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Convergence of Semi-Groups Associated with Continuous Additive Functionals of a Markov Process

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Introduction

Let X_t be a standard Markov process on a compact metric space K in duality with another process and having a sufficiently regular potential operator of the form $Uf(x)=\int g(x,y)f(y)\,dm(y)$. Here f is a Borel function and m is a non-negative Radon measure on K. Let $u_A(x)$ be the potential of a continuous additive functional (CAF) $A(t)$ of X_t. Then if $u_A(x)$ is finite, it can be written in the form $u_A(x)=E^x[A(\infty)]=\int g(x,y)\,dv(y)$ for some Radon measure $v\geqq 0$ on K. Moreover, to each such CAF $A(t)$ one can associate a new Markov process $\hat{X}_t$ whose transition function is defined by $\hat{P}_t f(x)=E^x[f(X_{\tau(t)})]$ where $\tau(t)$ is the inverse time of A. The potential operator of $\hat{X}_t$ is then given by $\hat{U}f(x)=\int g(x,y)f(y)\,dv(y)$. Suppose now that $\{A_n\}$ is a sequence of CAF's with continuous potentials such that $u_{A_n}(x)\to u_A(x)$ uniformly where $u_A(x)=E^x[A(\infty)]$ for some CAF A. The question then arises as to whether the associated semi-groups $\hat{P}_t^n$ converge in some sense to $\hat{P}_t$, the semi-group associated with A. In this paper we show that under certain conditions on U the following convergence holds: Let $F=\operatorname{supp} A$ be the fine support of A and P_F the hitting operator associated with F. Then if F is closed, we have $\hat{P}_t^n P_F f\to\hat{P}_t f$ uniformly for each continuous function f on K and $t>0$. It follows that if $F_n=\operatorname{supp} A_n$ satisfies $F_n\subset F$ for all n, then $\hat{P}_t^n f\to\hat{P}_t f$ uniformly.

If E is a topological space, $C(E)$ and $B(E)$ will denote the Banach spaces of bounded continuous functions and Borel functions respectively under the norm $\|f\|=\sup\{|f(x)|: x\in E\}$.

1. Convergence of Semi-Groups

Let B be a Banach space with norm $\|\ \|$. By a sub-markov semi-group of operators on B we mean a family $\{P_t\}_{t>0}$ of endomorphisms of B bounded by one in norm and satisfying the semi-group property $P_{t+s}=P_t P_s$ for $s,t>0$. If E is a topological space, a sub-markov transition function on E is a family of Borel kernals $\{P_t\}_{t>0}$ which forms a sub-markov semi-group of operators on $B(E)$. In addition we assume that for $f\in C(E)$ and $x\in E$ the function $t\to P_t f(x)$ is right continuous for $t>0$ and $\lim_{t\to 0} P_t f(x)$ exists. One can then define the resolvent operators $U_\alpha f(x)=\int_0^\infty e^{-\alpha t}P_t f(x)\,dt$ for $\alpha>0$ and $f\in B(E)$. We assume throughout that U_α: $B(E)\to C(E)$ so that U_α is a bounded linear operator on $C(E)$ of norm

$\|U_\alpha\| \leqq \alpha^{-1}$. It follows from Fubini's theorem and the semi-group property that for $s \geqq 0$ and $f \in C(E)$,

$$e^{-\alpha s} P_s U_\alpha f(x) = \int_0^\infty e^{-\alpha(s+t)} P_{s+t} f(x)\, dt = \int_s^\infty e^{-\alpha t} P_t f(x)\, dt \in C(E)$$

and that the family $\{U_\alpha\}_{\alpha>0}$ satisfies the resolvent equation: $U_\alpha - U_\beta = (\beta-\alpha) U_\beta U_\alpha$ for $\alpha, \beta > 0$ (cf. [3, p. 260]).

The following theorem is similar to a result proven by Trotter [5] under slightly different conditions. Note here that no assumption is made on the limit resolvent (e.g. that its image be dense). The proof given below can be easily adapted to give a new proof of Trotter's theorem. Convergence of operators is to be interpreted as strong operator convergence in the Banach space $C(E)$. We use the notation $\operatorname{Im} U_\beta = U_\beta(C(E))$.

(1.1) **Theorem.** *Let $\{P_t^n\}$ be a sequence of sub-markov transition functions on E with resolvents $\{U_\alpha^n\}$ and suppose that for some $\beta>0$, $\lim_n U_\beta^n = U_\beta$ exists. Set $B_1 = \operatorname{Cl}(\operatorname{Im} U_\beta)$. Then there is a sub-markov semi-group of operators $\{P_t\}$ on B_1 such that $P_t^n f \to P_t f$ for all $f \in B_1$, $t>0$.*

Proof. Using the formula $U_\alpha^n = \sum_{k=1}^\infty (\beta-\alpha)^k (U_\beta^n)^k$ valid for $|\beta-\alpha|<\beta$ where the convergence is uniform in n in the uniform operator topology, it follows as in [5, p. 270] that $\lim_n U_\beta^n = U_\alpha$ exists for all $\alpha>0$ and that the operators $\{U_\alpha\}_{\alpha>0}$ on $C(E)$ satisfy the resolvent equation with $\|U_\alpha\| \leqq \alpha^{-1}$ for all $\alpha>0$. Moreover, $\{U_\alpha\}$ is strongly continuous on $B_1 = \operatorname{Cl}(\operatorname{Im} U_\beta)$ and therefore by the Hille-Yosida theorem there is a strongly continuous sub-Markov semi-group $\{P_t\}_{t>0}$ on B_1 such that $U_\alpha f = \int_0^\infty e^{-\alpha t} P_t f\, dt$ for all $\alpha>0$, $f \in B_1$ (cf. [3, p. 261]). We show now that for each $u \in C(R^+)$ and $f \in B_1$ the sequence $\left\{\int_0^\infty u(t)\, e^{-\beta t} P_t^n f\, dt\right\}_n$ converges to $\int_0^\infty u(t)\, e^{-\beta t} P_t f\, dt$ in $B(E)$. Indeed, if u is a finite sum of the form $u(t) = \sum_i \beta_i e^{-\alpha_i t}$ for constants β_i and $\alpha_i>0$, then the convergence follows from the hypothesis. If $u, v \in C(R^+)$ are arbitrary, then for $n>0$ and $f \in B_1$ we have

$$\begin{aligned}
&\left\|\int_0^\infty u(t)\, e^{-\beta t} P_t^n f\, dt - \int_0^\infty u(t)\, e^{-\beta t} P_t f\, dt\right\| \leqq \left\|\int_0^\infty (u(t)-v(t))\, e^{-\beta t} P_t^n f\, dt\right\| \\
&\quad + \left\|\int_0^\infty v(t)\, e^{-\beta t} P_t^n f\, dt - \int_0^\infty v(t)\, e^{-\beta t} P_t f\, dt\right\| + \left\|\int_0^\infty (v(t)-u(t))\, e^{-\beta t} P_t f\, dt\right\| \\
&\quad \leqq 2\int_0^\infty |u(t)-v(t)|\, \|f\|\, e^{-\beta t}\, dt + \left\|\int_0^\infty v(t)\, e^{-\beta t} P_t^n f\, dt - \int_0^\infty v(t)\, e^{-\beta t} P_t f\, dt\right\|.
\end{aligned}$$

By a form of the Stone-Weierstrass theorem the exponential functions $\{e^{-\alpha t}: \alpha>0\}$ are total in $C_0(R^+)$, the continuous functions vanishing at ∞. Since $\int_0^\infty e^{-\beta t}\, dt = \beta^{-1} < \infty$ we can find functions v, linear combinations of exponentials, such that the first term of the last line in the above inequality is arbitrarily small.

The second term becomes small on letting $n\to\infty$ by hypothesis, and therefore the stated convergence holds. Consider now the function $u(t)=I_{[s,\infty)}(t)$, the indicator function of $[s,\infty)$. We can find functions $v\in C(R^+)$ such that $\int_0^\infty |u(t)-v(t)|\, e^{-\beta t}\,dt$ is arbitrarily small, and exactly the same argument as above shows that the sequence $\left\{\int_s^\infty P_t^n f\,dt\right\}$ converges to $\int_s^\infty e^{-\beta t} P_t f\,dt$ in $B(E)$. But

$$\int_s^\infty e^{-\beta t} P_t^n f\,dt = e^{-\beta s} P_s^n U_\beta^n f \quad \text{for all } n,$$

and since $\{P_t\}$ is strongly continuous on B_1 it follows that $\int_s^\infty e^{-\beta t} P_t f\,dt = e^{-\beta s} P_s U_\beta f$ and therefore $P_s^n U_\beta^n f\to P_s U_\beta f$ for each $f\in B_1$ and $s>0$.

Now

$$\begin{aligned}\|P_s^n U_\beta f - P_s U_\beta f\| &\leqq \|P_s^n U_\beta f - P_s^n U_\beta^n f\| + \|P_s^n U_\beta^n f - P_s U_\beta f\|\\ &\leqq \|P_s^n\|\, \|U_\beta f - U_\beta^n f\| + \|P_s^n U_\beta^n f - P_s U_\beta f\| \to 0 \quad \text{as } n\to\infty.\end{aligned}$$

In other words, $P_s U_\beta f = \lim_n P_s^n U_\beta f$. Moreover, since $\|P_s^n\|\leqq 1$ for all n, P_s extends to an endomorphism of B_1 with norm $\leqq 1$ and such that $P_s^n f\to P_s f$ for all $f\in B_1$. The proof of Theorem (1.1) is complete.

(1.3) *Remarks.* (i) The resolvent equation implies that the image space $\operatorname{Im} U_\alpha$ is independent of $\alpha>0$, so that $B_1=\mathrm{Cl}(\operatorname{Im} U_\alpha)$ for all $\alpha>0$.

(ii) The convergence stated in the theorem implies that $U_\alpha^n f\to \int_0^\infty e^{-\alpha t} P_t f\,dt = U_\alpha f$ for all $f\in B_1$. Therefore if $B_1=C(E)$ we have that P_t: $C(E)\to C(E)$ where $\{P_t\}_{t>0}$ has resolvent $U_\alpha=\lim_n U_\alpha^n$ and $P_t^n f\to P_t f$ for all $f\in C(E)$. On the other hand, if $B_1=C(E)$ then the resolvent $\{U_\alpha\}_{\alpha>0}$ is strongly continuous, i.e. $\alpha U_\alpha\to I$ as $\alpha\to\infty$, or $\lim_{t\to 0} P_t=I$ (cf. [3, p. 260]).

(iii) Suppose now the potential operators $U^n f=\int_0^\infty P_t^n f\,dt$ exist as bounded linear operators on $C(E)$. Then if $U=\lim U^n$ exists the same proof as above shows that $P_t^n f\to P_t f$ for all $f\in\mathrm{Cl}(\operatorname{Im} U)=B_1$ where P_t: $B_1\to B_1$, $t>0$, is a sub-markov semi-group of operators on B_1. Moreover, if it is known that $Uf(x)=\int_0^\infty P_t f(x)\,dt$ for some sub-markov transition function $\{P_t\}_{t>0}$ on E, then $P_t^n f\to P_t f$ for all $f\in\mathrm{Cl}(\operatorname{Im} U)$.

As an application of the preceeding theorem we consider the following: Let K be a compact space and $\{P_t^n\}$ a sequence of sub-markov transition functions on K of the form $P_t^n f(x)=E_n^x[f(X_t^n)]$ where X_t^n is a Markov process on K. Suppose the resolvent operators U_α^n: $B(K)\to C(K)$ for all $\alpha>0$ and that $\lim_n U_\alpha^n=U_\alpha$ exists in $C(K)$ for some $\alpha>0$, hence all $\alpha>0$. Let $B_1=\mathrm{Cl}(\operatorname{Im} U_\alpha)$ separate the points of K. Then by a theorem of Ray (cf. [3, p. 266] or [4]) there is a semi-group $\{P_t\}_{t>0}$ of sub-markov transition functions on E whose resolvent is $\{U_\alpha\}_{\alpha>0}$, and from (1.3, iii) $P_t^n f\to P_t f$ for all $f\in B_1$. In the next section we will study a situation where an explicit characterisation of B_1 can be given.

2. Convergence of Continuous Additive Functionals

In this section we assume that X_t is a standard Markov process on a compact metric space K in duality with another process $\tilde{X}_t$ (cf. [1, Chap. VI]). Thus the potential operators U and $\tilde{U}$ are of the form $Uf(x)=\int g(x,y)f(y)\,dm(y)$ and $\tilde{U}f(y)=\int f(x)g(x,y)\,dm(x)$. Here $m\geqq 0$ is a Radon measure on K and the kernal function $g(x,y)$ satisfies the usual hypotheses of duality. We suppose that both U and $\tilde{U}$ map continuous functions into continuous functions. In addition the following properties are assumed to be satisfied by g:

(A) $g\colon K\times K\to[0,\infty]$ is bounded and continuous outside each neighborhood of the diagonal $\Delta=\{(x,x)\colon x\in K\}$ and for each $x\in K$ the function $y\to g(x,y)$ is unbounded.

These conditions are satisfied by many of the familiar processes, in particular the kernal function defining Brownian motion in R^n for $n\geqq 3$.

In what follows all measures will be nonnegative Radon measures on K. If ν is such a measure, define the operator U_ν on $B(K)$ by $Uf(x)=\int g(x,y)f(y)\,d\nu(y)$. Note that if $U_\nu 1(x)$ is bounded, then ν does not charge points, i.e. $\nu\{x\}=0$ for all $x\in K$.

(2.1) **Proposition.** *Let ν be a measure and suppose that the function*

$$x\to\int g(x,y)\,d\nu(y)=U_\nu 1(x)$$

is continuous on K. Then U_ν is strong Feller, i.e. $U_\nu\colon B(K)\to C(K)$. Moreover, given $\varepsilon>0$ there exists $\delta>0$ such that $\int_{S_\delta(x)} g(x,y)\,d\nu(y)<\varepsilon$ for all $x\in K$. Here $S_\delta(x)$ is the closed ball of radius δ and center x.

Proof. If $x_n\to x_0$, then $g(x_n,\cdot)\to g(x_0,\cdot)$ a.e. from property (A) of g and the fact that ν does not charge points. Since $\int g(x_n,y)\,d\nu(y)\to\int g(x_0,y)\,d\nu(y)$ it follows from Scheffé's lemma that $g(x_n,\cdot)\to g(x_0,\cdot)$ in $L^1(\nu)$. Thus for $f\in B(K)$ we have

$$\left|\int g(x_n,y)f(y)\,d\nu(y)-\int g(x_0,y)f(y)\,d\nu(y)\right|\leqq\|f\|\int|g(x_n,y)-g(x_0,y)|\,d\nu(y)\to 0$$

as $n\to\infty$ so that $U_\nu f$ is continuous and therefore U_ν is strong Feller. Suppose the last statement in the proposition were not true. Then from the compactness of K we could find sequences $\delta_n\downarrow 0$ and $x_n\to x_0$ such that $\int_{S_{\delta_n}(x_n)} g(x_n,y)\,dy(y)\geqq\varepsilon>0$ for all n and some $\varepsilon>0$. Since ν does not charge x_0, we can find $\delta>0$ such that $\int_{S_\delta(x_0)} g(x_0,y)\,d\nu(y)<\varepsilon/2$ and from the continuity of the function $x\to\int_{S_\delta(x_0)} g(x,y)\,d\nu(y)$ the inequality holds in a neighborhood V of x_0. But for n large, $S_{\delta_n}(x_n)\subset V$ and therefore $\int_{S_{\delta_n}(x_n)} g(x_n,y)\,d\nu(y)<\varepsilon/2$ giving the desired contradiction.

By an abuse of terminology we will say that a measure ν is regular if the function $x\to U_\nu 1(x)$ is continuous on K. The following result is crucial for further developments.

(2.2) **Theorem.** *Let $\{\nu_n\}$ be a sequence of regular measures such that $U_{\nu_n}1\to U_\nu 1$ uniformly for some (regular) measure ν. Then $U_{\nu_n}f\to U_\nu f$ uniformly for all $f\in C(K)$.*

We will break up the proof into several parts.

(2.3) **Lemma.** *Let $f(x,y)$ be a continuous function on $K\times K$ and $\{v_n\}$ a uniformly bounded sequence of measures such that $\int f(x,y)\,dv_n(y)\to\int f(x,y)\,dv(y)$ pointwise for some measure v. Then $\int f(x,y)\,dv_n(y)\to\int f(x,y)\,dv(y)$ uniformly on K.*

Proof. It follows from the uniform continuity of f that the map $x\to f_x(y)=f(x,y)$ is continuous from K to $C(K)$. Since K is compact, given $\varepsilon>0$ we can find a finite open covering $\{G_{x_i}\}$ of K with $G_{x_i}\ni x_i$, $i=1,2,\ldots,n$, such that $\|f_x-f_{x_i}\|<\varepsilon$ whenever $x\in G_{x_i}$. Then for any i,

$$\begin{aligned}|\textstyle\int f(x,y)\,dv(y)-\int f(x,y)\,dv_n(y)|&\leqq\textstyle\int|f(x,y)-f(x_i,y)|\,dv(y)\\&\quad+|\textstyle\int f(x_i,y)\,dv(y)-\int f(x_i,y)\,dv_n(y)|+\int|f(x_i,y)-f(x,y)|\,dv_n(y)\\&\leqq\|f_x-f_{x_i}\|\,v(K)+|\textstyle\int f(x_i,y)\,dv(y)-\int f(x_i,y)\,dv_n(y)|+\|f_{x_i}-f_x\|\,v_n(K).\end{aligned}$$

By hypothesis we can choose N such that $|\int f(x_i,y)\,dv(y)-\int f(x_i,y)\,dv_n(y)|<\varepsilon$ for all i whenever $n>N$. If $x\in K$ then $x\in G_{x_{i_0}}$ for some i_0 and therefore $\|f_x-f_{x_{i_0}}\|<\varepsilon$. Hence $|\int f(x,y)\,dv(y)-\int f(x,y)\,dv_n(y)|\leqq\varepsilon\,v(K)+\varepsilon+\varepsilon\,v_n(K)\leqq\varepsilon(1+2M)$ for all $n>N$ where M is a bound for $v_n(K)$, $v(K)$.

(2.4) **Lemma.** *Let $\{v_n\}$ be a sequence of regular measures such that $U_{v_n}1\to U_v1$ uniformly for some measure v. Then given $\varepsilon>0$, there is a $\delta>0$ such that $\int_{S_\delta(x)} g(x,y)\,dv_n(y)<\varepsilon$ for all n and $x\in K$.*

Proof. Given $\varepsilon>0$, choose $\delta_1>0$ such that $\int_{S_{\delta_1}(x)} g(x,y)\,dv(y)<\varepsilon$ for all $x\in K$ (Proposition (2.1)). Let $\alpha\in C^+(R)$ be such that $0\leqq\alpha\leqq1$, support $(\alpha)\subseteq[-\delta_1,\delta_1]$ and $\alpha=1$ in a neighborhood of 0. Define $\alpha(x,y)=\alpha(\rho(x,y))\leqq1$ where ρ is the metric on K. Then for $x\in K$ $\operatorname{supp}_y\alpha(x,y)\subseteq S_{\delta_1}(x)$ and therefore $\int g(x,y)\,\alpha(x,y)\,dv(y)<\varepsilon$ for all $x\in K$. Set $\beta(x,y)=1-\alpha(x,y)$ so that $\alpha+\beta=1$. Then

$$U_{v_n}1(x)=\int g(x,y)\,\alpha(x,y)\,dv_n(y)+\int g(x,y)\,\beta(x,y)\,dv_n(y)\to U_v1(x)$$

by hypothesis. Now the function $f(x,y)=g(x,y)\,\beta(x,y)$ is a continuous function on $K\times K$ since $\beta=0$ in a neighborhood of the diagonal Δ. On the other hand it follows from the hypothesis and the duality assumption stated at the beginning of this section that $v_n\to v$ vaguely (see [1, p. 268]). Thus for each $x\in K$

$$\int f(x,y)\,dv_n(y)\to\int f(x,y)\,dv(y)$$

and from Lemma (3.3) the convergence is uniform on K. It now follows that

$$\int g(x,y)\,\alpha(x,y)\,dv_n(y)\to\int g(x,y)\,\alpha(x,y)\,dv(y)<\varepsilon$$

uniformly on K. Choose $\delta>0$ such that $S_\delta(x)\subseteq\{y\colon\alpha(x,y)=1\}$ for all $x\in K$. Then

$$\int_{S_\delta(x)} g(x,y)\,dv_n(y)\leqq\int g(x,y)\,\alpha(x,y)\,dv_n(y)<\varepsilon$$

for all $x\in K$ and sufficiently large n, and using (2.1) we can find a $\delta_1>0$ such that $\int_{S_{\delta_1}(x)} g(x,y)\,dv_n(y)<\varepsilon$ for all n, $x\in K$.

Proof of Theorem. Let $f \in C(K)$ and $\varepsilon > 0$ be given. From Lemma (2.4) and its proof we can find a continuous function $\alpha \in C^+(K \times K)$ such that

$$\left|\int g(x,y)\,\alpha(x,y)\,f(y)\,d\nu_n(y)\right| < \varepsilon$$

for all n and $x \in K$ and such that the function $\beta(x,y) = 1 - \alpha(x,y)$ has the property that $g(x,y)\,\beta(x,y)$ is continuous on $K \times K$. The vague convergence of ν_n to ν implies that of $f(y)\,d\nu_n(y)$ to $f(y)\,d\nu(y)$ and therefore

$$\int g(x,y)\,\beta(x,y)\,f(y)\,d\nu_n(y) \to \int g(x,y)\,\beta(x,y)\,d\nu(y)$$

pointwise, hence uniformly by Lemma (2.3). But then for all $x \in K$

$$\begin{aligned}
&\left|\int g(x,y)\,f(y)\,d\nu_n(y) - \int g(x,y)\,f(y)\,d\nu(y)\right| \\
&\quad \leqq \left|\int g(x,y)\,\alpha(x,y)\,f(y)\,d\nu_n(y)\right| + \left|\int g(x,y)\,\alpha(x,y)\,f(y)\,d\nu(y)\right| \\
&\qquad + \left|\int g(x,y)\,\beta(x,y)\,f(y)\,d\nu_n(y) - \int g(x,y)\,\beta(x,y)\,f(y)\,d\nu(y)\right|.
\end{aligned}$$

By altering the function α if need be, we can guarantee that the first and second terms on the right side of the inequality are less than ε, uniformly in n and x. The third term approaches zero uniformly in $x \in K$ as $n \to \infty$. The proof of Theorem (2.2) is complete.

Let now $A(t)$ be a CAF of X_t and $\tau(t) = \inf\{s\colon A(s) > t\}$ the inverse time associated with $A(t)$. Then the process $X_{\tau(t)}$ is also a Markov process with transition function defined by $\hat{P}_t f(x) = E^x[f(X_{\tau(t)})]$ and resolvent operator

$$\hat{U}_\alpha f(x) = E^x \int_0^\infty e^{-\alpha t} f(X_{\tau(t)})\,dt, \qquad \alpha \geqq 0.$$

See (2.11), p. 212 of [1]. It follows from the right continuity of the functions $t \to f(X_{\tau(t)})$ for $f \in C(K)$ that $\alpha\,\hat{U}_\alpha f(x) \to E^x[f(X_{\tau(0)})]$ as $\alpha \to \infty$. Also $\tau(0) = T_F$, the hitting time of the set $F = \operatorname{supp} A$ = fine support of A. See [1, Chap. V, Sec. 3]. On the other hand, it follows from a result of Lion [2, p. 425] that if $\hat{U}_\alpha\colon C(K) \to C(K)$ for all $\alpha > 0$, then the function $P_F f(x)$ is continuous if and only if F is closed and in this case $\alpha\,\hat{U}_\alpha f(x) \to P_F f(x)$ uniformly on K as $\alpha \to \infty$.

(2.5) **Lemma.** *Suppose* $\hat{U} = \hat{U}_0\colon C(K) \to C(K)$ *and that* $F = \operatorname{supp} A$ *is closed. Then* $\mathrm{Cl}(\mathrm{Im}\,\hat{U}) = P_F(C(K))$.

Proof. If $f = \hat{U} g$ for $g \in C(K)$, then it follows from the strong Markov property and the fact that $T_F + \tau(t) \circ \theta_{T_F} = \tau(t)$ a.s. that $P_F f = f$ and hence $P_F h = h$ for all $h \in \mathrm{Cl}(\mathrm{Im}\,\hat{U})$. On the other hand, if $f \in C(K)$, then from the previous discussion we have that $\alpha\,\hat{U}_\alpha f(x) \to P_F f(x)$ uniformly on K as $\alpha \to \infty$. Using the fact that $\mathrm{Im}\,\hat{U}_\alpha$ is independent of α, it follows that $P_F f \in \mathrm{Cl}(\mathrm{Im}\,\hat{U}_\alpha)$ for all $\alpha > 0$, and therefore $P_F f \in \mathrm{Cl}(\mathrm{Im}\,\hat{U})$ since $\lim_{\alpha \to 0} \hat{U}_\alpha = \hat{U}$.

We come now to the main result of this development. Call a CAF $A(t)$ regular if the function $x \to E^x[A(\infty)] = u_A(x)$ is continuous. Then if A is regular, A is a regular potential and since K is compact, $u_A(x) = U_\nu 1(x)$ for some measure ν. See [1, p. 271]. In fact, under our hypotheses every continuous excessive function $u(x)$ can be written in the form $u(x) = E^x[A(\infty)] = U_\nu 1(x)$ for some CAF $A(t)$ and measure ν.

(2.6) **Theorem.** *Let* $\{A_n\}$ *be a sequence of regular* CAF's *such that* $u_{A_n}(x) \to u(x)$ *uniformly. Then* $u(x) = E^x[A(\infty)]$ *for some regular* CAF $A(t)$. *Let* $F = \operatorname{supp} A$ *and suppose that* F *is closed. Then* $\hat{P}_t^n P_F f \to \hat{P}_t f$ *uniformly for all* $f \in C(K)$ *and* $t > 0$ *where* $\hat{P}_t^n (\hat{P}_t)$ *is the sub-markov transition function associated with* A_n *(resp.* A*).*

Proof. Since the uniform limit of continuous excessive functions is excessive, $u(x) = E^x[A(\infty)]$ for some regular CAF $A(t)$. On the other hand, we can write $u_{A_n}(x) = U_{\nu_n} 1(x)$ and $u(x) = U_\nu 1(x)$ for some (uniquely determined) measures ν_n, ν and therefore $U_{\nu_n} 1(x) \to U_\nu 1(x)$ uniformly on K. It now follows from (1.3, iii), (2.2), (2.5) and the fact that U_{ν_n} is the potential operator associated with $\hat{P}_t^n$ that $\hat{P}_t^n P_F f \to \hat{P}_t P_F f$ uniformly as $n \to \infty$ for all $t > 0$. Finally, note that

$$\hat{P}_t P_F f(x) = E^x[f(X_{\tau(t) + T_F \circ \theta_{\tau(t)}})] = E^x[f(X_{\tau(t)})] = \hat{P}_t f(x)$$

since $T_F \circ \theta_{\tau(t)} = 0$ a.s. as $X_{\tau(t)} \in F$ a.s. and $F = F^r$ (here F^r is the set of points regular for F. Cf. [1, p. 61]).

(2.7) **Corollary.** *Same assumptions as in* (2.6). *Suppose* $F_n = \operatorname{supp} A_n$ *satisfies* $F_n \subset F$ *for all* n. *Then* $\hat{P}_t^n f \to \hat{P}_t f$ *uniformly for all* $f \in C(K)$ *and* $t > 0$.

Proof. This follows immediately from Theorem (2.6) and the fact that $\hat{P}_t^n P_F f(x) = E^x[f(X_{\tau_n(t) + T_F \circ \theta_{\tau_n(t)}})]$: Since $F_n \subset F$, $X_{\tau_n(t)} \in F$ a.s. and since $F = F^r$, $T_F \circ \theta_{\tau_n(t)} = 0$ a.s. and therefore $\hat{P}_t^n P_F f(x) = \hat{P}_t^n f(x)$ for all n and $t > 0$.

We conclude with the following remarks. Recall that we have assumed the potential operator of X_t to be of the form $Uf(x) = \int g(x, y) f(y)\, dm(y)$. Therefore, if $f(x)$ is any bounded excessive function for X_t, the function

$$f_n(x) = n\big(f(x) - P_{1/n} f(x)\big) \geqq 0$$

satisfies

$$Uf_n(x) = \int_0^\infty P_t f_n(x)\, dt = n \int_0^\infty \big(P_t f(x) - P_{t+1/n} f(x)\big)\, dt = n \int_0^{1/n} P_t f(x)\, dt \uparrow f(x)$$

from Proposition (3.4), p. 161 of [1]. Suppose now that $A(t)$ is a regular CAF with closed fine support. Then $u_A(x) = E^x[A(\infty)] = \int g(x, y)\, d\nu(y)$ for some measure ν. Moreover, if we define the continuous functions

$$f_n(y) = n\big(U_\nu 1(y) - P_{1/n} U_\nu 1(y)\big) \geqq 0$$

then $\int g(x, y) f_n(y)\, dm(y) \uparrow \int g(x, y)\, d\nu(y)$ pointwise and hence uniformly by Dini's theorem. If now A_n is the CAF associated with the measure $f_n(y)\, dm(y)$, then $u_{A_n}(x) \to u_A(x)$ uniformly and therefore we have convergence of the associated transition functions in the sense described in Theorem (2.6). In other words, any CAF A satisfying the aforementioned hypotheses can be approximated in the sense of Theorem (2.6) by CAF's having potentials of the form $\int g(x, y) f(y)\, dm(y)$ with $f \in C(K)$.

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