t)]$$

for all x, $H \in b\mathscr{M}_t$, $G \in b\mathscr{K}$, $Z \in b\mathscr{L}$. But this follows from Proposition (2.5) and the observation that

$$E^y[GZ] = E^y[GE^y[Z|\mathscr{K}]] = E^y[GEZ]$$

for all $y \in E_\Delta$.

The next theorem states that, roughly speaking, conditional on the history $\mathscr{K}$ of X, Y is a process with independent increments. We will make this more precise in Theorem (2.22) by choosing a proper version of "the conditional probability given $\mathscr{K}$".

(2.18) **Proposition.** *For any integer* $n \geqq 1$, $0 \leqq t_0 < t_1 < \cdots < t_n$, *and* $f_1, \ldots, f_n \in b\mathscr{F}^*$ *we have, almost surely,*

$$E\left[\prod_{j=1}^n f_j(Y_{t_j} - Y_{t_{j-1}})\right] = \prod_{j=1}^n E[f_j(Y_{t_j} - Y_{t_{j-1}})].$$

The proof follows from Proposition (2.17) by induction on n. The following corollary is easy to derive by using Propositions (2.18) and (2.17) along with (1.2a, b, d).

(2.19) **Corollary.** *Let*

$$M_t^\lambda = E[e^{i(\lambda, Y_t)}], \qquad \lambda \in F,\ t \geqq 0,$$

where (λ, y) *is the usual inner product in* $F = \overline{\mathbb{R}}^m$. *Then* $\{M_t^\lambda;\ t \geqq 0\}$ *is a multiplicative functional* of X *for each fixed* $\lambda \in F$.

The following is what we needed to make the rough statement of (2.18) precise.

(2.20) **Proposition.** *There exists a regular version of* $P^x\{\cdot|\mathscr{K}\}$ *on* $\mathscr{L}$ *which is further independent of* x; *that is, there exists a function* $P_\omega(A)$ *on* $\Omega \times \mathscr{L}$ *such that*

a) *for any* $A \in \mathscr{L}$, $\omega \to P_\omega(A)$ *is in* $\mathscr{K}$;

b) *for any* $\omega \in \Omega$, $A \to P_\omega(A)$ *is a probability measure on* $\mathscr{L}$;

c) *for any* $B \in \mathscr{K}$ *and* $x \in E_\Delta$,

$$\int_B P_\omega(A)\, P^x(d\omega) = P^x(A \cap B).$$

Proof. By Theorem (3.7) of Çinlar [2] there exists a regular version $P^x\{\cdot|\mathscr{K}\}$ on $\mathscr{L}$ for any fixed x. That this version can further be selected to be independent of x is what is being claimed. The proof is exactly the same as in [2] except that, in the notation of Lemma (3.2) of [2], we need to choose

$$G_{s,t}(w, B) = (\hat{P}^x\{\hat{Y}_t - \hat{Y}_s \in B|\hat{\mathscr{K}}\})(w)$$

so that it is independent of x. That this is possible follows from (2.11).

We will write P_ω to denote the conditional probability (evaluated at ω) whose existence is shown above. When suppressing ω, we write simply P. We note that, referring to Notation (2.16), we have

$$(EZ)(\omega) = \int_\Omega Z(\omega')\, P_\omega(d\omega') \tag{2.21}$$

for any $Z \in b\mathscr{L}$ almost surely for all ω. Thus, E is the "expectation" operator corresponding to P.

(2.22) **Theorem.** *For any fixed* $\omega \in \Omega$, *the stochastic process* $(\Omega, \mathscr{M}, \mathscr{M}_t, Y_t, P_\omega)$ *has independent increments.*

The proof is immediate from Proposition (2.18) and (2.21). Appealing to the theory of additive processes (cf. Doob [3; Chapter VIII] and Ito [5; Section 4]) we obtain the following analog of Lévy's decomposition for processes with independent increments. We shall omit the proof and only remark that a) independence with respect to P means conditional independence given $\mathscr{K}$ with respect to P^x for each x, and b) if $Z \in \mathscr{L}_t$ is constant with respect to P then $Z \in \mathscr{K}_t$.

(2.23) **Theorem.** *We have*

$$Y_t = A_t + Y_t^f + Y_t^c + Y_t^d, \qquad t \geqq 0, \tag{2.24}$$

where $\sigma(Y_t^f;\ t \geqq 0)$, $\sigma(Y_t^c;\ t \geqq 0)$, $\sigma(Y_t^d;\ t \geqq 0)$ *are conditionally independent given* $\mathscr{K}$ *with respect to* P^x *for each* x. *This decomposition is unique up to the addition of functionals of* X. *The components satisfy the following:*

a) $A = \{A_t;\ t \geqq 0\}$ *is an additive functional of* X.

b) $Y^f = \{Y_t^f;\ t \geqq 0\}$ *is a pure jump process;* (X, Y^f) *is a* MAP; *if* T *is a jump time of* Y^f, *then* $T \in \mathscr{K}$.

c) $Y^c=\{Y_t^c;\, t\geqq 0\}$ *is continuous;* (X, Y^c) *is a* MAP. *Thus,* Y^c *is a Gaussian process over* $(\Omega, \mathscr{L}, P)$.

d) $Y^d=\{Y_t^d;\, t\geqq 0\}$ *is a stochastically continuous process with independent increments over* $(\Omega, \mathscr{L}, P)$; (X, Y^d) *is a* MAP.

(2.25) **Corollary.** *Let* M_t^λ *be defined as in* (2.19). *Then, for each* $t\geqq 0$, $\lambda\in F$, $\omega\in\Omega$,

$$M_t^\lambda(\omega)=\Big\{\prod_{\tau_j(\omega)\leqq t} F_j^\lambda(\omega)\Big\}\Big\{\exp\Big[i(\lambda, A_t(\omega))-\tfrac{1}{2}(\lambda, C_t(\omega)\,\lambda) + \int_F \Big(e^{i(\lambda,x)}-1-\frac{i(\lambda,x)}{1+|x|^2}\Big)\frac{1+|x|^2}{|x|^2}\,B_t(\omega, dx)\Big]\Big\} \tag{2.26}$$

where

a) *for each* j, $\tau_j\in\mathscr{K}$ *and each* F_j^λ *is a characteristic function in* λ *and is* $\mathscr{K}$-*measurable for fixed* λ;

b) $A=\{A_t;\, t\geqq 0\}$ *is an additive functional of* X;

c) $C=\{C_t;\, t\geqq 0\}$ *is a continuous additive functional of* X *taking values in the space of non-negative definite symmetric operators on* $F=\overline{\mathbb{R}}^m$; *then* $(\lambda, C\lambda)$ *is the usual quadratic form;*

d) *for each* $t\geqq 0$ *and* $\omega\in\Omega$, $A\to B_t(\omega, A)$ *is a finite measure on* $\mathscr{F}$; *and for fixed* $A\in\mathscr{F}$, $B(A)=\{B_t(A);\, t\geqq 0\}$ *is a numerical valued non-decreasing continuous additive functional of* X.

In the decomposition (2.24) A is completely determined by X; the jumps of Y^f are fixed by X but the amounts of jumps themselves may depend on other variables; Y^c is continuous and is obtained from a multidimensional Brownian motion which is independent of X through a random time and scale transformation, the rules of the transformation being determined by X; Y^d is, up to a normalizing continuous additive functional, a pure jump process and is obtained from a Poisson random measure by certain transformations whose laws are governed by X again. In sections 4 and 5, the structures of Y^f and Y^d will be examined further.

Ezhov and Skorohod [4] give a decomposition by using (2.18) directly without the aid of the version P of the conditional probability $P^x\{\cdot|\mathscr{K}\}$. Then they assume $t\to M_t^\lambda$ to be continuous and give something close to our Corollary (2.25). As a result, the component Y^f is missing in their decomposition, and $t\to A_t$ must be continuous. They do, however, give a complete decomposition in the case where X is a regular step process (with each x a holding point). They use characteristic functions exclusively.

3. Strong Markov Property

As before $(X, Y)=(\Omega, \mathscr{M}, \mathscr{M}_t, X_t, Y_t, \theta_t, P^x)$ is a fixed Markov additive process, $X=(\Omega, \mathscr{M}, \mathscr{M}_t, X_t, \theta_t, P^x)$ is a standard Markov process with state space $(E, \mathscr{E})$, Y takes values in $(F, \mathscr{F})=(\overline{\mathbb{R}}^m, \overline{\mathscr{R}}^m)$. The families of σ-algebras $\{\mathscr{K}_t\}$, $\{\mathscr{L}_t\}$ are as before. We remind that $s(\mathscr{G}_t)$ denotes the set of all stopping times with respect to the increasing family of σ-algebras $\{\mathscr{G}_t\}$, and that P and E refer to the conditional probability and expectation discussed in (2.20) and (2.21).

It was shown in Theorem (2.22) that the process $\{Y_t; P_\omega\}$ has independent increments for any fixed $\omega \in \Omega$. Thus, Proposition (2.18) remains true if $t_1, \ldots, t_n$ there are replaced by $T_1, \ldots, T_n$ where each $T_i: \Omega \to [0, +\infty]$ is in $\mathscr{K}$ and $T_1 < \cdots < T_n$ almost surely. We will be able to say somewhat more on this later.

(3.1) **Proposition.** *For any* $x \in E_\Delta$, $Z \in b\mathscr{L}$, $T \in s(\mathscr{M}_t)$ *we have* $\theta_T \in \mathscr{M}_{T+t}/\mathscr{L}_t$ *for each t and*

$$E^x[Z \circ \theta_T] = E^x[E^{X(T)}[Z]].$$

Proof. Since X is strong Markov, for each $x \in E_\Delta$, if we write $\mu = P^x X_T^{-1}$ then $P^\mu = P^x \theta_T^{-1}$. Further, the completion of $\mathscr{L}_T$ in $\mathscr{L}$ with respect to the family $\{P^\mu;$ μ a finite measure on $\mathscr{E}_\Delta\}$ is equal to $\mathscr{L}_T$. Thus, $\theta_T \in \mathscr{M}_{T+t}/\mathscr{L}_t$ for each t, and therefore $\theta_T \in \mathscr{M}/\mathscr{L}$. Now,

$$\begin{aligned} E^x[Z \circ \theta_T] &= \int_\Omega Z \circ \theta_T \, dP^x = \int_\Omega Z \, d(P^x \theta_T^{-1}) = \int_\Omega Z \, dP^\mu \\ &= \int_{E_\Delta} E^y(Z) \, \mu(dy) = \int_{E_\Delta} E^y(Z) \, d(P^x X_T^{-1}) \\ &= \int_\Omega E^{X(T)}(Z) \, dP^x = E^x[E^{X(T)}(Z)] \end{aligned}$$

as claimed. (We have used the fact that $y \to E^y(Z)$ is in $\mathscr{E}_\Delta^*$; this follows from Proposition (2.4).)

Following shows that (X, Y) has the strong Markov property in the sense of Definition (1.4).

(3.2) **Theorem.** *For any* $T \in s(\mathscr{M}_t)$, $f \in b\mathscr{E}_\Delta^*$, $g \in b\mathscr{F}^*$

$$E^x[f(X_t \circ \theta_T) \, g(Y_t \circ \theta_T) | \mathscr{M}_T] = E^{X(T)}[f(X_t) \, g(Y_t)]$$

for all $x \in E_\Delta$ *and* $t \geqq 0$.

Proof. For $\Lambda \in \mathscr{M}_T$ define $T_\Lambda = T$ on Λ and $T_\Lambda = +\infty$ on Λ^c. Then $T_\Lambda \in s(\mathscr{M}_t)$ also. Noting that $\theta_\infty \, \omega = \omega_\Delta$ and $X_t(\omega_\Delta) = \Delta$, $\zeta(\omega_\Delta) = 0$, $Y_t(\omega_\Delta) = Y_\zeta(\omega_\Delta) = 0$ we have

$$E^x[f(X_t \circ \theta_{T_\Lambda}) \, g(Y_t \circ \theta_{T_\Lambda})] = E^x[f(X_t \circ \theta_T) \, g(Y_t \circ \theta_T); \; \Lambda] + f(\Delta) \, g(0) \, P^x(\Lambda^c)$$

$$E^x[E^{X(T_\Lambda)}[f(X_t) \, g(Y_t)]] = E^x[E^{X(T)}(f(X_t) \, g(Y_t)); \; \Lambda] + f(\Delta) \, g(0) \, P^x(\Lambda^c).$$

By Proposition (3.1) the left sides of these equations are equal.

This yields by the usual induction methods, using the monotone class theorem, the following

(3.3) **Corollary.** *For any* $T \in s(\mathscr{M}_t)$, $Z \in b\mathscr{L}$ *and all* $x \in E_\Delta$,

$$E^x[Z \circ \theta_T | \mathscr{M}_T] = E^{X(T)}[Z].$$

Next is a result concerning $\{\mathscr{K}_t\}$ stopping times.

(3.4) **Proposition.** *For any* $T \in s(\mathscr{K}_t)$, $Z \in b\mathscr{L}$, $x \in E_\Delta$,

$$E^x[Z \circ \theta_T | \mathscr{M}_T \vee \mathscr{K}] = (EZ) \circ \theta_T.$$

Proof. Let $\mathscr{H}_T$ be the completion in $\mathscr{K}$ of $\sigma(X_{T+t}; t \geqq 0)$ with respect to $\{P^\mu;$ μ a finite measure on $\mathscr{E}_\Delta\}$. Then, since $T \in s(\mathscr{K}_t)$, the completion of $\sigma(\mathscr{K}_T \cup \mathscr{H}_T)$

in $\mathscr{K}$ with respect to $\{P^\mu\}$ is equal to $\mathscr{K}$ itself (cf. BG III.4.20 for a proof). Since $\mathscr{K}_T \subset \mathscr{M}_T$, it suffices to show that

$$E^x[GH(Z\circ\theta_T)] = E^x[GH(EZ\circ\theta_T)] \tag{3.5}$$

for $G\in b\mathscr{M}_T$, $H=\prod_{j=1}^{n} f_j(X_{T+t_j})\in\mathscr{H}_T$ with $f_j\in b\mathscr{E}_\Delta^*$, $j=1,\dots,n$, $0\leqq t_1<\dots<t_n$. We can write $H=F\circ\theta_T$ where $F\in b\mathscr{K}$. Then, since $FZ\in b\mathscr{L}$, by Corollary (3.3),

$$E^x[G(F\circ\theta_T)(Z\circ\theta_T)] = E^x[GE^{X(T)}[FZ]]. \tag{3.6}$$

On the other hand,

$$E^y[FZ] = E^y[FE^y[Z|\mathscr{K}]] = E^y[F(EZ)]; \tag{3.7}$$

and by strong the Markov property for X, since $F(EZ)\in b\mathscr{K}$,

$$E^{X(T)}[F(EZ)] = E^x[F\circ\theta_T(EZ)\circ\theta_T|\mathscr{M}_T]. \tag{3.8}$$

Now (3.6), (3.7), (3.8) yield (3.5).

(3.9) **Corollary.** *For any $Z\in b\mathscr{L}$, $T\in s(\mathscr{K}_t)$*

$$E^x(Z\circ\theta_T|\mathscr{K}) = (EZ)\circ\theta_T$$

for all x.

If Y is perfect then $Y_{T+t} = Y_T + Y_t\circ\theta_T$ for all t almost surely for each $T\in s(\mathscr{M}_t)$. This put together with Proposition (3.4) gives the following

(3.10) **Corollary.** *Suppose Y is perfect and let $T, S\in s(\mathscr{K}_t)$, $f\in b\mathscr{F}^*$. Then*

$$E^x[f(Y_S\circ\theta_T)] = (Ef(Y_S))\circ\theta_T.$$

Note that, when Y is perfect, $Y_S\circ\theta_T = Y_{S\circ\theta_T}(\theta_T) = Y_{T+S\circ\theta_T} - Y_T$.

If Y is not known to be perfect, results above do not enable us to handle expressions such as $E^x[f(X_{T+t})\,g(Y_{T+t})|\mathscr{M}_T]$ for $T\in s(\mathscr{M}_t)$. Following are some results in that direction.

Assume Y is real valued and non-decreasing. Then $M^\lambda = \{M_t^\lambda;\, t\geqq 0\}$, defined by

$$M_t^\lambda = E[e^{-\lambda Y(t)}], \qquad t\geqq 0, \tag{3.11}$$

is a multiplicative functional of X in the sense of BG III.1.1. Since $Y_0=0$, $M_0^\lambda=1$ almost surely. Hence, for fixed $\lambda\geqq 0$, M^λ is exact (in the sense of BG III.4.13), regular (in the sense of BG III.4.11), and therefore (cf. BG III.4.12) has the strong Markov property:

$$E^x[M_{T+t}^\lambda f(X_{T+t})] = E^x[M_T^\lambda E^{X(T)}[M_t^\lambda f(X_t)]] \tag{3.12}$$

for all $x\in E_\Delta$, $t\geqq 0$, $f\in b\mathscr{E}_\Delta^*$, and $T\in s(\mathscr{M}_t)$. Of these properties, exactness especially will be used below.

(3.13) **Proposition.** *For any $\lambda\geqq 0$, $f\in b\mathscr{E}^*$, and $T\in s(\mathscr{M}_t)$,*

$$E^x[f(X_{T+t})\,e^{-\lambda Y(T+t)}] = E^x[e^{-\lambda Y(T)}E^{X(T)}[f(X_t)\,e^{-\lambda Y_t}]]$$

for all $x\in E_\Delta$ and $t\geqq 0$.

Proof. For each n define

$$T_n = \begin{cases} \frac{k}{2^n} & \text{on } \left\{\frac{k-1}{2^n} \leqq T < \frac{k}{2^n}\right\}, \\ +\infty & \text{on } \{T=+\infty\}. \end{cases}$$

Then $T_n \in s(\mathscr{M}_t)$ for each n, and $T_n \downarrow T$ as $n \uparrow \infty$.

Suppose first that f is non-negative and continuous. Then $t \to f(X_t)$ is right-continuous, and $t \to Y_t$ is right-continuous by (1.2a). Thus, for $\alpha > 0$,

$$\begin{aligned} E^x \int_0^\infty & e^{-\alpha t} f(X_{T+t}) \exp(-\lambda Y_{T+t})\, dt \\ &= \lim_n E^x \int_0^\infty e^{-\alpha t} f(X_{T_n+t}) \exp(-\lambda Y_{T_n+t})\, dt \\ &= \lim_n \sum_k E^x \left[\int_0^\infty e^{-\alpha t} f(X_{t+k/2^n}) \exp(-\lambda Y_{t+k/2^n});\ T_n = \frac{k}{2^n}\right] \\ &= \lim_n \sum_k E^x \left[\exp(-\lambda Y_{k/2^n}) \left(\int_0^\infty e^{-\alpha t} f(X_t)\, e^{-\lambda Y_t}\, dt\right) \circ \theta_{k/2^n};\ T_n = \frac{k}{2^n}\right] \\ &= \lim_n E^x \left[\exp(-\lambda Y_{T_n})\, E^{X(T_n)} \int_0^\infty e^{-\alpha t} f(X_t)\, e^{-\lambda Y_t}\, dt\right] \end{aligned}$$

where we used (1.2d) and applied Proposition (2.5) after noting that $\{T_n = k/2^n\} \in \mathscr{M}_{k/2_n}$.

As $n \to \infty$, $T_n \downarrow T$ and $Y_{T_n} \downarrow Y_T$ by the right continuity of Y. On the other hand,

$$g^\alpha(x) = E^x \int_0^\infty e^{-\alpha t} f(X_t)\, e^{-\lambda Y_t}\, dt = E^x \int_0^\infty e^{-\alpha t} f(X_t)\, M_t^\lambda\, dt$$

where M^λ is as defined in (3.11). Since $M_0^\lambda = 1$, by BG III.4.10, g^α is α-excessive and hence $t \to g^\alpha(X_t)$ is right-continuous almost surely. Thus, $g^\alpha(X_{T_n}) \to g^\alpha(X_T)$ as $n \to \infty$; and we have

$$\begin{aligned} E^x \int_0^\infty & e^{-\alpha t} f(X_{T+t})\, e^{-\lambda Y_{T+t}}\, dt \\ &= E^x \left[e^{-\lambda Y(T)} E^{X(T)} \int_0^\infty e^{-\alpha t} f(X_t)\, e^{-\lambda Y_t}\, dt\right]. \end{aligned}$$

Hence the functions $t \to E^x[f(X_{T+t})\, e^{-\lambda Y(T+t)}]$ and

$$t \to E^x\left[e^{-\lambda Y(T)} E^{X(T)}[f(X_t)\, e^{-\lambda Y_t}]\right]$$

have the same Laplace transform. By the right continuity of these two functions and the uniqueness of Laplace transforms, it follows that they are identical. This shows that (3.13) holds for f continuous non-negative; and therefore (3.13) holds for any $f \in b\mathscr{E}^*$.

(3.14) *Remark.* For $T \in s(\mathscr{K}_t)$, $E[e^{-\lambda Y(T)}] = M_T^\lambda$. However, this is not true for general $T \in s(\mathscr{M}_t)$. Thus (3.13) does not follow from (3.12).

(3.15) **Proposition.** *For any* $x \in E_\Delta$, $f \in b\mathscr{E}_\Delta^*$, $t \geqq 0$, $\lambda \geqq 0$, *and* $T \in s(\mathscr{M}_t)$,

$$E^x[f(X_{T+t})\, e^{-\lambda Y(T+t)} | \mathscr{M}_T] = e^{-\lambda Y(T)} E^{X(T)}[f(X_t)\, e^{-\lambda Y_t}].$$

Proof. The method of proof used in (3.2) may be used to complete this proof once we show that

$$E^x[f(X_{T+t})\, e^{-\lambda Y(T+t)}] = E^x[e^{-\lambda Y(T)} E^{X(T)}[f(X_t)\, e^{-\lambda Y_t}]] \tag{3.16}$$

for all $T \in s(\mathscr{M}_t)$ and $\lambda \geqq 0$, $f \in b\mathscr{E}_\Delta^*$, and $x \in E_\Delta$.

If $f \in b\mathscr{E}^*$, then this follows from Proposition (3.13). On the other hand, for $f = I_{E_\Delta}$, going through the same proof as in (3.13) and using the exactness (cf. BG III.4.13) of the multiplicative functional M^λ to show

$$\lim_n E^{X(T_n)}\left[\int_0^\infty e^{-\alpha t}\, e^{-\lambda Y_t}\, dt\right] = E^{X(T)} \int_0^\infty e^{-\alpha t}\, e^{-\lambda Y_t}\, dt$$

(cf. BG III.4.23 to see that this holds even though M_∞^λ is possibly non-zero in our case), we see that (3.16) still holds. Since any function $f \in b\mathscr{E}_\Delta^*$ is a linear combination of a function in $b\mathscr{E}^*$ and I_{E_Δ}, and since the set of functions in $b\mathscr{E}_\Delta^*$ for which (3.16) holds is a vector space, (3.16) holds for all $f \in b\mathscr{E}_\Delta^*$.

(3.17) **Proposition.** *For any* $x \in E_\Delta$, $t \geqq 0$, $Z \in b\mathscr{K}$, $T \in s(\mathscr{M}_t)$,

$$E^x[Z \circ \theta_T\, e^{-\lambda Y(T+t)} | \mathscr{M}_T] = e^{-\lambda Y(T)} E^{X(T)}[Z\, e^{-\lambda Y(t)}].$$

Proof. It is sufficient to show this for $Z = \prod_1^n f_j(X_{t_j})$ where $0 \leqq t_0 < t_1 < \cdots < t_n$; $f_j \in b\mathscr{E}_\Delta^*$. For $t_n \leqq t$, this follows by induction on n from Proposition (3.15). Suppose next that $0 \leqq t_0 < \cdots < t_j \leqq t < t_{j+1} < \cdots < t_n$. Then, using the strong Markov property of X for the stopping time $T+t$,

$$\begin{aligned}
&E^x\left[\prod_1^n f_i \circ X_{T+t_i}\, e^{-\lambda Y(T+t)} | \mathscr{M}_T\right] \\
&\quad = E^x\left[\prod_1^j f_i \circ X_{T+t_i}\, e^{-\lambda Y(T+t)}\, E^{X(T+t)}\left[\prod_{j+1}^n f_i \circ X_{t_i - t}\right] \middle| \mathscr{M}_T\right]
\end{aligned}$$

which now falls in the previous case. Thus, this last expression is

$$\begin{aligned}
&e^{-\lambda Y(T)}\, E^{X(T)}\left[\prod_1^j f_i \circ X_{t_i}\, E^{X(t)}\left[\prod_{j+1}^n f_i \circ X_{t_i - t}\right] e^{-\lambda Y_t}\right] \\
&\quad = e^{-\lambda Y(T)}\, E^{X(T)}\left[\prod_1^n f_i \circ X_{t_i}\, e^{-\lambda Y_t}\right].
\end{aligned}$$

This completes the proof.

4. Structure of Y^f

In this section we will consider the second component in the decomposition (2.24). By the first statement of Theorem (2.23), the σ-algebras generated by different components are conditionally independent given $\mathscr{K}$; and by (b) of the same theorem, (X, Y^f) is a MAP. In considering this component, then, there is no loss of generality in assuming that all the other terms in (2.24) are zero.

Let, then, (X, Y) be a MAP so that $Y = Y^f$; that is, Y is a pure jump process taking values in $(F, \mathscr{F}) = (\mathbb{R}^m, \mathscr{R}^m)$, each jump time of Y is fixed by X, the amounts of jumps themselves depending on X and on further variables. As before, X is standard, the σ-algebras $\mathscr{K}_t \subset \mathscr{L}_t \subset \mathscr{M}_t$ are as before.

Let $\{\tau_j\}$ be the finite or countably infinite set of times of discontinuity of the mapping $t \to Y_t$; and for each j, let U_j be the amount of jump at τ_j, that is, $U_j = Y_{\tau_j} - Y_{\tau_j -}$. Then we can write

(4.1) $$Y_t = \sum_j U_j I_{\{\tau_j \leqq t\}}, \qquad t \geqq 0.$$

Since a centered separable process with independent increments is "almost surely" bounded on any finite interval (cf. Doob [3, p. 411]) for any finite interval $J \subset [0, \infty]$ we have that $\sum_j U_j I_{\{\tau_j \in J\}}$ converges almost surely.

By Theorem (2.23), each $\tau_j \in \mathscr{K}$ and the U_j are conditionally independent given $\mathscr{K}$. The next result makes this dependence on X more precise.

(4.2) **Proposition.** *Each* $\tau_j \in s(\mathscr{K}_t)$, *each* $U_j \in \mathscr{L}_{\tau_j}$.

Proof. For each $t \geqq 0$, $\lambda \in F$ we have

$$M_t^\lambda = E[e^{i\lambda Y_t}] = \prod_{\tau_j \leqq t} E[e^{i\lambda U_j}]$$

by (4.1) and the conditional independence of U_j given $\mathscr{K}$. Each τ_j is a time of discontinuity of $t \to M_t^\lambda$ for some $\lambda > 0$, therefore the τ_j are the times of discontinuity of the function $t \to M_t$ where

$$M_t = \int_{|\lambda| \leqq 1} |M_t^\lambda| \, d\lambda, \qquad t \geqq 0.$$

Since the mapping $(\lambda, \omega) \to M_t^\lambda(\omega)$ is in $\mathscr{F} \times \mathscr{K}_t$, by Fubini's theorem

(4.3) $$M_t \in \mathscr{K}_t$$

for each $t \geqq 0$.

It is clear that $t \to M_t$ is right continuous and has left-hand limits. For $\varepsilon > 0$ define

(4.4) $$T = \inf\{t \colon M_{t-} - M_t \geqq \varepsilon\};$$

put $T_0 = 0$ and $T_{n+1} = T_n + T \circ \theta_{T_n}$. It follows from (4.3) and (4.4) that $T \in s(\mathscr{K}_t)$, and thus each of its iterates is also in $s(\mathscr{K}_t)$. Letting $\varepsilon \to 0$ we obtain the desired result for the τ_j.

Since $\mathscr{K}_t \subset \mathscr{L}_t$ for any $t \geqq 0$, each τ_j is also in $s(\mathscr{L}_t)$. Thus $\mathscr{L}_{\tau_j}$ is well defined. That $U_j \in \mathscr{L}_{\tau_j}$ is evident from (4.1).

The following gives a further decomposition of the component we are studying. Here we call the attention to the definition of *natural* given in (1.5).

(4.5) **Theorem.** *Let X be a Hunt process. We can then write*

(4.6) $$Y_t = Y_t^q + Y_t^n, \qquad t \geqq 0,$$

where the following hold:

a) $Y^q = \{Y_t^q;\, t \geqq 0\}$ *is quasi-left-continuous and not natural;* (X, Y^q) *is a* MAP;

b) $Y^n = \{Y_t^n;\, t \geqq 0\}$ *is natural and not quasi-left-continuous;* (X, Y^n) *is a* MAP;

c) Y^q *and* Y^n *are conditionally independent given* $\mathscr{K}$.

Proof. For each $t \geqq 0$ let

$$Y_t^q = \sum_{s \leqq t} (Y_s - Y_{s-})\, I_{\{X_{s-} \neq X_s\}} \tag{4.7}$$

$$Y_t^n = Y_t - Y_t^q.$$

a) Let $\{T_n\} \subset s(\mathscr{M}_t)$, $T_n \uparrow T \in s(\mathscr{M}_t)$. Since Y^q is continuous except at its jump times, $Y^q(T_n) \uparrow Y^q(T)$ except on the set

$$\Gamma = \bigcup_j \{T_n < T \text{ for all } n;\ T_n \uparrow T;\ T = \tau_j^q;\ T < \zeta\}$$

where the τ_j^q are the jump times of Y^q. But by (4.7), every jump time of Y^q is also a jump time of X; therefore, $\lim X(T_n) \neq X(T)$ on the set Γ. Thus, since X is quasi-left-continuous, we must have $P^x(\Gamma) = 0$ for every x. Hence, Y^q is quasi-left-continuous, and obviously not natural.

b) By (4.7), Y^n has no discontinuity in common with X; so, it is natural. Let τ be a jump time of Y^n. By Proposition (4.2), $\tau \in s(\mathscr{K}_t)$. Since X is a Hunt process and $X_{\tau-} = X_\tau$ almost surely by (4.7), the stopping time τ is accessible; that is, for each $x \in E$, there is a sequence $\{T_n\} \subset s(\mathscr{K}_t)$ such that $T_n \uparrow \tau$ a.s. P^x and $T_n < \tau$ for all n a.s. P^x on $\{\tau > 0\}$; (cf. Meyer [6; no. 120 p. 147, no. 114 p. 146] where the term used is "previsible," our use of the term is in the sense of BG IV.4.5). Then $\lim_n Y(T_n) = Y_{\tau-} \neq Y(\tau)$ a.s. P^x on $\{\tau > 0\}$. Hence Y is not quasi-left-continuous.

c) Given $\mathscr{K}$, the jump times of Y are fixed by Proposition (4.3), and further, the magnitudes of jumps are conditionally independent. Hence $\sigma(Y_t^n;\, t \geqq 0)$ and $\sigma(Y_t^q;\, t \geqq 0)$ are conditionally independent given $\mathscr{K}$.

The regularity conditions (1.2a), (1.2b), (1.2c) and homogeneity (1.2d) are easy to check for Y^q and Y^n. The condition (1.2e) for each of the Y^q and Y^n follows from their independence given $\mathscr{K}$. Hence, (X, Y^q) and (X, Y^n) are both MAP's.

In the remainder of this section we assume $F = \mathbb{R}$, $\mathscr{F} = \mathscr{R}$ and Y non-decreasing. The following theorem characterizes the quasi-left-continuous part completely.

(4.8) **Theorem.** *Let X be a Hunt process for which there is a reference measure, and suppose Y is real valued, non-decreasing, quasi-left-continuous. Then, each τ_j is a time of discontinuity for X and the conditional distribution of U_j given $\mathscr{K}$ depends only on X_{τ_j-} and X_{τ_j}. More precisely,*

$$M_t^\lambda = E[e^{-\lambda Y_t}] = \prod_{s \leqq t} F^\lambda(X_{s-}, X_s) \tag{4.9}$$

where

a) *for each $\lambda \geqq 0$, $F^\lambda(x, x) = F^\lambda(x, \Delta) = 1$ for all $x \in E_\Delta$;*

b) *$\lambda \to F^\lambda(x, y)$ is completely monotonic for all $x, y \in E_\Delta$;*

c) *$(x, y) \to F^\lambda(x, y)$ is in $\mathscr{E}_\Delta \times \mathscr{E}_\Delta$ for any $\lambda \geqq 0$.*

Proof. Fix $\lambda>0$ and define

(4.10) $$A_t=-\log M_t^\lambda, \quad t\geqq 0.$$

Then, $\{A_t\}$ is a pure jump type additive functional of X. Since Y is quasi-left-continuous, the monotone convergence theorem implies that the same is true for $\{A_t\}$. Under our hypothesis that X is a Hunt process with a reference measure, every quasi-left-continuous, non-decreasing pure jump type additive functional is of the form

$$A_t=\sum_{s\leqq t} f(X_{s-}, X_s)\, I_{\{X_{s-}\neq X_s\}}$$

where $f\colon E^2\to[0,\infty)$ is in $\mathscr{E}^2$ (cf. Watanabe [8]). Using (4.10) now gives (4.9).

This theorem shows that a quasi-left-continuous process Y^q jumps only when X jumps, the amount of jump depending only on the values of the left-hand and right-hand limits of X at the time of jump. Next is a characterization of a natural process Y^n. It shows that Y^n is the limit of a sequence of natural processes, a typical one of which, say Y^0, behaves as follows. Corresponding to Y^0 there is an α-excessive function $x\to g^\lambda(x)$ so that $Y^0(\omega)$ jumps at time t if and only if $s\to g^\lambda\circ X_s(\omega)$ has a jump at t and $s\to X_s(\omega)$ is continuous at t. The amount by which Y^0 jumps at a jump point τ is a random variable whose conditional Laplace transform given $\mathscr{H}$ is the amount by which $s\to g^\lambda\circ X_s$ jumps at τ. Following is the precise statement.

(4.11) **Theorem.** *Let X be a Hunt process, Y real valued, non-decreasing, natural. Then there exists a sequence of functions g_n^λ, each g_n^λ being α_n-excessive for some $\alpha_n\geqq 0$, such that*

(4.12) $$M_t^\lambda=E[e^{-\lambda Y_t}]=\lim_{n\to\infty}\prod_{s\leqq t}\exp\{[g_n^\lambda\circ X_{s-}-(g_n^\lambda\circ X_s)_-]\, I_{\{X_{s-}=X_s\}}\}.$$

Proof. Let $A_t=-\log M_t^\lambda$. Then $A=\{A_t;\ t\geqq 0\}$ is an additve functional of X which is of pure jump type, non-decreasing, natural. Let

$$B^k(t)=\sum_{s\leqq t}(A_s-A_{s-})\, I_{\{A_s-A_{s-}\in[\frac{1}{k+1},\frac{1}{k})\}}; \quad t\geqq 0.$$

Then, from Meyer [7, Propositions 2 and 3], it follows that every jump time of B^k is a terminal $\{\mathscr{H}_t\}$ stopping time without any regular points, and thus, B^k can be written as a sum of countably many additive functionals of X each of which is a non-decreasing natural pure jump process and has a bounded α-potential for some $\alpha\geqq 0$. Thus, we can write

(4.13) $$A_t=\sum_n A_t^n, \quad t\geqq 0,$$

where, for each n, A^n is natural non-decreasing pure jump type additive functional of X having the α-potential

$$u_n(x)=E^x\left[\int_0^\infty e^{-\alpha t}\, dA_t^n\right]$$

bounded for some fixed $\alpha=\alpha_n\geqq 0$.

For fixed n, u_n is α_n-excessive and

$$(u_n \circ X_s)_- = \lim_{t \uparrow s} u_n \circ X_t \geqq u_n \circ X_{s-} \quad \text{a.s.} \tag{4.14}$$

(the limit defining $(u_n \circ X_s)_-$ existing almost surely). Using the fact that every jump time of A^n is accessible, and that the strong Markov property holds for A^n, we can show that (cf. BG IV.4.29 for a proof)

$$\begin{aligned} \lim_m (A^n(T) - A^n(T_m)) &= \lim_m u_n \circ X_{T_m} - u_n \circ X_T \\ &= (u_n \circ X_T)_- - u_n \circ X_T \end{aligned} \tag{4.15}$$

for each jump time T of A^n and $\{T_m\} \subset s(\mathscr{K}_t)$, $T_m \uparrow T$, $T_m < T$ for all m on $\{T > 0\}$ almost surely. Thus, we can write

$$A_t^n = \sum_{s \leqq t} [(u_n \circ X_s)_- - u_n \circ X_s] \, I_{\{X_s = X_{s-}\}}. \tag{4.16}$$

Define $g_n^\lambda = u_1 + \cdots + u_n$; $\beta_n = \max(\alpha_1, \ldots, \alpha_n)$. Then g_n is β_n-excessive, and from (4.13), (4.14), (4.15) we have

$$A_t = \lim_{n \to \infty} \sum_{s \leqq t} [(g_n^\lambda \circ X_s)_- - g_n^\lambda \circ X_s] \, I_{\{X_s = X_{s-}\}}. \tag{4.17}$$

This is equivalent to (4.12).

(4.18) **Corollary.** *Let X be a Hunt process. If X is continuous, then $Y = Y^n$. If X is a regular step process (in the sense of* BG II.5.5), *then $Y = Y^q$.*

Proof. The first statement follows trivially from (4.7). To prove the second, in view of Theorem (4.5), we will suppose that Y is natural and X is a regular step process and show that $Y_t = 0$ almost surely for all $t \geqq 0$.

Define A, A^n as in the proof of Theorem (4.11). Each jump time T of A^n is then accessible. Let $\{T_m\} \subset s(\mathscr{K}_t)$ be such that $T_m \uparrow T$ and $T_m < T$ for all m on $\{T > 0\}$ a.s. P^x. Then, since $X_T = X_{T-}$ a.s. P^x and since every point in E is a holding point for X, we have $X_{T_m}(\omega) = X_T(\omega)$ for large enough m for all ω except possibly over a set of P^x-measure zero. Thus, the right-hand side of (4.15) vanishes and we have $A^n(T) - A^n(T-) = 0$ a.s. P^x. Since such a sequence $\{T_m\}$ exists for each x, we have

$$A^n(T) - A^n(T-) = 0$$

almost surely. This implies that $A_t^n = 0$ almost surely for all $t \geqq 0$ and by (4.17) $A_t = 0$ almost surely. Since $\lambda > 0$ this can happen only if $Y_t = 0$ almost surely for all $t \geqq 0$.

The same proof also yields the following

(4.19) **Corollary.** *Suppose the state space E of X is discrete; then, Y is quasi-left-continuous.*

We remark, in closing, that both of the Theorems (4.8) and (4.11) have obvious converses that may be used in constructing the compoent Y^f.

5. Structure of Y^d

In this section we are interested in the last term of the decomposition (2.24). In view of the first and the last statements of Theorem (2.23) we may, and do, assume that $Y = Y^d$.

We are then considering a Markov additive process

$$(X, Y) = (\Omega, \mathcal{M}, \mathcal{M}_t, X_t, Y_t, \theta_t, P^x) \quad \text{where} \quad X = (\Omega, \mathcal{M}, \mathcal{M}_t, X_t, \theta_t, P^x)$$

is standard and has state space $(E, \mathscr{E})$ and where $\{Y_t; P_\omega\}$ is a stochastically continuous process taking values in $(F, \mathscr{F}) = (\mathbb{R}^m, \mathscr{R}^m)$ and having independent increments for each ω; (cf. Theorem (2.23) and (2.20), (2.21) for the description of P_ω).

If $\{T_n\}$ is a sequence of stopping times in $s(\mathscr{L}_t)$ with $\lim T_n = T \in s(\mathscr{L}_t)$, then we must have

$$P\{\lim_n Y(T_n) = Y(T)\} = 1$$

identically. Thus, $\lim_n Y(T_n) = Y(T)$ almost surely for any sequence $\{T_n\} \subset s(\mathscr{L}_t)$ with $\lim T_n = T$; that is, Y is quasi-left-continuous on $[0, +\infty]$ for $\{\mathscr{L}_t\}$ stopping times.

If X is continuous then Y is natural trivially. Next let X be a regular step process, that is, each $x \in E$ is a holding point. Then, almost surely, $t \to X_t(\omega)$ has countably many discontinuities. $\{Y_t; P_\omega\}$ being stochastically continuous, P_ω-probability of $t \to Y_t$ having a jump at one of these times of discontinuity is zero. Therefore, $t \to X_t$ and $t \to Y_t$ have no discontinuities in common almost surely. Thus, if X is a regular step process, then Y is natural. However, Y may fail to be natural in general: Suppose E is discrete, $x_0 \in E$ is instantaneous, and x_0 is regular for the set $\{x_0\}$; then it is possible to define a process Y whose all jumps are concentrated on the set $\{t: X_t = x_0\}$; but these times are also jump times for X since E is discrete and x_0 instantaneous. (See Proposition (5.12) for such processes.)

In the remainder of this section we will show under certain conditions that we can represent Y minus a suitable centering term in the form

$$\int A_t \, dN$$

where $\{A_t\}$ is a functional of X and N is a Poisson random measure roughly independent of X. In certain cases it is further possible to express Y_t as

$$Y_t = \sum_{x \in D} Z^x(L_t^x)$$

where $D \subset E$ is countable, each Z^x has stationary independent increments independent of X, and $L^x = \{L_t^x\}$ is the local time at x. This means that Y is the "sum" of a collection of processes with *stationary* independent increments which are defined over disjoint parameter sets, parameter set of each being a random set $\{t: X_t = x\}$.

By Corollary (2.25) we have

$$(5.1) \quad (E \exp[i(\lambda, Y_t)])(\omega) = \exp\left[\int \left(e^{i(\lambda, y)} - 1 - \frac{i(\lambda, y)}{1+|y|^2}\right) \frac{1+|y|^2}{|y|^2} B_t(dy, \omega)\right]$$

for all $t \geqq 0$, $\lambda \in F$ where

a) for each t and ω, $A \to B_t(A, \omega)$ is a finite measure on $\mathscr{F}$,

b) for each $A \in \mathscr{F}$, $B(A) = \{B_t(A); t \geqq 0\}$ is a non-decreasing continuous additive functional of X.

Let $B_t = B_t(F)$, $t \geqq 0$, and define

(5.2) $$R(\omega) = \inf\{t\colon B_t(\omega) > 0\}, \qquad \omega \in \Omega$$

(we set $R(\omega) = +\infty$ if the set in braces is empty). Then, $R \in s(\mathscr{K}_t)$ clearly and the set of its regular points, namely

(5.3) $$D = \{x \in E\colon P^x\{R = 0\} = 1\},$$

is called the *support* of B. We remark that R is almost surely equal to the hitting time of the set D.

For each $\omega \in \Omega$, $t \to B_t(\omega)$ is non-decreasing continuous and thus induces a measure $dB_t(\omega)$ on the Borel subsets of $[0, +\infty)$. The support of $dB_t(\omega)$ is the set

(5.4) $$\mathbf{J}(\omega) = \{t\colon B_{t+\varepsilon}(\omega) - B_{t-\varepsilon}(\omega) > 0 \text{ for all } \varepsilon > 0\}$$

which is also the closure of the set

(5.5) $$\mathbf{I}(\omega) = \{t\colon B_{t+\varepsilon}(\omega) - B_t(\omega) > 0 \text{ for all } \varepsilon > 0\}.$$

Furthermore, $\mathbf{J}(\omega) - \mathbf{I}(\omega)$ is at most countable and therefore all the mass of $dB_t(\omega)$ is carried by $\mathbf{I}(\omega)$. In view of (5.1), this implies that

(5.6) $$P_\omega\Big\{\sum_{s \in [0, \infty) - \mathbf{I}(\omega)} |Y_s - Y_{s-}| = 0\Big\} = 1,$$

and we have the following

(5.7) **Proposition.** *For each $\omega \in \Omega$, the stochastic process $\{Y_t;\ t \in \mathbf{J}(\omega)\}$ defined over $(\Omega, \mathscr{L}, P_\omega)$ is a process with independent increments and without fixed discontinuities. We can write*

(5.8) $$Y_t = \lim_{n \uparrow \infty} \left[\sum_{s \in [0, t] \cap \mathbf{I}} (Y_s - Y_{s-}) I_{(\frac{1}{n}, n)}(Y_s - Y_{s-}) - \int_{(\frac{1}{n}, n)} \frac{y}{|y|^2} B_t(dy)\right]$$

for all t almost surely.

Since the set

(5.9) $$\mathbf{K} = \{t\colon X_t \in D\}$$

is such that $\mathbf{I} \subset \mathbf{K} \subset \mathbf{J}$ almost surely by BG V.3.8, in view of (5.8) we have also proved the following.

(5.10) **Corollary.** *Almost surely*

$$Y_t = \lim_{n \uparrow \infty} \left[\sum_{s \leqq t} (Y_s - Y_{s-}) I_D(X_s) I_{(\frac{1}{n}, n)}(Y_s - Y_{s-}) - \int_{(\frac{1}{n}, n)} \frac{y}{|y|^2} B_t(dy)\right]$$

for all t.

Corollary (5.10) is the analog of BG V.3.9 which, in this instance, states that

(5.11) $$B_t=\int_0^t I_D(X_s)\,dB_s$$

almost surely for all t.

The next two propositions characterize the process Y completely in the case where D consists of a single point. We will use these results as building blocks for the more general cases. A rough explanation of the next proposition is the following. By (5.7), Y is a process with independent increments and with a parameter set $\mathbf{J}$ which is roughly the set of times t for which $X_t=x_0$, $x_0\in E$ is fixed and x_0 is regular for $\{x_0\}=D$. The picture is easier to see if x_0 is a holding point; then $\mathbf{J}$ is a countable union of disjoint closed intervals, say $\mathbf{J}=\bigcup_n \mathbf{J}_n$. Then $\{Y_t;\, t\in\mathbf{J}_n\}$ is a process with *stationary* independent increments over $(\Omega,\mathscr{L},P)$; further, if $\mathbf{J}_n(\omega)=[a,b]$, $\mathbf{J}_{n+1}(\omega)=[c,d]$, then $Y_b(\omega)=Y_c(\omega)$. Now, let us hitch these processes $\{Y_t;\, t\in J_n\}$ onto each other so that the point b of $\mathbf{J}_n(\omega)$ coincides with the point c of $\mathbf{J}_{n+1}(\omega)$. The resulting process Z has *stationary* independent increments over the interval $[0,B_\infty(\omega)]$ and further Z is *independent* of X given B_∞. Following is the precise statement; we do not assume x_0 to be a holding point.

(5.12) **Proposition.** *Suppose D consists of a single point, say $D=\{x_0\}$, $x_0\in E$, and suppose Y is perfect. Define*

(5.13) $$\tau_u(\omega)=\inf\{s:\ B_s(\omega)>u\}$$

(5.14) $$Z_u(\omega)=Y_{\tau_u(\omega)}(\omega)$$

for all $u\geqq 0$ and $\omega\in\Omega$. Then the following hold.

a) *Almost surely, $Z_0=0$, $t\to Z_t$ is right continuous and has lefthand limits, $Z_t=Z_{B_\infty}$ for all $t\geqq B_\infty$.*

b) *For each $u\geqq 0$, $\tau_u\in s(\mathscr{K}_t)\subset s(\mathscr{M}_t)$ and $Z_u\in\mathscr{M}_{\tau_u}$.*

c) *$Z_{u+v}=Z_u+Z_v\circ\theta_{\tau(u)}$ for all $u,v\geqq 0$ almost surely.*

d) *$Y_t=Z_{B_t}$ for all $t\geqq 0$ almost surely.*

e) *For $0=t_0<t_1<\cdots<t_n$ and $A_1,\dots,A_n\in\mathscr{F}$ we have*

$$P^x\{Z(t_j)-Z(t_{j-1})\in A_j;\ j=1,\dots,n\,|\,\mathscr{K}\}=\prod_{j=1}^n P^y\{Z(t_j-t_{j-1})\in A_j\}$$

on $\{\tau(t_n)<\infty\}$ independent of $x,y\in E$.

Proof. Let τ_u be defined by (5.13). Since D consists of the single point x_0, B is a local time at x_0. We may, and do, assume that B is perfect and that $t\to B_t(\omega)$ is continuous and non-decreasing for all ω. By BG V.2.3,

(5.15) $$\tau_{t+s}=\tau_t+\tau_s\circ\theta_{\tau_t}$$

for all $t,s\geqq 0$ almost surely (see also BG V.3.14ff.). Further, for any ω, $t\to\tau_t(\omega)$ is strictly increasing on $[0,B_\infty)$ and $\tau_t(\omega)=\infty$ on $[B_\infty,\infty]$.

a) Let $R=T_D$ be the hitting time of $D=\{x_0\}$. Then $\tau_0=R$ by (5.13); $B_R=B_0=0$ by the continuity of B, and $Y_R=Y_0=0$ almost surely by (5.1) since $B_R=0$. Hence

$Z_0=0$ almost surely. Right continuity of Z follows by the right continuity of $t \to Y_t$ and $t \to \tau_t$; similarly for the existence of left-hand limits. Finally, for $t \geqq B_\infty$, $\tau_t = +\infty$ and thus $Z_t = Y_\infty$.

b) For each u, $\tau_u \in s(\mathscr{K}_t)$ by BG V.2.3; $\tau_u \in s(\mathscr{M}_t)$ also and $Y_{\tau_u} \in \mathscr{M}_{\tau_u}$ since Y is progressively measurable.

c) By the perfectness of Y, (1.2d) holds when t, s are replaced by stopping times. Using (5.15),

$$Y(\tau_{u+v}) = Y(\tau_u + \tau_v \circ \theta_{\tau_u}) = Y_{\tau_u} + Y_{\tau_v \circ \theta_{\tau_u}}(\theta_{\tau_u}) = Y_{\tau_u} + Y_{\tau_v} \circ \theta_{\tau_u}$$

almost surely.

d) Since $\tau(B_t) = t + R \circ \theta_t$, $Z(B_t) = Y(t + R \circ \theta_t) = Y_t + Y_R \circ \theta_t = Y_t$ almost surely because $Y_R = 0$ almost surely by (a) above.

e) It is clear that, for each $A \in \mathscr{F}$, $B(A) = \{B_t(A);\, t \geqq 0\}$ has as support the same set $D = \{x_0\}$. Thus, each $B(A)$ is a local time for X at x_0; cf. BG V.3.12 for definition. By BG V.3.13, then there exists a constant $H(A)$ for each $A \in \mathscr{F}$ so that

(5.16) $$B_t(A, \omega) = H(A)\, B_t(\omega)$$

for all $t \geqq 0$ almost surely. Since $A \to B_t(A, \omega)$ is a measure, $A \to H(A)$ is also a measure on $\mathscr{F}$ and obviously $0 \leqq H(A) \leqq 1$. Putting (5.16) in (5.1) we have

(5.17) $$E \exp[i(\lambda, Y_t)] = \exp[h(\lambda)\, B_t]$$

where

(5.18) $$h(\lambda) = \int_F \left(e^{i(\lambda, y)} - 1 - \frac{i(\lambda, y)}{1+|y|^2}\right) \frac{1+|y|^2}{|y|^2} H(dy), \qquad \lambda \in F.$$

For $u, v \geqq 0$, $\lambda \in F$ we have, by Corollary (3.9) and (5.12c) above,

$$\begin{aligned} E^x[e^{i(\lambda, (Z(u+v) - Z(u)))} | \mathscr{K}] &= E[\exp[i(\lambda, Y_{\tau_v} \circ \theta_{\tau_u})]] \\ &= (E\, e^{i(\lambda, Y(\tau_v))}) \circ \theta_{\tau_u} \end{aligned}$$

for all x. But, in view of (5.17), since $\tau_s \in s(\mathscr{K}_t)$, this last expression is equal to

$$(\exp[h(\lambda)\, B_{\tau_v}]) \circ \theta_{\tau_u}.$$

On the set $\{\tau_v < \infty\}$, $B_{\tau_v} = v$; and obviously $\tau_v < \infty$ on $\{\tau_{u+v} < \infty\}$. Thus,

(5.19) $$E^x[e^{i(\lambda, (Z(u+v) - Z(u)))} | \mathscr{K}] = e^{v h(\lambda)} \quad \text{on } \{\tau_{u+v} < \infty\}.$$

This completes the proof since

$$P^x\{Z(t_j) - Z(t_{j-1}) \in A_j, j = 1, \ldots, n | \mathscr{K}\} = \prod_{j=1}^n P^x\{Z(t_j) - Z(t_{j-1}) \in A_j | \mathscr{K}\}$$

using Theorem (2.22) since $\tau(t_j) \in s(\mathscr{K}_t)$ for each j.

We note that, if $P^{x_0}\{\tau_u < \infty\} = 1$, then $\{Z; P^x\}$ is a process with stationary independent increments in the ordinary sense. Otherwise Z is obtained by stopping a process with stationary independent increments at B_∞. The following converse to Proposition (5.12) follows easily from it. We omit the proof and merely point out that, by BG V.3.13, any local time at x_0 is a constant multiple of *the* local time L^{x_0} at x_0 (i.e. a continuous additive functional of X with support $D = \{x_0\}$

and normalized so that

$$E^x(e^{-R}) = E^x\left(\int_0^\infty e^{-t}\, dL_t^{x_0}\right)$$

for all y where R is as defined by (5.2) with $L_t^{x_0}$ replacing B_t).

(5.20) **Proposition.** *Suppose the support D of B consists of a single point $x_0 \in E$. Then there exists a* MAP $(\hat{X}, \hat{Y}) = (\hat{\Omega}, \hat{\mathscr{M}}, \hat{\mathscr{M}}_t, \hat{X}_t, \hat{Y}_t, \hat{\theta}_t, \hat{P}^x)$ *equivalent to* (X, Y), *i.e. they have the same finite dimensional distributions, such that*

(5.21) $$\hat{Y}_t = Z(L_t)$$

where

a) *Z is a process over $(\hat{\Omega}, \hat{\mathscr{M}}, \hat{P}^x)$ with stationary independent increments and without a Gaussian component for any x,*

b) *Z is independent of $\hat{X}$, i.e., $\sigma(Z_s;\, s \geqq 0)$ and $\sigma(\hat{X}_s, s \geqq 0)$ are independent;*

c) *L is the local time for $\hat{X}$ at x_0.*

(5.22) *Remark.* By BG V.3.19, in (5.12), the process $\{\tau_t;\, P^{x_0}\}$ itself has stationary independent increments. Thus, roughly speaking, when $D = \{x_0\}$, Y is obtained from a Lévy process Z with stationary independent increments by a random time change $Y_t = Z(L_t)$ where L_t is the time inverse of a process τ (i.e. $L_t = \inf\{s\colon \tau_s > t\}$) with non-negative stationary independent increments. This is a converse to subordination where the random time itself (rather than its inverse as in our case) has stationary independent increments.

For the construction of a process Y of the type we are considering in (5.20) then, all we need is a point $x_0 \in E$ which is regular for $\{x_0\}$ (so that the local time at x_0 exists) and a probability measure H on $\mathscr{F}$ with $H(\{0\}) = 0$. It is, for example, possible to do this when X is the Brownian motion on $\mathbb{R}$ and $x_0 = 0$. In this case the process Y constructed through (5.20) has a parameter set $\mathbf{J}(\omega)$ which is perfect, nowhere dense, closed, and has Lebesgue measure zero. The same is true if X is a process with E discrete, x_0 instantaneous but not fictitious.

Below we shall generalize (5.12) and (5.20) to the case where D is countable and later to the case where D is uncountable.

(5.23) **Lemma.** *Suppose D is countable. Then, for each $x \in D$ the local time $L^x = \{L_t^x;\, t \geqq 0\}$ for X at x exists and we can write*

(5.24) $$B_t(A, \omega) = \sum_{x \in D} H(x, A)\, L_t^x(\omega)$$

almost surely for all $t \geqq 0$ and $A \in \mathscr{F}$; here $A \to H(x, A)$ is a measure on $\mathscr{F}$ for each fixed $x \in D$.

Proof. For any $A \in \mathscr{F}$, $B_t = B_t(A) + B_t(A^c)$ for all $t \geqq 0$ and each one of $\{B_t;\, t \geqq 0\}$, $\{B_t(A);\, t \geqq 0\}$, $\{B_t(A^c);\, t \geqq 0\}$ is a continuous additive functional of X. Since D is countable, this implies by BG V.3.11 that

(5.25) $$B_t(A) = \int_0^t G(X_s, A)\, dB_s$$

almost surely for each $t \geqq 0$ and $A \in \mathscr{F}$ where $x \to G(x, A)$ is in $\mathscr{E}$ and $0 \leqq G(x, A) \leqq 1$.

On the other hand, for each $x \in D$, $\int_0^t I_{\{x\}}(X_s)\,dB_s$ defines a continuous additive functional of X whose support is $\{x\}$; thus, this is a local time at x and by the uniqueness theorem for local times (cf. BG V.3.13)

$$\int_0^t I_{\{x\}}(X_s)\,dB_s = b^x L_t^x$$

for some constant b^x. Writing $H(x, A) = b^x G(x, A)$ for $x \in D$, noting that $\sum_{x \in D} b^x L_t^x = B_t$, and further that $\int_0^t G(X_s, A)\,dL_s^x = G(x, A)\,L_t^x$ we obtain the representation (5.24). Since $A \to B_t(A, \omega)$ is a measure on $\mathscr{F}$, the same is true of $A \to H(x, A)$ for each $x \in D$ by (5.24) and the monotone convergence theorem.

(5.26) **Theorem.** *Suppose D is countable and Y is perfect, and let L^x be the local time for X at x for each $x \in D$. Define, for each $u \geqq 0$ and $x \in D$,*

$$\tau_u^x = \inf\{s\colon\ L_s^x > u\}, \tag{5.27}$$

$$Z_u^x = \lim_{n \uparrow \infty} \left[\sum_{s \leqq \tau_u^x} (Y_s - Y_{s-})\, I_{\{x\}}(X_s)\, I_{\left(\frac{1}{n}, n\right)}(Y_s - Y_{s-}) - u \int_{\left(\frac{1}{n}, n\right)} \frac{y}{|y|^2}\, H(x, dy) \right] \tag{5.28}$$

on $\{\tau_u^x < \infty\}$ and on $\{\tau_u^x = \infty\}$ the same expression except that the u in the second term on the right is replaced by L_∞^x. Then for each $x \in D$ the following holds:

a) *almost surely, $Z_0^x = 0$, $t \to Z_t^x$ is right-continuous and has lefthand limits, $Z_t^x = Z_{L_\infty^x}^x$ for all $t \geqq L_\infty^x$;*

b) *for each $u \geqq 0$, $\tau_u^x \in s(\mathscr{K}_t)$ and $Z_u^x \in \mathscr{M}_{\tau_u^x}$;*

c) *$Z_{u+v}^x = Z_u^x + Z_v^x \circ \theta_{\tau_u^x}$ for all $u, v \geqq 0$ almost surely;*

d) *given L_∞^x, Z^x is independent of $\mathscr{K}$ with respect to any probability P^y;*

e) *Z^x is a process with stationary independent increments with respect to $P^y\{\cdot \mid L_\infty^x\}$.*

Furthermore, given $\sigma(L_\infty^x; x \in D)$, the processes Z^x, $x \in D$, are conditionally independent of each other, and

$$Y_t = \sum_{x \in D} Z^x(L_t^x) \tag{5.29}$$

almost surely for each $t \geqq 0$.

Proof. By Corollary (5.10), when D is countable we can write

$$Y_t = \sum_{x \in D} Y_t^x \tag{5.30}$$

almost surely by defining

$$Y_t^x = \lim_{n \uparrow \infty} \left[\sum_{s \leqq t} (Y_s - Y_{s-})\, I_{\{x\}}(X_s)\, I_{\left(\frac{1}{n}, n\right)}(Y_s - Y_{s-}) - L_t^x \int_{\left(\frac{1}{n}, n\right)} \frac{y}{|y|^2}\, H(x, dy) \right]. \tag{5.31}$$

Let $\mathbf{K}^x(\omega) = \{t\colon X_t(\omega) = x\}$ for each $x \in D$. Then, just as in (5.8), we have

$$\{Y_t^x;\ t \in \mathbf{J}^x(\omega)\}$$

defined over $(\Omega, \mathscr{L}, P_\omega)$ as a process with independent increments where $\mathbf{J}^x(\omega)$ is defined as in (5.4) by replacing B_t by L_t^x. And, each $\mathbf{K}^x(\omega)$ differs from $\mathbf{J}^x(\omega)$ by at most countably many points and the $\mathbf{K}^x(\omega)$, $x\in D$, are obviously disjoint. Since $\{Y_t; P_\omega\}$ has independent increments, this implies that Y_t^x, $x\in D$, are independent (P_ω) random variables for each t. Since each Y^x has independent increments, this is equivalent to saying $\{Y_t^x; t\geqq 0\}$, $x\in D$, are P_ω-independent processes.

To complete the proof we need only note that for each $x\in D$, (X, Y^x) is a MAP for which (5.1) becomes

$$E(e^{i(\lambda, Y_t^x)})=\exp\left[\int\left(e^{i(\lambda, y)}-1-\frac{i(\lambda, y)}{1+|y|^2}\right)\frac{1+|y|^2}{|y|^2}B_t^x(dy)\right]$$

where, in view of Lemma (5.23) above

$$B_t^x(A)=H(x, A)\,L_t^x.$$

Thus, Proposition (5.20) holds for Y^x and the properties listed for Z^x in (a)-(e) above hold.

Conditional independence of the Z^x, $x\in D$, given $\{L_\infty^x; x\in D\}$ follows from (5.26d) above since the Y^x are conditionally independent given $\mathscr{K}$. Finally, (5.29) follows from (5.30) and (5.20d).

Following analog of (5.20) is easy to see; we omit the proof.

(5.32) **Theorem.** *Suppose D is countable. Then there exists a* MAP $(\hat{X}, \hat{Y})=(\hat{\Omega}, \hat{\mathscr{M}}, \hat{\mathscr{M}}_t, \hat{X}_t, \hat{Y}_t, \hat{\theta}_t, \hat{P}^x)$ *equivalent to* (X, Y) *such that*

$$\hat{Y}_t=\sum_{x\in D} Z^x(L_t^x) \tag{5.33}$$

where

a) $\sigma(Z_t^x; t\geqq 0)$, $x\in D$, *are independent of each other and of* $\sigma(\hat{X}_t; t\geqq 0)$;

b) *for each* $x\in D$, $Z^x=\{Z_t^x; t\geqq 0\}$ *is a process with stationary independent increments and without a Gaussian component over* $(\hat{\Omega}, \hat{\mathscr{M}}, \hat{P}^y)$ *for any* y;

c) *for each* $x\in D$, $L^x=\{L_t^x; t\geqq 0\}$ *is the local time for* $\hat{X}$ *at* x.

Next we will discuss the situation in cases where D is not countable. For this purpose we first introduce the following definitions and a useful result.

Let $(G, \mathscr{G})$ be a measurable space and m a σ-finite non-negative measure on $\mathscr{G}$. Let $(W, \mathscr{B}, \hat{P})$ be a probability space and suppose, for each $A\in\mathscr{G}$, we have a map $w\to M(A, w)$ from W into $[0, \infty]$ which is in $\mathscr{B}$. $M=\{M(A); A\in\mathscr{G}\}$ is called a *random measure* if $A\to M(A, w)$ is a measure on $\mathscr{G}$ for $\hat{P}$-almost all w. It is called *additive* if $M(A_1), \ldots, M(A_n)$ are independent whenever $A_1, \ldots, A_n\in\mathscr{G}$ are disjoint. An additive random measure M on $(G, \mathscr{G})$ is called a *Poisson measure* with mean m if $M(A)$ has the Poisson distribution with parameter $m(A)$ for each $A\in\mathscr{G}$ (if $m(A)=+\infty$ then $M(A)=+\infty$ a.s.). Following is a characterization which is easy to prove.

(5.34) **Lemma.** *Let M be a random measure on $(G, \mathscr{G})$ defined over the probability space $(W, \mathscr{B}, \hat{P})$. Then, M is a Poisson measure with mean measure m if and only if*

$$\hat{E}\left[\exp\left(-\int_G f(x)\,M(dx)\right)\right]=\exp\left(-\int_G (1-e^{-f(x)})\,m(dx)\right)$$

for every $f\in\mathscr{G}_+$.

In the notation of Theorem (5.26) assume $L^x_\infty = +\infty$ for each $x \in D$ almost surely; then each one of the processes $\{Z^x; P^y\}$ is a process with stationary independent increments. Thus, (cf. Ito [5], Section 4), there is a Poisson measure M^x on $(\mathbb{R}_+ \times F, \mathscr{R}_+ \times \mathscr{F})$, with mean m^x, defined over $(\Omega, \mathscr{L}, P^y)$ such that

$$(5.35) \qquad Z^x_t = \lim_{n \uparrow \infty} \int_{1/n}^{n} y\, M^x((0,t] \times dy) - \frac{y}{1+|y|^2}\, m^x((0,t] \times dy).$$

Furthermore, since the Z^x, $x \in D$, are independent the random measure M on $(G, \mathscr{G}) = (\mathbb{R}_+ \times E \times F, \mathscr{R}_+ \times \mathscr{E} \times \mathscr{F})$ defined by

$$M(A_1 \times A_2 \times A_3) = \sum_{x \in D \cap A_2} M^x(A_1 \times A_3)$$

is also a Poisson measure over $(\Omega, \mathscr{L}, P^y)$. Further, it is clear from the definition of Z^x that we have

$$(5.36) \qquad M(A) = \sum_j I_A(L^{X(\tau_j)}_{\tau_j}, X_{\tau_j}, Y_{\tau_j} - Y_{\tau_j-})$$

almost surely for each $A \in \mathscr{G}$, where $\{\tau_j\}$ are the times of jumps for the process Y. The expression (5.36) generalizes over to the case where the support D of B is uncountable subject, of course, to our ability to define the local times L^x for each $x \in D$ and to express B as an integral of these local times. Here is the main result.

(5.37) **Theorem.** *Assume the following:*

(i) *X has a reference measure;*

(ii) *each $x \in D$ is regular for $\{x\}$;*

(iii) *the mapping $(t, x, \omega) \to L^x_t(\omega)$ of $[0, s] \times D \times \Omega$ into $\mathbb{R}_+$ is in $\mathscr{R}^*_{[0,s]} \times \mathscr{D} \times \mathscr{K}_s$ for each $s \geqq 0$ where $\mathscr{D}$ is the trace of $\mathscr{E}^*$ on D;*

(iv) *there exists a measure μ on $\mathscr{D}$ so that*

$$B_t(\omega) = \int_D \mu(dx)\, L^x_t(\omega)$$

almost surely for each t.

Let $\{\tau_j\}$ be the jump times of Y and define

$$(5.38) \qquad M(A) = \sum_j I_A(L^{X(\tau_j)}_{\tau_j}, X_{\tau_j}, Y_{\tau_j} - Y_{\tau_j-})$$

for each $A \in \mathscr{R}^*_+ \times \mathscr{D} \times \mathscr{F}$. Then the following hold:

a) *Given $\sigma(L^x_\infty;\, x \in D)$, M is conditionally independent of $\mathscr{K}$ with respect to P^y for any $y \in E$;*

b) *M is a Poisson random measure on $(G, \mathscr{G}) = (\mathbb{R}_+ \times D \times F,\ \mathscr{R}^*_+ \times \mathscr{G} \times \mathscr{F})$ defined over $(\Omega, \mathscr{L}, P_\omega)$.*

c) *for each $A \in \mathscr{E}$, $t \geqq 0$* we have

$$(5.39) \qquad \begin{aligned} &\sum_{s \leqq t} (Y_s - Y_{s-})\, I_A(X_s)\, I_{(\frac{1}{n}, n)}(Y_s - Y_{s-}) \\ &\qquad = \int_G y\, I_{(\frac{1}{n}, n)}(y)\, I_{A \cap D}(x)\, I_{[0, L^x_t]}(s)\, M(ds, dx, dy) \end{aligned}$$

almost surely; in particular,

(5.40) $$Y_t=\lim_{n\uparrow\infty}\int_G I_{(\frac{1}{n},n)}(y)\, I_{[0,L_t^x]}(s)\left[y\,M(ds,dx,dy)-\frac{y}{1+|y|^2}\,m(ds,dx,dy)\right]$$

almost surely for all $t\geqq 0$.

Proof. By assumption (ii), the local time L^x for X at x exists for each $x\in D$; so that (iii) makes sense and this in turn implies that the integral in (iv) makes sense.

The function $L_{\tau_j}^{X(\tau_j)}$ is the composition of the mappings $\omega\to(\tau_j(\omega),\omega)$, $(t,\omega)\to(X_t(\omega),t,\omega)$ and $(x,t,\omega)\to L_t^x(\omega)$ which are in $\mathscr{L}/\mathscr{R}_+^*\times\mathscr{L}$, $\mathscr{R}_+^*\times\mathscr{L}/\mathscr{E}\times\mathscr{R}_+^*\times\mathscr{L}\subset\mathscr{R}_+^*\times\mathscr{L}/\mathscr{D}\times\mathscr{R}_+^*\times\mathscr{L}$, and $\mathscr{D}\times\mathscr{R}^*\times\mathscr{L}/\mathscr{R}_+^*$ respectively (we used (iii) here). Thus, $\omega\to L_{\tau_j}^{X(\tau_j)}$ is in $\mathscr{L}$. Further, $\tau_j\in s(\mathscr{L}_t)$ so that $X_{\tau_j}\in\mathscr{L}_{\tau_j}\subset\mathscr{L}$, and obviously $Y_{\tau_j}-Y_{\tau_j-}\in\mathscr{L}$. Thus, each term of the sum in (5.38) is in $\mathscr{L}$ and hence $M(A)\in\mathscr{L}$ for each $A\in\mathscr{G}$; that is, M is a random measure.

a) For any $f\in\mathscr{G}_+$, we have by (5.38)

(5.41) $$\begin{aligned}\int_G f\,dM&=\sum_j f(L_{\tau_j}^{X(\tau_j)},X_{\tau_j},Y_{\tau_j}-Y_{\tau_j-})\\&=\sum_j f\circ g(\tau_j,Y_{\tau_j}-Y_{\tau_j-})\\&=\int f\circ g\,dN\end{aligned}$$

where

(5.42) $$N(A)=\sum_j I_A(\tau_j,Y_{\tau_j}-Y_{\tau_j-}),\qquad A\in\mathscr{R}_+^*\times\mathscr{F};$$

and

(5.43) $$g(t,y)=(L_t^{X(t)},X_t,y).$$

It is clear that N is a random measure on $(\mathbb{R}_+\times F,\mathscr{R}_+^*\times\mathscr{F})$ and defined over $(\Omega,\mathscr{L},P_\omega)$; further, since Y has independent increments, N is a Poisson random measure with mean measure

(5.44) $$n(dt,dy)=\frac{|y|^2}{1+|y|^2}\,B(dt,dy,\omega)$$

(where we are writing $B(dt,A,\omega)$ for the measure induced by $t\to B_t(A,\omega)$). Hence, by (5.41) and Lemma (5.34)

(5.45) $$E[\exp(-\int f\,dM)]=\exp[-\int(1-e^{-f\circ g})\,dn]=\exp[-\int_G(1-e^{-f})\,dm]$$

where

(5.46) $$m=n\,g^{-1}.$$

By Lemma (5.34), then, M is a Poisson measure on $(G,\mathscr{G})$ over $(\Omega,\mathscr{L},P_\omega)$ with mean measure m.

b) By our assumption (i) Lemma (5.53) below holds and we have

$$B_t(A;\omega)=\int_0^t H(X_s,A)\,dB_s$$

for all $A \in \mathscr{F}$ (where $x \to H(x, A)$ is in $\mathscr{E}$ and $A \to H(x, A)$ is a measure on $\mathscr{F}$) almost surely for all $t \geqq 0$. Further, by assumption (iv), we have

$$B_t(A; \omega) = \int_D \mu(dx)\, H(x, A)\, L_t^x(\omega) \tag{5.47}$$

where we used Fubini's theorem along with the fact that support of L^x is $\{x\}$.

Using (5.43), (5.44), (5.46), (5.47) together we get, for $A_1 \in \mathscr{R}_+$, $A_2 \in \mathscr{D}$, $A_3 \in \mathscr{F}$,

$$\begin{aligned} m(A_1 \times A_2 \times A_3) &= n(\{(t, y)\colon L_t^{X(t)} \in A_1, X_t \in A_2, y \in A_3\}) \\ &= \int I_{A_1}(L_t^{X(t)})\, I_{A_2}(X_t)\, n(dt, A_3) \\ &= \int_{A_2} \mu(dx) \int_{A_3} H(x, dy) \frac{|y|^2}{1+|y|^2} \int I_{A_1}(L_t^x)\, dL_t^x. \end{aligned}$$

The last integral is equal to $\int_0^{L_\infty^x} I_{A_1}(s)\, ds$; thus, writing $G(x, A_3)$ for the middle term we have

$$m(A_1 \times A_2 \times A_3) = \int_{A_2} \mu(dx)\, G(x, A_3)\, \lambda(A_1 \cap [0, L_\infty^x]) \tag{5.48}$$

where λ is the Lebesgue measure on $\mathscr{R}_+$. This along with (5.45) and Lemma (5.34) proves the statements (a) and (b) of Theorem (5.37). Statement (c) follows easily from (5.38).

(5.49) *Remark.* If D is countable, then the conclusions of Theorem (5.37) hold without the assumptions (i)–(iv).

(5.50) **Corollary.** *Suppose the conditions* (i)–(iv) *of* (5.37) *hold and further assume that* $L_\infty^x = +\infty$ *for all* $x \in D$ *almost surely. Define* M *as in* (5.38). *Then,* M *is a Poisson measure with mean*

$$m(ds, dx, dy) = ds\, \mu(dx)\, G(x, dy)$$

defined over $(\Omega, \mathscr{L}, P^y)$ *for any* y. *Further,* M *is independent of* $\mathscr{K}$ *with respect to any* P^y.

The proof is obvious and we omit it.

(5.51) *Remark.* Let X be the Brownian motion on $(E, \mathscr{E}) = (\mathbb{R}, \mathscr{R})$. Then X has a reference measure so that (i) holds. Each $x \in \mathbb{R}$ is regular for $\{x\}$ so that (ii) holds and L^x exists. Further, then the assumptions of BG V.3.30 are satisfied and we can select L_t^x so that $(x, t) \to L_t^x(\omega)$ is continuous from which (iii) follows. That (iv) also holds is known (cf. BG VI.4.21). Furthermore, in this case $L_\infty^x = +\infty$ for all x almost surely. Hence, the conclusions of Theorem (5.37) and Corollary (5.50) hold in this case.

(5.52) *Remark.* Somewhat more generally, suppose $(E, \mathscr{E}) = (\mathbb{R}, \mathscr{R})$ and assume (i) and (ii) of (5.37) hold. Then, (iii) and (iv) also follow if the conditions of Theorem BG V.3.30 hold (cf. BG V.3.30 and BG VI.4.21). Thus, for such a process the conclusions of (5.37) hold. Brownian motion is one such process.

Following is the lemma which was appealed to in the proof of (5.37).

(5.53) **Lemma.** *Suppose* X *has a reference measure. Then we can write*

$$B_t(A) = \int_0^t H(X_s, A)\, dB_s \tag{5.54}$$

almost surely for any $t \geqq 0$ *and* $A \in \mathscr{F}$ *; here*

a) $x \to H(x, A)$ *is in* $\mathscr{E}$ *for fixed* $A \in \mathscr{F}$,

b) $A \to H(x, A)$ *is a measure on* $\mathscr{F}$ *for fixed* $x \in E$,

c) $0 \leqq H(x, A) \leqq 1$ *for all* $x \in E$ *and* $A \in \mathscr{F}$.

Proof. For any $A \in \mathscr{F}$, $B_t = B_t(A) + B_t(A^c)$ for all $t \geqq 0$ and each of $\{B_t;\ t \geqq 0\}$, $\{B_t(A);\ t \geqq 0\}$, $\{B_t(A^c);\ t \geqq 0\}$ is a continuous additive functional of X. Under the hypothesis that X has a reference measure this implies that for each $A \in \mathscr{F}$ there exists a Borel measurable function $x \to H(x, A)$, $0 \leqq H(x, A) \leqq 1$ such that (5.54) holds for all t almost surely (cf. BG V.2.6).

Choosing $H(x, A)$ first for "rectangles" A with rational end points in a proper manner, and then extending it to all $A \in \mathscr{F}$ we obtain a unique measure $A \to H(x, A)$ for each x. The function $x \to H(x, A)$ is in $\mathscr{E}$ by selection when A is a rectangle with rational end points; therefore it is in $\mathscr{E}$ for all $A \in \mathscr{F}$ by the monotone class theorem.

Let $\{\tau_j\}$ be the times of jumps of Y and let $\{\alpha_j\}$ be the corresponding jumps. It follows from the preceding Lemma that, given X_{τ_j}, α_j is conditionally independent of all the τ_j, α_i $(i \neq j)$, and $\mathscr{K}$.

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