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Wave-Length and Amplitude for a Stationary Gaussian Process after a High Maximum

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1. Introduction

Let $\{\xi(t), t \in R\}$ be a stationary, zero-mean, Gaussian process with covariance function r and assume $r(0)=1$, $-r''(0)=\lambda_2$. The object of this paper is to study the distribution of the two wave-characteristics wave-length and amplitude, i.e. the horizontal and vertical distances between "a randomly chosen" local maximum and the following minimum, especially when the maximum is very high. The main tool is a random process

$$\xi_u(t) = u\,r(t) - \eta_u(\lambda_2 r(t) + r''(t)) + \Delta(t)$$

(where η_u is a certain random variable, and Δ is a certain non-stationary Gaussian process), which can be interpreted as the original stationary process $\xi(t)$ conditioned by the presence of a local maximum with height u at $t=0$. The condition is to be taken in the horizontal window sense of Kac and Slepian [5], i.e.

$$P(\xi_u(t) \leqq x) = \lim_{h, h' \to 0} P\left\{ \xi(t) \leqq x \;\middle|\; \begin{array}{l} \xi(s) \text{ has a local maximum with height in} \\ (u, u+h) \text{ for some } s \text{ in } (-h', 0) \end{array} \right\}.$$

It is shown by Lindgren [7] that this gives the "ergodic", (i.e. the long-run, in a single realization) distribution of $\xi(\cdot)$ at a distance t from the local u-maxima of ξ.

Let the wave-length $\tau_u > 0$ be the time for the first local minimum of ξ_u, and let $\delta_u = u - \xi_u(\tau_u)$ be the corresponding amplitude. Then the distributions of τ_u and δ_u coincide with the ergodic distributions of the distances (horizontal and vertical) between local u-maxima of $\xi(t)$ and the following minimum, cf. Lindgren [9], Theorem 1.2 and 2.3.

The exact distribution of (τ_u, δ_u) is difficult to obtain in the general case, and one has to rely on approximations (see Lindgren [9] for fairly accurate moment bounds), or turn to asymptotic results as $u \to \pm\infty$. The case $u \to -\infty$ has been treated previously by Lindgren [8], who shows that $(|u|\,\tau_u, |u|^3\,\delta_u)$ has a non-trivial limit distribution as $u \to -\infty$, provided the covariance function r possesses a finite sixth order spectral moment.

Here we will concentrate upon the case $u \to +\infty$. Then the dominant term in $\xi_u(t)$ is $u\,r(t)$, and the behaviour of the process after a very high maximum is well determined by the behaviour of its covariance function; the process "follows its covariance function". Especially, $\xi_u(t)$ can have a local minimum only at points where $r'(t)$ is relatively small. Thus it is necessary to distinguish between the following three cases, where we say that r has a stationary point at t_0 if $r'(t_0)=0$:

i) r has a first local minimum at $t_0 > 0$ and has no stationary points in $(0, t_0)$.

ii) r has a first local minimum at $t_0>0$ and has at least one stationary point in $(0, t_0)$.

iii) r has no stationary points in $(0, \infty)$.

In case i) we will prove that, after suitable normalizations, $\tau_u - t_0$ and $\delta_u - u(1-r(t_0))$ are asymptotically normal, while in case ii) there will be a positive probability that τ_u falls near some of the stationary points less than t_0. Then the asymptotic normal distributions have to be modified. This is done in Section 3 and 4 respectively.

In case iii) then, for every fixed $t>0$, the probability that $\xi_u(\cdot)$ is strictly decreasing in $(0, t)$, will tend to 1 and hence $\tau_u \to \infty$ in probability as $u \to \infty$. In Section 5, 6 we investigate the rate with which this happens, and in Section 7 we give some results for the amplitude δ_u.

2. Some Definitions and General Results

Suppose the covariance function r is four times continuously differentiable with $r(0)=1$, $-r''(0)=\lambda_2$, $r^{\mathrm{IV}}(0)=\lambda_4$, and assume

$$r^{\mathrm{IV}}(t)=\lambda_4+O(|\log|t||^{-a}) \quad \text{as } t\to 0 \text{ for some } a>1, \tag{2.1}$$
$$r(t)\to 0 \quad \text{as } t\to\infty.$$

Then the process ξ can be supposed to have, with probability 1, twice continuously differentiable sample paths, and local maxima are easily defined in terms of the sample derivative $\xi'(t)$. (For analytic properties of sample paths as well as many valuable references on crossing problems and local maxima, see the book by Cramér and Leadbetter [3].)

Now define the functions

$$\begin{aligned} C(s,t) &= r(s-t)-[\lambda_2(\lambda_4-\lambda_2^2)]^{-1}\{\lambda_2\lambda_4 r(s)r(t)+\lambda_2^2 r(s)r''(t) \\ &\quad +(\lambda_4-\lambda_2^2)r'(s)r'(t)+\lambda_2^2 r''(s)r(t)+\lambda_2 r''(s)r''(t)\} \\ c(s,t) &= \frac{\partial^2 C(s,t)}{\partial s\,\partial t} = -r''(s-t)-[\lambda_2(\lambda_4-\lambda_2^2)]^{-1}\{\lambda_2\lambda_4 r'(s)r'(t) \\ &\quad +\lambda_2^2 r'(s)r'''(t)+(\lambda_4-\lambda_2^2)r''(s)r''(t)+\lambda_2^2 r'''(s)r'(t)+\lambda_2 r'''(s)r'''(t)\}, \end{aligned}$$

and let $\{\Delta(t), t\in R\}$ be a non-stationary, zero-mean, Gaussian process with the covariance function C. (That C is non-negative definite follows from Lemma 2 of [7].) The process Δ can be chosen to have, with probability 1, twice continuously differentiable sample paths, the derivatives of which, $\delta(t)=\Delta'(t)$, constitute a non-stationary, zero-mean, Gaussian process with the covariance function c. This can be proved in a similar way as Lemma 1.1 of [8] if one makes use of a weaker condition for the existence of sample derivatives given by e.g. Leadbetter and Weissner [6].

Also let η_u be a random variable (r.v.), defined on the same probability space as Δ, independent of the process Δ, and with the density

$$q_u^*(y)=\begin{cases} 0 & (y<-u/\beta) \\ \dfrac{\lambda_2\beta(u/\beta+y)\exp(-\lambda_2\beta y^2/2)}{\sqrt{2\pi}\,\Psi(u\sqrt{\lambda_2/\beta})} & (y\geqq -u/\beta), \end{cases} \tag{2.2}$$

where

$$\begin{aligned}\beta &= (\lambda_4 - \lambda_2^2)/\lambda_2,\\ \Psi(x) &= \phi(x) + x\,\Phi(x),\end{aligned} \tag{2.3}$$

and ϕ and Φ are the standard normal density and distribution functions.

Now define the process ξ_u and its derivative ξ'_u by

$$\xi_u(t) = u\,r(t) - \eta_u\big(\lambda_2\,r(t) + r''(t)\big) + \Delta(t), \tag{2.4}$$

$$\xi'_u(t) = u\,r'(t) - \eta_u\big(\lambda_2\,r'(t) + r'''(t)\big) + \delta(t), \tag{2.5}$$

and the wave-length and amplitude

$$\begin{aligned}\tau_u &= \text{first local minimum of } \xi_u(t)\\ &= \text{first upcrossing zero of } \xi'_u(\mathrm{t})\\ \delta_u &= u - \xi_u(\tau_u).\end{aligned} \tag{2.6}$$

As was mentioned in the introduction, ξ_u and τ_u, δ_u can be used to describe the ergodic (and horizontal-window conditional) properties of ξ after a local maximum with height u:

Proposition 2.1. a) $\xi'_u\,(t) < 0$ *for all sufficiently small positive t, and* ξ_u *has a local maximum with height u at* 0 *(a.s.).*

b) *Given that ξ has a local maximum with height u at t_0 (in h.w. sense), $\xi(t_0+t)$ and $\xi'(t_0+t)$ have the same distributions as $\xi_u(t)$ and $\xi'_u(t)$.*

c) *The wave-length and amplitude of ξ after a local maximum with height u (in h.w. sense) have the same distributions as τ_u and δ_u.*

Proof. Rewriting $\xi_u(t)$ as

$$\xi_u(t) = u \cdot \frac{\lambda_4\,r(t) + \lambda_2\,r''(t)}{\lambda_4 - \lambda_2^2} + \Delta(t) - \zeta \cdot \frac{\lambda_2\,r(t) + r''(t)}{\lambda_4 - \lambda_2^2},$$

where the r.v. $\zeta = \lambda_2\,\beta\,(u/\beta + \eta_u)$ has the density $q_u(z) = q_u^*\big((z - \lambda_2\,u)/\lambda_2\,\beta\big)/\lambda_2\,\beta\ (z>0)$, we recognize (2.4) and (2.5) as the processes given by (1.1) and (1.2) in [8]. Then part a) follows from the proof of Lemma 1.1 in [9]. Part b) and part c) are essentially the ergodic Theorems 1.1 and 1.2 in [9]. We only have to ascertain that $\{\xi(t), t \in R\}$ is an ergodic process. But since $r(t) \to 0$ as $t \to \infty$, the spectral distribution of ξ can have no discrete part, and this implies that ξ is ergodic, see [3, Chap. 7].

Lemma 2.1. *The r.v. $\eta_u \sqrt{\lambda_2\,\beta}$ has an asymptotic standard normal distribution as $u \to \infty$, and its density tends to ϕ with dominated convergence.*

Proof. The function $x/\Psi(x)$ increases to 1 as $x \to \infty$, and therefore the density $(2\pi)^{-\frac{1}{2}} \exp(-x^2/2)(u\sqrt{\lambda_2/\beta} + x)/\Psi(u\sqrt{\lambda_2/\beta})$ of $\eta_u \sqrt{\lambda_2\,\beta}$ tends to $\phi(x)$ with dominated convergence.

The lemma implies that the term $\eta_u\big(\lambda_2\,r'(t) + r'''(t)\big)$ in $\xi'_u(t)$ is of moderate order as $u \to \infty$, and so is the $\delta(t)$-term in any bounded interval (since δ has continuous sample paths). Hence we expect that, for large u, $\xi'_u(t)$ can be zero only if $r'(t)$ is close to zero. To express it more precisely, let I be any bounded, measurable set of non-negative times, and define

$$I_\varepsilon = I \cap \{t;\ t \geqq \varepsilon\}.$$

Lemma 2.2. *If* $\inf_{t\in I_\varepsilon}|r'(t)|>0$ *for all* $\varepsilon>0$, *then* $P(\tau_u\in I)\to 0$ *as* $u\to\infty$.

Proof. There is an $\varepsilon>0$ such that $r'(t)<0$ for $0<t\leqq\varepsilon$. Since

$$P(\tau_u\in I)\leqq P(\tau_u\leqq\varepsilon)+P(\tau_u\in I_\varepsilon),$$

it is sufficient to prove that a) $P(\tau_u\leqq\varepsilon)\to 0$, b) $P(\tau_u\in I_\varepsilon)\to 0$.

a) r' and r''' are continuously differentiable with $r'(0)=r'''(0)=0$, $r''(0)=-\lambda_2<0$, so that

$$\inf_{0\leqq s\leqq\varepsilon} s^{-1}|r'(s)|>0, \qquad \sup_{0\leqq s\leqq\varepsilon} s^{-1}|\lambda_2 r'(s)+r'''(s)|<\infty.$$

Since, furthermore, δ has continuously differentiable sample paths with $\delta(0)=0$ (a.s.), we have that $\sup_{0\leqq s\leqq\varepsilon} s^{-1}|\delta(s)|$ is a well-defined, finite r.v. Thus, for $0<t\leqq\varepsilon$,

$$\xi_u'(t)\leqq -u\,t\cdot\inf\left|\frac{r'(s)}{s}\right|+|\eta_u|\,t\cdot\sup\left|\frac{\lambda_2 r'(s)+r'''(s)}{s}\right|+t\cdot\sup\left|\frac{\delta(s)}{s}\right|,$$

which is strictly negative if

$$|\eta_u|<\frac{u\cdot\inf|r'(s)/s|-\sup|\delta(s)/s|}{\sup|(\lambda_2 r'(s)+r'''(s))/s|}.$$

This occurs with high probability if u is large, and therefore $P(\tau_u\leqq\varepsilon)=1-P(\xi_u'(t)<0$ for $0<t\leqq\varepsilon)$ is arbitrarily close to zero for large u.

b) Take T so large that $I_\varepsilon\subset[\varepsilon, T]$. Then

$$m=\inf_{I_\varepsilon}|r'(t)|>0, \qquad M=\sup_{[\varepsilon,T]}|\lambda_2 r'(t)+r'''(t)|<\infty,$$

and we can estimate $|\xi_u'(t)|$ in terms of $|\eta_u|$ and $\sup_{[\varepsilon,T]}|\delta(t)|$. Thus, for $|\eta_u|\leqq u\,m/2M$, we get

$$\inf_{I_\varepsilon}|\xi_u'(t)|\geqq u\,m-|\eta_u|\,M-\sup_{I_\varepsilon}|\delta(t)|\geqq u\,m/2-\sup_{I_\varepsilon}|\delta(t)|,$$

and

$$P\big(\sup_{[\varepsilon,T]}|\delta(t)|<u\,m/2\big)\leqq P\big(\inf_{I_\varepsilon}|\xi_u'(t)|>0\big)+P(|\eta_u|>u\,m/2M),$$

$$P(\tau_u\in I_\varepsilon)\leqq 1-P\big(\inf_{I_\varepsilon}|\xi_u'(t)|>0\big)\leqq 1-P\big(\sup_{[\varepsilon,T]}|\delta(t)|<u\,m/2\big)+P(|\eta_u|>u\,m/2M).$$

The last probability on the right hand side tends to zero as $u\to\infty$. Since the sample paths of δ are continuous, they are bounded over the compact interval $[\varepsilon, T]$, and thus the first probability tends to one. This proves that $\lim_{u\to\infty}P(\tau_u\in I_\varepsilon)=0$.

3. Asymptotic Normality in Case I

We specify the conditions for case i), in addition to (2.1).

C1. There is a time $t_0>0$ and a positive integer k_0 such that the covariance function r is $2k_0$ times continuously differentiable near t_0 and

$$\begin{aligned} r'(t)&<0 \quad \text{for } 0<t<t_0,\\ r^{(j)}(t_0)&=0 \quad \text{for } j=1,2,\dots,2k_0-1,\\ r^{(2k_0)}(t_0)&>0. \end{aligned}$$

Thus r' has its first upcrossing zero at t_0, and we will show that τ_u tends to t_0 in probability as $u\to\infty$. The asymptotic distribution of (τ_u, δ_u) can be expressed in terms of two independent normal r.v., ψ_0 and χ_0, defined as follows. Let η be $N(0, 1/\sqrt{\lambda_2 \beta})$ and independent of $(\Delta(t_0), \delta(t_0))$, and define

$$\begin{aligned}\psi_0 &= \eta(\lambda_2 r(t_0)+r''(t_0))-\Delta(t_0),\\ \chi_0 &= \eta r'''(t_0)-\delta(t_0).\end{aligned} \tag{3.1}$$

Then (ψ_0, χ_0) is bivariate normal, with mean zero and the covariances

$$\begin{aligned}V(\psi_0) &= (\lambda_2 r(t_0)+r''(t_0))^2 V(\eta)+V(\Delta(t_0))\\ &= (\lambda_2 r(t_0)+r''(t_0))^2/\lambda_2 \beta + C(t_0, t_0) = 1-r^2(t_0),\\ \operatorname{Cov}(\psi_0, \chi_0) &= r'''(t_0)(\lambda_2 r(t_0)+r''(t_0))\, V(\eta)+\operatorname{Cov}(\Delta(t_0), \delta(t_0))\\ &= r'''(t_0)(\lambda_2 r(t_0)+r''(t_0))/\lambda_2 \beta + \left.\frac{\partial C(s,t)}{\partial t}\right|_{s=t=t_0} = 0,\\ V(\chi_0) &= r'''(t_0)^2 V(\eta)+V(\delta(t_0))\\ &= r'''(t_0)^2/\lambda_2 \beta + c(t_0, t_0) = \lambda_2 - r''(t_0)^2/\lambda_2.\end{aligned}$$

Here we have made repeated use of the fact that $r'(t_0)=0$.

Theorem 3.1. *If condition* C1 *is fulfilled with t_0 and k_0 then, as $u\to\infty$*

$$\{u(\tau_u - t_0)^{2k_0-1}, \delta_u - u(1-r(t_0))\} \xrightarrow{\mathscr{L}} \left\{\frac{(2k_0-1)!}{r^{(2k_0)}(t_0)}\chi_0, \psi_0\right\}.$$

($\xrightarrow{\mathscr{L}}$ means convergence in law.) The theorem simply says that $u(\tau_u - t_0)^{2k_0-1}$ and $\delta_u - u(1-r(t_0))$ are asymptotically normal and independent with the variances $\{(2k_0-1)!/r^{(2k_0)}(t_0)\}^2(\lambda_2 - r''(t_0)^2/\lambda_2)$ and $1-r^2(t_0)$ respectively.

Proof. We first show that $\tau_u \xrightarrow{\mathscr{P}} t_0$, i.e. that, for all small $\varepsilon>0$,

$$P(t_0-\varepsilon \leqq \tau_u \leqq t_0+\varepsilon)\to 1.$$

But if ε is small enough, the closed interval $[0, t_0-\varepsilon]$ fulfills the requirements of Lemma 2.2, so that $P(\tau_u \leqq t_0-\varepsilon)\to 0$. Furthermore

$$\xi_u'(t_0+\varepsilon) = u r'(t_0+\varepsilon) - \eta_u(\lambda_2 r'(t_0+\varepsilon)+r'''(t_0+\varepsilon))+\delta(t_0+\varepsilon),$$

and since the covariance derivative $r'(t_0+\varepsilon)$ is strictly positive for all small positive ε, we conclude that $P(\xi_u'(t_0+\varepsilon)>0)\to 1$ as $u\to\infty$. But ξ_u' has continuous sample paths (a.s.), so that $\xi_u'(t_0+\varepsilon)>0$ implies that $\tau_u<t_0+\varepsilon$, except on a set of probability zero. Thus, as asserted,

$$P(t_0-\varepsilon\leqq\tau_u\leqq t_0+\varepsilon)\geqq P(\xi_u'(t_0+\varepsilon)>0)-P(\tau_u\leqq t_0-\varepsilon)\to 1.$$

Now, as we know that $V=\tau_u - t_0$ tends to zero in probability, we can expand the functions r, r', r'', and r''', as well as the random functions Δ and δ in Taylor series near t_0. If we write k instead of k_0 and employ the symbol $o_p(1)$ for any r.v. $\xrightarrow{\mathscr{P}} 0$ as $u \to \infty$, then $V = o_p(1)$, and

$$r'(\tau_u) = \frac{V^{2k-1}}{(2k-1)!} \{r^{(2k)}(t_0) + o_p(1)\} = o_p(1),$$
$$r'''(\tau_u) = r'''(t_0) + o_p(1). \tag{3.2}$$

Since δ is continuously differentiable (a.s.) we also get

$$\delta(\tau_u) = \delta(t_0) + o_p(1). \tag{3.3}$$

Inserting (3.2) and (3.3) into the definition (2.5) of $\xi_u'(\tau_u)$, and using that $\eta_u \cdot o_p(1)$ is $o_p(1)$, yields

$$\xi_u'(\tau_u) = \frac{u V^{2k-1}}{(2k-1)!} \{r^{(2k)}(t_0) + o_p(1)\} - \{\eta_u r'''(t_0) - \delta(t_0)\} + o_p(1).$$

Up to now we have not used that $\xi_u'(\tau_u) = 0$. Doing so now we get, by Lemma 2.1,

$$u V^{2k-1} = \frac{(2k-1)!}{r^{(2k)}(t_0) + o_p(1)} \{\eta_u r'''(t_0) - \delta(t_0)\} + o_p(1) \xrightarrow{\mathscr{L}} \frac{(2k-1)!}{r^{(2k)}(t_0)} \chi_0 \tag{3.4}$$

as asserted.

The rest of the theorem is now straightforward:

$$\begin{aligned}\delta_u - u(1 - r(t_0)) &= u - \xi_u(\tau_u) - u(1 - r(t_0)) \\ &= u\, r(t_0) - u\, r(\tau_u) + \eta_u(\lambda_2 r(\tau_u) + r''(\tau_u)) - \Delta(\tau_u).\end{aligned} \tag{3.5}$$

Here

$$u\, r(\tau_u) = u\, r(t_0) + \frac{u V^{2k}}{(2k)!} \{r^{(2k)}(t_0) + o_p(1)\} = u\, r(t_0) + o_p(1)$$

according to (3.4). Since $\lambda_2 r(\tau_u) + r''(\tau_u) = \lambda_2 r(t_0) + r''(t_0) + o_p(1)$, and $\Delta(\tau_u) = \Delta(t_0) + o_p(1)$, we see that (3.5) equals

$$\eta_u(\lambda_2 r(t_0) + r''(t_0)) - \Delta(t_0) + o_p(1) \xrightarrow{\mathscr{L}} \psi_0. \tag{3.6}$$

Thus both τ_u and δ_u have the due limiting distributions. If we combine (3.4) and (3.6), we easily obtain the bivariate statement of the theorem.

4. Modified Asymptotic Normality in Case II

In this case the covariance function r has a (series of) "terrace" point(s) before its first local minimum, and at these terrace points the covariance derivative r' has a tangency of zero[1]. Even if the sample derivative $\xi_u'(t)$ given by (2.5) closely follows the function $u\, r'(t)$, the question whether it will cross the zero level or not near the terrace point depends on the sign of $-\eta_u r'''(t) + \delta(t)$.

We specify the conditions for case ii).

C2. There is a finite number of times, $0 < t_1 < t_2 < \cdots < t_n < t_0$, and positive integers, $k_1, k_2, \ldots, k_n, k_0$, such that r has continuous derivatives up to order

[1] The covariance function $r(t) = \frac{\sin t}{t} \cdot \cos^2 \frac{t}{2}$ has a terrace point at $t = \pi$ and its first minimum at a point $t > \pi$.

$2k_i+1$ near t_i, $i=1,2,\ldots,n$ and up to order $2k_0$ near t_0. Furthermore

$$\begin{aligned}
r'(t)&<0 \quad \text{for } 0<t<t_0,\ t\neq t_1,\ldots,t_n\\
r^{(j)}(t_0)&=0 \quad \text{for } j=1,\ldots,2k_0-1\\
r^{(2k_0)}(t_0)&>0\\
\left.\begin{aligned} r^{(j)}(t_i)&=0 \quad \text{for } j=1,\ldots,2k_i\\ r^{(2k_i+1)}(t_i)&<0 \end{aligned}\right\}& \text{ for } i=1,\ldots,n.
\end{aligned}$$

With (3.1) we introduced two independent normal r.v. ψ_0 and χ_0 equal to $\eta(\lambda_2 r(t_0)+r''(t_0))-\Delta(t_0)$ and $\eta r'''(t_0)-\delta(t_0)$, where $\eta_u \xrightarrow{\mathscr{L}} \eta$ and η is $N(0, 1/\sqrt{\lambda_2\beta})$ and independent of all the $\Delta(t_j)$ and $\delta(t_j)$. Now let

$$\begin{aligned}
\psi_i&=\eta(\lambda_2 r(t_i)+r''(t_i))-\Delta(t_i), \qquad i=0,1,\ldots,n\\
\chi_i&=\eta r'''(t_i)-\delta(t_i), \qquad i=0,1,\ldots,n.
\end{aligned} \tag{4.1}$$

Thus $(\boldsymbol{\psi},\boldsymbol{\chi})=(\psi_0,\psi_1,\ldots,\psi_n,\chi_0,\chi_1,\ldots,\chi_n)$ is $(2n+2)$-variate normal with mean zero and with the covariances

$$\begin{aligned}
\operatorname{Cov}(\psi_i,\psi_j)&=(\lambda_2 r(t_i)+r''(t_i))(\lambda_2 r(t_j)+r''(t_j))/\lambda_2\beta+C(t_i,t_j)\\
&=r(t_i-t_j)-r(t_i)\,r(t_j),\\
\operatorname{Cov}(\psi_i,\chi_j)&=(\lambda_2 r(t_i)+r''(t_i))\,r'''(t_j)/\lambda_2\beta+\left.\frac{\partial C(s,t)}{\partial t}\right|_{\substack{s=t_i\\ t=t_j}}=-r'(t_i-t_j),\\
\operatorname{Cov}(\chi_i,\chi_j)&=r'''(t_i)\,r'''(t_j)/\lambda_2\beta+c(t_i,t_j)\\
&=-r''(t_i-t_j)-r''(t_i)\,r''(t_j)/\lambda_2.
\end{aligned}$$

It should be observed that ψ_i and χ_i are independent and have the variances $1-r(t_i)^2$ and $\lambda_2-r''(t_i)^2/\lambda_2$ as before.

If we recall the proof of Theorem 3.1 and try to use ψ_i,χ_i in a limit theorem, we have to modify the procedure. Since $\xi_u'(t)$ is zero near one of the stationary points t_j only if $\eta_u r'''(t_j)-\delta(t_j)$ is negative, we have actually not normal but conditional normal r.v.

We devise the following method to pick up the right time and the right r.v. (ψ_j,χ_j). Cover the times $t_0,t_1,\ldots,t_n$ with disjoint ε-intervals

$$I_j^\varepsilon=[t_j-\varepsilon, t_j+\varepsilon], \qquad j=0,1,\ldots,n.$$

Usually ε is held fixed, and then we suppress it. Let the indicator variable κ^* be defined by

$$\kappa^*=\begin{cases} j & \text{if } \tau_u\in I_j, \quad (j=0,1,\ldots,n)\\ 0 & \text{if } \tau_u\notin\bigcup_0^n I_i. \end{cases}$$

A corresponding indicator variable κ for the contemplated limit distribution is defined by

$$\kappa=\begin{cases} j & \text{if } \chi_j<0,\ \chi_i\geqq 0 \quad \text{for } i=1,\ldots,j-1\\ 0 & \text{if } \chi_i\geqq 0 \quad \text{for } i=1,\ldots,n. \end{cases}$$

Then the r.v. κ has the distribution

$$P(\kappa=j)=p_j=P(\chi_j<0, \chi_i\geqq 0 \text{ for } i=1,\dots,j-1),$$
$$P(\kappa=0)=p_0=1-\sum_1^n p_j,$$

where p_1, p_2, and p_3 can be expressed in terms of elementary functions. In general, p_j is an integral of a j-variate normal density, see Gupta [4].

If we write

$$\varepsilon_j=\begin{matrix} 0 & \text{if } j=0 \\ 1 & \text{if } j=1,\dots,n, \end{matrix}$$

then $2k_{\kappa^*}+\varepsilon_{\kappa^*}$ and $2k_\kappa+\varepsilon_\kappa$ are the orders of the first non-vanishing derivatives of r at the randomly chosen times t_{κ^*} and t_κ respectively. It is then clear how to observe the r.v.

$$u(\tau_u-t_{\kappa^*})^{2k_{\kappa^*}+\varepsilon_{\kappa^*}-1} \quad \text{and} \quad \delta_u-u(1-r(t_{\kappa^*}))=u\,r(t_{\kappa^*})-\xi_u(\tau_u),$$

since once we have the value of τ_u we can pick up the t_{κ^*}-value and rise the difference $\tau_u-t_{\kappa^*}$ to the appropriate power. Similarly we can observe

$$\frac{(2k_\kappa+\varepsilon_\kappa-1)!}{r^{(2k_\kappa+\varepsilon_\kappa)}(t_\kappa)}\chi_\kappa \quad \text{and} \quad \psi_\kappa$$

by taking the first negative χ_κ in the sequence $\chi_1,\dots,\chi_n$, (or if they are all positive, taking χ_0) and the corresponding ψ_κ.

Theorem 4.1. *If condition* C2 *is fulfilled with* $t_0, t_1, \dots, t_n$ *and* $k_0, k_1, \dots, k_n$ *then, as* $u\to\infty$,

$$\{u(\tau_u-t_{\kappa^*})^{2k_{\kappa^*}+\varepsilon_{\kappa^*}-1}, \delta_u-u(1-r(t_{\kappa^*}))\} \xrightarrow{\mathscr{L}} \left\{\frac{(2k_\kappa+\varepsilon_\kappa-1)!}{r^{(2k_\kappa+\varepsilon_\kappa)}(t_\kappa)}\chi_\kappa, \psi_\kappa\right\}.$$

Remark. The theorem implies that $\tau_u \xrightarrow{\mathscr{L}} t_\kappa$, which gives the probabilities with which τ_u falls near the different t_j. The full theorem also says something about the distance between τ_u and the t_j it happens to be near.

Proof. From Lemma 2.2 it follows that, for every $\varepsilon>0$,

$$P\left(\tau_u\notin\bigcup_0^n I_i^\varepsilon\right)\to 0 \quad \text{as } u\to\infty.$$

To get a comprehensible and short notation write (c.f. (4.1))

$$\begin{aligned} \psi_i^u &= \eta_u(\lambda_2 r(t_i)+r''(t_i))-\Delta(t_i) \\ \chi_i^u &= \eta_u r'''(t_i)-\delta(t_i) \end{aligned} \qquad i=0,1,\dots,n$$

so that

$$(\psi^u, \chi^u)=(\psi_0^u,\dots,\psi_n^u,\chi_0^u,\dots,\chi_n^u)\xrightarrow{\mathscr{L}}(\psi,\chi). \tag{4.2}$$

Also, for $j=0,1,\ldots,n$, define the events

$$A_0: \quad \{\chi_i \geqq 0, i=1,\ldots,n\}$$
$$A_j: \quad \{\chi_j<0, \chi_i \geqq 0, i=1,\ldots,j-1\}$$
$$A_j(x,y): \quad \left\{\frac{(2k_j+\varepsilon_j-1)!}{r^{(2k_j+\varepsilon_j)}(t_j)}\chi_j<x, \psi_j<y\right\}$$
$$B_j: \quad \{\tau_u \in I_j\}$$
$$B_j(x,y): \quad \{u(\tau_u-t_j)^{2k_j+\varepsilon_j-1}<x, \delta_u-u(1-r(t_j))<y\}.$$

A little reflection then shows that the theorem is equivalent to

$$\lim_{u\to\infty} P(B_j \wedge B_j(x,y)) = P(A_j \wedge A_j(x,y)), \qquad j=0,1,\ldots,n. \tag{4.3}$$

The point in the proof is that we can express the conditions for the events B_j and $B_j(x,y)$ in terms of certain relations for the variables ψ_i^u, χ_i^u $(i=1,\ldots,j)$, which are very similar to the relations which define the events A_j and $A_j(x,y)$.

We concentrate upon the case $1\leqq j\leqq n$. The case $j=0$ is quite analogous, and the details are left out. To start with, we derive some bounds (4.7a) and (4.7b) for the random functions

$$\begin{aligned} &ur(t_j)-\xi_u(t_j+h)=ur(t_j)-ur(t_j+h)+\eta_u(\lambda_2 r(t_j+h)+r''(t_j+h))-\Delta(t_j+h),\\ &-\xi_u'(t_j+h)=-ur'(t_j+h)+\eta_u(\lambda_2 r'(t_j+h)+r'''(t_j+h))-\delta(t_j+h). \end{aligned} \tag{4.4}$$

Starting with the non-random terms, we notice that, for any $\theta>0$, there is an $\varepsilon>0$ such that, for $|h|\leqq\varepsilon$, $j=1,\ldots,n$,

$$\begin{aligned} &M_j^-(h)\leqq ur(t_j)-ur(t_j+h)\leqq M_j^+(h),\\ &m_j^-(h)\leqq -ur'(t_j+h)\leqq m_j^+(h), \end{aligned} \tag{4.5}$$

where

$$M_j^+(h)=-(1+\theta\cdot\operatorname{sign}(h))\frac{u\,h^{2k_j+1}}{(2k_j+1)!}r^{(2k_j+1)}(t_j)$$

$$M_j^-(h)=-(1-\theta\cdot\operatorname{sign}(h))\frac{u\,h^{2k_j+1}}{(2k_j+1)!}r^{(2k_j+1)}(t_j)$$

$$m_j^+(h)=-(1+\theta)\frac{u\,h^{2k_j}}{(2k_j)!}r^{(2k_j+1)}(t_j)$$

$$m_j^-(h)=-(1-\theta)\frac{u\,h^{2k_j}}{(2k_j)!}r^{(2k_j+1)}(t_j).$$

In order to obtain bounds for the random terms in (4.4), fix a $T\geqq t_0+\varepsilon$ and let $\theta'>0$ be arbitrary. Then there is an M such that the event

$$N: \quad \{|\eta_u|\leqq M, \sup_{0\leqq t\leqq T}|\delta(t)|\leqq M, \sup_{0\leqq t\leqq T}|\delta'(t)|\leqq M\}$$

has a probability $P(N) \geqq 1-\theta'$. Considering only outcomes in N, we get

$$\begin{aligned}\eta_u(\lambda_2 r(t_j+h)+r''(t_j+h))-\Delta(t_j+h)&=\psi_j^u+h\cdot G_j(h),\\ \eta_u(\lambda_2 r'(t_j+h)+r'''(t_j+h))-\delta(t_j+h)&=\chi_j^u+h\cdot H_j(h),\end{aligned} \tag{4.6}$$

where

$$\begin{aligned}|G_j(h)|&=|\eta_u(\lambda_2 r'(t_j+h')+r'''(t_j+h'))-\delta(t_j+h')|\\ &\leqq M(\sup|\lambda_2 r'|+\sup|r'''|+1)\leqq K,\\ |H_j(h)|&\leqq M(\sup|\lambda_2 r''|+\sup|r^{\mathrm{IV}}|+1)\leqq K,\end{aligned}$$

with some K depending on θ'.

Adding (4.5) and (4.6), we obtain the following estimates for (4.4), valid for all outcomes in N, $|h|\leqq\varepsilon$, $j=1,\ldots,n$:

$$\psi_j^u+M_j^-(h)-|h|K\leqq ur(t_j)-\xi_u(t_j+h)\leqq\psi_j^u+M_j^+(h)+|h|K, \tag{4.7a}$$

$$\chi_j^u+m_j^-(h)-|h|K\leqq-\xi_u'(t_j+h)\leqq\chi_j^u+m_j^+(h)+|h|K. \tag{4.7b}$$

As is easily shown by differentiation, the lower bound functions in (4.7) are uniformly bounded from below (remember that $r^{(2k_j+1)}(t_j)<0$), in the sense that there is a constant $K'>0$ such that, for all $h, j=1,\ldots,n$

$$\begin{aligned}M_j^-(h)-|h|K&\geqq-K'/u,\\ m_j^-(h)-|h|K&\geqq-K'/u.\end{aligned} \tag{4.8}$$

Now we can proceed to the announced equivalences. First suppose that the event $B_j\wedge B_j(x,y)$ occurs. If $j>0$, the only interesting case is $x\geqq0$. For all outcomes in N, the event B_j implies that $\xi_u'(t)<0$ for all $t\in\bigcup_{i=1}^{j-1}I_i$, and that $\xi_u'(t)>0$ for some $t\in I_j$, which in turn, together with (4.7b) and (4.8) gives $\chi_i^u\geqq-\xi_u'(t_i)>0$, $i=1,\ldots,j-1$ and $\chi_j^u-K'/u\leqq-\xi_u'(t_j+h)<0$, i.e.

$$\chi_i^u>0 \quad \text{for } i=1,\ldots,j-1 \quad \text{and} \quad \chi_j^u\leqq K'/u. \tag{4.9}$$

If $x>0$ and the event $B_j(x,y)$ also occurs, and especially $u(\tau_u-t_j)^{2k_j}<x$, then $\xi_u'(t_j+h)$ is zero for some $|h|<h_x=(x/u)^{1/2k_j}$. But since $m_j^+(h)+|h|K$ decreases as h tends to zero, we have that $\chi_j^u+m_j^+(h_x)+h_xK<0$ implies that $\chi_j^u+m_j^+(h)+|h|K<0$ for all $|h|<h_x$. The upper bound in (4.7b) then gives that $\xi_u'(t_j+h)>0$ for such h-values. Thus the event $B_j(x,y)$ implies that $\chi_j^u+m_j^+(h_x)+h_xK\geqq0$ or equivalently

$$\frac{(2k_j)!}{r^{(2k_j+1)}(t_j)}\chi_j^u<(1+\theta)x-\left(\frac{x}{u}\right)^{1/2k_j}K''. \tag{4.10}$$

But the event $B_j\wedge B_j(x,y)$ also implies that $ur(t_j)-\xi_u(\tau_u)<y$ and this, together with (4.8) and the lower bound in (4.7a), gives

$$\psi_j^u\leqq y+K'/u. \tag{4.11}$$

We sum up the inequalities (4.9)–(4.11) and obtain

$$P(B_j \wedge B_j(x,y)) \leqq P\left(\chi_i^u > 0, i=1,\dots,j-1 \wedge \chi_j^u \leqq K'/u \wedge \frac{(2k_j)!}{r^{(2k_j+1)}(t_j)} \chi_j^u < (1+\theta)x - \left(\frac{x}{u}\right)^{1/2k_j} K'' \wedge \psi_j^u \leqq y + K'/u\right) + P(N^*).$$

Letting $u \to \infty$ we get from (4.2)

$$\limsup_{u\to\infty} P(B_j \wedge B_j(x,y)) \leqq P(A_j \wedge A_j((1+\theta)x, y)) + \theta'. \tag{4.12}$$

A reverse inequality can be derived in a similar way, again considering only outcomes in N. The relations (4.7b) and (4.8) give that if $\chi_i^u > K'/u$ for $i=1,\dots,j-1$ and $\chi_j^u < 0$, then $\xi_u'(t_i+h) < 0$, $|h| \leqq \varepsilon$, $i=1,\dots,j-1$, and $\xi_u'(t_j) > 0$, so that

$$\tau_u \in I_j \quad \text{or} \quad \tau_u \notin \bigcup_0^n I_i. \tag{4.13}$$

If, furthermore, the lower bound in (4.7b) is positive for $h = -h_x = -(x/u)^{1/2k_j}$, i.e. if

$$\frac{(2k_j)!}{r^{(2k_j+1)}(t_j)} \chi_j^u < (1-\theta)x - \left(\frac{x}{u}\right)^{1/2k_j} K''',$$

then the derivative $\xi_u'(t_j+h)$ is negative at $h \leqq -h_x$ and positive at $h=0$, so its first zero must fall in the interval $(-h_x, 0)$ i.e.

$$u(\tau_u - t_j)^{2k_j} < x. \tag{4.14}$$

A bound similar to (4.8) can be obtained for M_j^+ to the effect that $M_j^+(h) + |h|K \leqq h_x K^4$ if $|h| \leqq h_x$. Therefore the upper bound in (4.7a) gives that, if

$$\psi_j^u < y - \left(\frac{x}{u}\right)^{1/2k_j} K^4 = y - h_x K^4,$$

then

$$\delta_u - u(1-r(t_j)) = u r(t_j) - \xi_u(\tau_u) < y. \tag{4.15}$$

Summing the implications leading to (4.13)–(4.15), we obtain

$$P(B_j \wedge B_j(x,y)) \geqq P\left(\chi_i^u > K'/u, i=1,\dots,j-1 \wedge \chi_j^u < 0 \wedge \frac{(2k_j)!}{r^{(2k_j+1)}(t_j)} \chi_j^u < (1-\theta)x - \left(\frac{x}{u}\right)^{1/2k_j} K''' \wedge \psi_j^u < y - \left(\frac{x}{u}\right)^{1/2k_j} K^4\right) - P(N^*) - P\left(\tau_u \notin \bigcup_0^n I_i\right),$$

and, if $u \to \infty$

$$\liminf_{u\to\infty} P(B_j \wedge B_j(x,y)) \geqq P(A_j \wedge A_j((1-\theta)x, y)) - \theta'. \tag{4.16}$$

Now a little reflection shows that the left hand limits in (4.12) and (4.16) do not depend on ε. The right hand bounds can be made arbitrarily close to each other by first taking small θ and θ', then a sufficiently large M and a small ε for the arguments to go through. Thus (4.3) follows for $j=1, \ldots, n$, and by similar arguments it can be shown for $j=0$.

5. Decreasing Covariance Function

$$-r''(t)/r'(t) \to 0 \quad \text{as } t \to \infty$$

We now turn to the case when $r'(t)$ is strictly negative for $t>0$, and $r(t)$ and its first four derivatives tend to zero as $t \to \infty$. Then, for every fixed but large t, the dominant term in $\xi_u'(t)$ (as defined by (2.5)) is $u r'(t)$, while $\eta_u(\lambda_2 r'(t)+r'''(t))$ is negligible. For large s and t the covariance function $c(s,t)$ of the non-stationary process δ is approximately $-r''(s-t)$, which we recognize as the covariance function of the original process derivative ξ'. Therefore we might expect $\delta(t)$ to behave almost like $\xi'(t)$ for large t-values, and it seems plausible that the distribution of τ_u (=the time for the first upcrossing zero of $\xi_u'(t)$) can be expressed in terms of the times for the upcrossing zeros of $u r'(t)+\xi'(t)$.

Define

$$\begin{aligned} \nu_t &= \text{the number of upcrossing zeros of } \xi_u'(s) \text{ in } (0,t], \\ \mu_t &= \text{the number of upcrossing zeros of } u r'(s)+\xi'(s) \text{ in } (0,t], \end{aligned} \tag{5.1}$$

and let $T(=T(u) \to \infty$ as $u \to \infty)$ be any function of u such that $E(\nu_T)$ and $E(\mu_T)$ has a finite limit, θ, say. Then it can be worth while to examine $P(\tau_u - T > x) = P(\nu_{T+x}=0)$ as $u \to \infty$.

Doing so we must impose new conditions upon r and specify the asymptotic behaviour of $-r''(t)/r'(t)$ for large t.

C3. a) $r'(t)<0$ for $t>0$,

b) $r'(t)=O(t^{-\gamma})$ as $t \to \infty$ for some $\gamma>1$,

c) there is a constant $C \geqq 0, \leqq \infty$ such that, for $k=1,2,3,4$, the function $(-1)^k r^{(k)}(t)$ is convex, positive and decreasing for large t, $t^k r^{(k)}(t) \to 0$ as $t \to \infty$, $-r^{(k)}(t)/r^{(k-1)}(t)$ increases (decreases) to C as $t \to \infty$;
if $C=0$ then $-r^{(k)}(t)/r^{(k-1)}(t)=O(t^{-1})$ as $t \to \infty$.

It should be noted that condition C3.c is fulfilled for $k \leqq 4$ if it is fulfilled for $k=4$.

According to the value of C we can speak about the algebraic case, ($C=0$, e.g. $r(t)=1/(1+t^2)$), the exponential case ($0<C<\infty$, e.g. $r(t)=\exp(-|t|)\cdot$ polynomial), and the over-exponential case ($C=\infty$, e.g. $r(t)=\exp(-t^2/2)$). This section deals with the algebraic case.

Theorem 5.1. *If r fulfills condition* C3, $-r''(t)/r'(t) \to 0$ *as* $t \to \infty$, *and* $T_\theta^0 = T_\theta^0(u) \to \infty$ *as* $u \to \infty$ *so that* $E(\mu_{T_\theta 0}) \to \theta$, *then*

$$E(\nu_{T_\theta 0+x}) \to \theta \quad \textit{for all } x \quad \textit{as} \quad u \to \infty.$$

Proof. We start with some definitions. Let

$$m_{uy}(t) = u r'(t) - y(\lambda_2 r'(t) + r'''(t)), \tag{5.2}$$

so that $m_{uy}(t)+\delta(t)$ is $\xi'_u(t)$ when η_u happens to take the value y. (Remember, η_u is random and has density q_u^* given by (2.2).) Also let

$$\nu_t(y)=\text{the number of upcrossing zeros of } m_{uy}(s)+\delta(s) \text{ in } (0,t]. \tag{5.3}$$

For the expected values of ν_t, $\nu_t(y)$, and μ_t we use the well-known formulas for crossings by non-stationary processes (see [3, Ch. 13]). Recall the definition (2.3): $\Psi(x)=\phi(x)+x\,\Phi(x)$, ϕ and Φ being the standard normal density and distributions functions. Then

$$\begin{aligned} E(\nu_t) &= \int\limits_{y=-u/\beta}^{\infty} E(\nu_t(y))\, q_u^*(y)\,dy \\ &= \int\limits_{y=-u/\beta}^{\infty} \int\limits_{s=0}^{t} \omega(s)\,\phi\big(m_{uy}(s)/\sigma(s)\big)\,\Psi\big(\eta_{uy}(s)\big)\,q_u^*(y)\,ds\,dy, \qquad (5.4) \\ E(\mu_t) &= \int\limits_{s=0}^{t} (\lambda_4/\lambda_2)^{1/2}\,\phi\big(ur'(s)/\sqrt{\lambda_2}\big)\,\Psi\big(ur''(s)/\sqrt{\lambda_4}\big)\,ds, \end{aligned}$$

where

$$\begin{aligned} &\sigma^2=\sigma^2(s)=V(\delta(s))=c(s,s), \\ &\gamma^2=\gamma^2(s)=V(\delta'(s))=\left.\frac{\partial^2 c(s,t)}{\partial s\,\partial t}\right|_{s=t}, \\ &\mu=\mu(s)=\mathrm{Cov}(\delta(s),\delta'(s))/\sigma(s)\,\gamma(s)=\left.\frac{\partial c(s,t)}{\partial s}\right|_{s=t}\Big/\sigma(s)\,\gamma(s), \\ &\omega=\omega(s)=\gamma(s)\sqrt{1-\mu^2(s)}/\sigma(s), \\ &\eta_{uy}=\eta_{uy}(s)=\{m'_{uy}(s)-\gamma(s)\,\mu(s)\,m_{uy}(s)/\sigma(s)\}/\gamma(s)\sqrt{1-\mu^2(s)} \\ &\quad=\{ur''-y(\lambda_2 r''+r^{\mathrm{IV}})\}/\gamma\sqrt{1-\mu^2}-\mu\{ur'-y(\lambda_2 r'+r''')\}/\sigma\sqrt{1-\mu^2}. \end{aligned}$$

By assumption C3 we immediately have that, as $s\to\infty$,

$$\begin{aligned} &\sigma^2(s)\to\lambda_2, \quad \gamma^2(s)\to\lambda_4, \quad \mu(s)\to 0, \\ &\omega(s)\to(\lambda_4/\lambda_2)^{1/2}. \end{aligned} \tag{5.5}$$

Also, $m_{uy}(s)/\sigma(s)\simeq ur'(s)/\sqrt{\lambda_2}$ and $\eta_{uy}(s)\simeq ur''(s)/\sqrt{\lambda_4}$, still for large s, at least when y is bounded, so that the two integrands in (5.4) can be expected to be similar for such s- and y-values. This suggests the following procedure for the proof of Theorem 5.1: first, find $T_-, y_-\to\infty$ as $u\to\infty$ such that the integrands in (5.4) are similar for $s\geqq T_-$, $|y|\leqq y_-$; then show that the integrals over $s<T_-$ and $|y|>y_-$ are negligible.

We build the proof from several lemmas.

Lemma 5.1 *For any* $t_0>0$ *as* $u\to\infty$

a) $E(\mu_{t_0})\to 0$,

b) $E(\nu_{t_0})\to 0$.

We already know, from Lemma 2.2, that $P(\nu_{t_0}=0)\to 1$, which, of course, does not imply that $E(\nu_{t_0})\to 0$.

Proof of Lemma 5.1. We estimate the integrals in (5.4) at first for one particular t_0^*, then for general t_0.

a) Take t_0^* so that $r''(t) \leqq 0$ for $0 \leqq t \leqq t_0^*$. Then, as $u \to \infty$,

$$E(\mu_{t_0^*}) = \int_0^{t_0^*} (\lambda_4/\lambda_2)^{\frac{1}{2}} \phi\left(\frac{u\,r'(t)}{\sqrt{\lambda_2}}\right) \Psi\left(\frac{u\,r''(t)}{\sqrt{\lambda_4}}\right) dt \leqq K \int_0^{t_0^*} \phi\left(\frac{u\,r'(t)}{\sqrt{\lambda_2}}\right) dt \to 0.$$

b) Let

$$a(t) = (\lambda_4 r'(t) + \lambda_2 r'''(t))/\lambda_2 \beta,$$
$$b(t) = (\lambda_2 r'(t) + r'''(t))/\lambda_2 \beta,$$

so that $m_{uy}(t) = u\,a(t) - \lambda_2 \beta (y + u/\beta)\, b(t)$, (remember $\beta = (\lambda_4 - \lambda_2^2)/\lambda_2$). Then

$$-a'(t) = \beta^{-1}(\lambda_4 - r^{IV}(t)) \left\{1 - \frac{\lambda_4}{\lambda_2} \cdot \frac{\lambda_2 + r''(t)}{\lambda_4 - r^{IV}(t)}\right\}.$$

Now, for any covariance function q, the limit $\lim_{t\to 0} t^{-2}(q(0) - q(t))$ exists $\leqq \infty$, see [3, § 9.3]. Thus, if we for short write

$$R(t) = \lambda_4 - r^{IV}(t),$$

we can conclude that $-a'(t) \sim K R(t) > 0$ as $t \to 0$, whether $\lim t^{-2} R(t)$ is finite or not. Since $b'(t) \sim 1$, this implies that $-m'_{uy}(t)$ is non-negative in an interval $(0, t_0^*)$ for any $y > -u/\beta$. From $\Psi(x) \leqq 1 + |x|$ and Ψ increasing, it follows that

$$\Psi(\eta_{uy}(t)) \leqq \Psi\left\{-\frac{\mu}{\sqrt{1-\mu^2}} \cdot \frac{m_{uy}(t)}{\sigma(t)}\right\} \leqq 1 + \frac{|\mu|}{\sqrt{1-\mu^2}} \cdot \frac{|m_{uy}(t)|}{\sigma(t)},$$

and

$$\begin{aligned} E(\nu_{t_0^*}) &= \int_{y=-u/\beta}^{\infty} \int_{t=0}^{t_0^*} \omega(t)\, \phi\left(\frac{m_{uy}(t)}{\sigma(t)}\right) \Psi(\eta_{uy}(t))\, q_u^*(y)\, dt\, dy \\ &\leqq \int_t \int_y \omega\, \phi(m_{uy}/\sigma)\, q_u^*(y)\, dy\, dt \\ &\quad + \int_t \int_y \frac{\gamma |\mu|}{\sigma} \cdot \frac{|m_{uy}|}{\sigma}\, \phi(m_{uy}/\sigma)\, q_u^*(y)\, dy\, dt. \end{aligned} \tag{5.6}$$

We will now show that the second integral in (5.6) tends to zero as u tends to infinity. The first integral is treated similarly.

Since $|\mu| \leqq 1$, the main problem is to estimate γ/σ and $|m_{uy}|/\sigma$. We start by showing that $\sigma^2(t)$ and $\gamma^2(t)$ are of the order $\int_0^t s R(s)\, ds$ and $R(t)$ respectively:

$$\sigma^2(t) \sim K_\sigma \int_{s=0}^{t} s R(s)\, ds, \quad \gamma^2(t) \sim K_\gamma R(t), \quad \text{as} \quad t \to 0. \tag{5.7}$$

Simple calculations show that $d\sigma^2(t)/dt = -2\beta\, \alpha'(t)\, b(t) \sim K_\sigma t R(t)$, from which (5.7) follows as far as $\sigma^2(t)$ is concerned. Differentiation also shows the relation for $\gamma^2(t)$ if this function happens to be differentiable. Otherwise, introduce the function V,

$$V(t)=-r''(t)-\lambda_2+\lambda_4 t^2/2,$$

and insert V, V', and V'' into the differential expression for $\gamma^2(t)$. Then the leading terms in $\gamma^2(t)$ will be $2V''(t)+a$ term of order t^2. Since $V''(t)=R(t)$ this shows the rest of (5.7).

Now, $a(t)\leqq 0$ and $b(t)\sim t$ implies that $|m_{uy}(t)|=-u\,a(t)+\lambda_2\,\beta(y+u/\beta)\,b(t)\geqq K(y+u/\beta)\,t$, and finally, with some specific constant K_0,

$$\begin{aligned}\frac{|m_{uy}(t)|}{\sigma(t)}&\geqq K_0(y+u/\beta)\,t\left\{\int_0^t s\,R(s)\,ds\right\}^{-\frac{1}{2}}\\ \frac{\gamma(t)}{\sigma(t)}&\leqq K\left\{R(t)\Big/\int_0^t s\,R(s)\,ds\right\}^{\frac{1}{2}}.\end{aligned}\tag{5.8}$$

For the density $q_u^*(y)$ we use the simple estimate

$$q_u^*(y)\leqq K\,\Psi(u\sqrt{\lambda_2/\beta})^{-1}(y+u/\beta).\tag{5.9}$$

We are now ready to estimate the second integral in (5.6). Change variable in the inner integral; let K_0 be the constant in (5.8) and put

$$z=K_0(y+u/\beta)\,t\left\{\int_0^t s\,R(s)\,ds\right\}^{-\frac{1}{2}},$$

and use that $z\,\phi(z)$ is bounded for $z\geqq 0$ and decreasing for $z\geqq 1$. Then (5.8) and (5.9) imply that

$$\begin{aligned}&\int_{t=0}^{t_0^*}\int_{y\geqq -u/\beta}\frac{\gamma|\mu|}{\sigma}\cdot\frac{|m_{uy}|}{\sigma}\,\phi(m_{uy}/\sigma)\,q_u^*(y)\,dy\,dt\\ &\leqq K\,\Psi(u\sqrt{\lambda_2/\beta})^{-1}\int_{t=0}^{t_0^*}\left\{\frac{R(t)}{\int_0^t s\,R(s)\,ds}\right\}^{\frac{1}{2}}\cdot\frac{\int_0^t s\,R(s)\,ds}{t^2}\\ &\quad\cdot\left\{\int_{z=0}^{1} z\,dz+\int_{z=1}^{\infty} z^2\,\phi(z)\,dz\right\}dt\\ &\leqq K'\,\Psi(u\sqrt{\lambda_2/\beta})^{-1}\int_{t=0}^{t_0^*}\left\{\frac{R(t)}{t}\right\}^{\frac{1}{2}}\cdot\left\{\frac{\int_0^t s\,R(s)\,ds}{t^3}\right\}^{\frac{1}{2}}dt.\end{aligned}$$

By the Schwartz's inequality

$$\begin{aligned}&\left\{\int_{t=0}^{t_0^*}\left\{\frac{R(t)}{t}\right\}^{\frac{1}{2}}\cdot\left\{t^{-3}\int_{s=0}^{t} s\,R(s)\,ds\right\}^{\frac{1}{2}}dt\right\}^2\\ &\leqq\int_{t=0}^{t_0^*}\frac{R(t)}{t}\,dt\cdot\int_{t=0}^{t_0^*} t^{-3}\int_{s=0}^{t} s\,R(s)\,ds\,dt\\ &\leqq\left\{\int_{t=0}^{t_0^*}\frac{R(t)}{t}\,dt\right\}^2,\end{aligned}$$

which gives the upper bound

$$K' \Psi\big(u\sqrt{\lambda_2/\beta}\big)^{-1} \int_{t=0}^{t_0^*} \frac{R(t)}{t}\, dt.$$

By the assumption (2.1) about $r^{\mathrm{IV}}(t)$, the integral $\int_0^{t_0^*} t^{-1} R(t)\, dt$ converges, and it follows that the bound tends to zero as u tends to infinity.

We now extend the results to general t_0.

a) Since $\inf_{(t_0^*, t_0)} |r'(t)|>0$, $\sup_{(t_0^*, t_0)} |r''(t)|<\infty$, we get

$$\int_{t_0^*}^{t_0} \phi\big(u\, r'(t)/\sqrt{\lambda_2}\big)\, \Psi\big(u\, r''(t)/\sqrt{\lambda_4}\big)\, dt \leqq \int_{t_0^*}^{t_0} \phi(K_1 u)(1+K_2 u)\, dt \to 0.$$

b) Taking inf and sup over $t \geqq t_0^*$ we have that $\sigma_- = \inf \sigma(t)>0$, $\sigma_+ = \sup \sigma(t)<\infty$, $\gamma_- = \inf \gamma(t)>0$, $\gamma_+ = \sup \gamma(t)<\infty$, $\varepsilon = \sup |\mu(t)|\,(1-\mu^2(t))^{-\frac{1}{2}}<\infty$. Since $|m_{uy}'(t)| \leqq K_1' u + K_2' |y|$ and $|m_{uy}(t)| \leqq K_1'' u + K_2'' |y|$ for all u and y, this implies that $\Psi(\eta_{uy}(t)) \leqq 1+|\eta_{uy}(t)| \leqq K_1 u + K_2 |y|$.

If we further notice that $\sup_{(t_0^*, t_0)} |\lambda_2 r'(t)+r'''(t)| / \inf_{(t_0^*, t_0)} |r'(t)|$ is finite, we get $|m_{uy}(t)| \geqq u\,|r'(t)| - |y| \cdot |\lambda_2 r'(t)+r'''(t)| \geqq K_3' u - K_4' |y| \geqq K_3 u$ for all y with $|y| \leqq K_4 u$, some $K_4>0$. Separating $|y| \geqq K_4 u$ and $\leqq K_4 u$, we then get

$$\begin{aligned}
&\int_{y=-u/\beta}^{\infty} \int_{t=t_0^*}^{t_0} \omega\, \phi(m_{uy}/\sigma)\, \Psi(\eta_{uy})\, q_u^*(y)\, dt\, dy \\
&\quad \leqq K_5 \int_{|y| \leqq K_4 u} \int_{t=t_0^*}^{t_0} \phi(K_6 u)(u+|y|)\, q_u^*(y)\, dt\, dy \\
&\quad\quad + K_7 \int_{|y| \geqq K_4 u} \int_{t=t_0^*}^{t_0} (u+|y|)\, q_u^*(y)\, dt\, dy \to 0 \quad \text{as} \quad u \to \infty,
\end{aligned}$$

and the lemma is proved.

Of course the conclusion of the lemma is true even if the region $(0, t_0)$ is permitted to increase slowly with u. Therefore we can base the rest of the proof of Theorem 5.1 on a comparison between $m_{uy}(t)/\sigma(t)$ and $\eta_{uy}(t)$ on one hand and $u\, r'(t)/\sqrt{\lambda_2}$ and $u\, r''(t)/\sqrt{\lambda_4}$ on the other for large t.

Lemma 5.2. *If $T = T_\theta^0(u)$ is as in Theorem 5.1, $y_- = T$, then*

$$\frac{m_{uy}(t)}{\sigma(t)} = \frac{u\, r'(t)}{\sqrt{\lambda_2}}\,(1+o(1))$$

$$\eta_{uy}(t) = \frac{u\, r''(t)}{\sqrt{\lambda_4}}\,(1+o(1))$$

where $o(1) \to 0$ uniformly in $|y| \leqq y_-$, as $u \to \infty$, $t \to \infty$.

Proof.

$$\begin{aligned}
m_{uy}(t) &= u\, r'(t)\, \{1 - R_{uy}(t)\}, \\
\eta_{uy}(t) &= u\, r''(t)\, \lambda_4^{-\frac{1}{2}}\, \{1 - S_{uy}(t)\},
\end{aligned} \tag{5.10}$$

where the residuals

$$R_{uy}(t)=\frac{y}{u}\,(\lambda_2+r'''(t)/r'(t)),$$

$$S_{uy}(t)=1-\frac{\sqrt{\lambda_4}}{\gamma\sqrt{1-\mu^2}}\left\{1-\frac{y}{u}\left(\lambda_2+\frac{r^{\mathrm{IV}}(t)}{r''(t)}\right)\right.$$
$$\left.-\frac{\mu\gamma}{\sigma}\left[\frac{r'(t)}{r''(t)}-\frac{y}{u}\left(\lambda_2\frac{r'(t)}{r''(t)}+\frac{r'''(t)}{r''(t)}\right)\right]\right\}$$

tend to zero uniformly in $|y|\leqq y_-$. Since $\mu(t)=O(r'(t))$ this assertion is proved if we can show that

a) $r'(t)^2/r''(t), r'(t)\,r'''(t)/r''(t)\to 0$ as $t\to\infty$; this is a consequence of the assumed convexity of $(-1)^k r^{(k)}(t)$.

b) $y_-/u\to 0$ as $u\to\infty$; $r'(t)=O(t^{-\gamma}), \gamma>1$ by condition C3, implies $-u r'(T)\leqq M u T^{-\gamma}$, and a little reflexion will show that $-u r'(T)\to\infty$, so that

$$y_-=T\leqq M' u^{1/\gamma} \tag{5.11}$$

and $y_-/u\to 0$. Since $\sigma^2(t)\to\lambda_2$, the lemma is proved.

The following lemma is important for the estimation of (5.4).

Lemma 5.3. *There exists a constant K, independent of u and y, such that, for all intervals I sufficiently remote from the origin*

$$\int_I \phi(m_{uy}(t)/\sigma(t))\,\Psi(\eta_{uy}(t))\,dt\leqq K(|I|+1),$$

where $|I|$ is the length of the interval.

Proof. Since $m'_{uy}(t)=u\,r''(t)\left\{1-\frac{y}{u}\,(\lambda_2+r^{\mathrm{IV}}(t)/r''(t))\right\}$ and $r^{\mathrm{IV}}(t)/r''(t)$ is monotonous for large t, $m'_{uy}(t)$ can change sign at most once in I. Let I_+ and I_- denote those parts of I in which $m'_{uy}\geqq 0$ and <0 respectively. With the same notation σ_+,σ_- etc. as in the proof of Lemma 5.1, we get

$$\begin{aligned}\int_{I_-}&\leqq\int_{I_-}\phi(m_{uy}/\sigma_+)\,\Psi(\varepsilon\,|m_{uy}|/\sigma)\,dt\\&\leqq\int_{I_-}\phi(m_{uy}/\sigma_+)\{1+\varepsilon\,|m_{uy}|/\sigma_-\}\,dt\leqq K_-|I_-|,\end{aligned}$$

since $\phi(x)$ and $x\,\phi(x)$ are bounded.

$$\begin{aligned}\int_{I_+}&\leqq\int_{I_+}\phi(m_{uy}/\sigma_+)\,\Psi\big(m'_{uy}/\gamma_-\sqrt{1-\varepsilon^2}+\varepsilon\,|m_{uy}|/\sigma_-\big)\,dt\\&\leqq\int_{I_+}\phi(m_{uy}/\sigma_+)\{m'_{uy}/\gamma_-\sqrt{1-\varepsilon^2}+1+\varepsilon\,|m_{uy}|/\sigma_-\}\,dt\\&\leqq K_-|I_+|+K\int\phi(x/\sigma_+)\,dx\leqq K_+(|I_+|+1).\end{aligned}$$

Since K_- and K_+ are independent of u, y, we are through.

We are now ready to choose the splitting point T_-: Let $\delta>0$ be a small number, its precise value being undetermined for a while, and take T_- so that

$$-u\,r'(T_-)=\sqrt{(2+\delta)\,\lambda_2\log T_-}\,. \tag{5.12}$$

(Such a T_- exists, since $-r'(t)$ decreases for large t.)

Lemma 5.4. *If* T_- *fulfills* (5.12) *then, as* $u\to\infty$

a) $E(\mu_{T_-})\to 0$,

b) $E(\nu_{T_-})\to 0$.

Proof. By Lemma 5.1 we need only to examine the integrals (5.4) over (t_0, T_-).

a)

$$\begin{aligned}
&\int_{t_0}^{T_-}\phi\big(u\,r'(t)/\sqrt{\lambda_2}\big)\,\Psi\big(u\,r''(t)/\sqrt{\lambda_4}\big)\,dt\\
&\quad\leqq\int_{t_0}^{T_-}\phi\big(u\,r'(t)/\sqrt{\lambda_2}\big)\{1+u\,r''(t)/\sqrt{\lambda_4}\}\,dt\\
&\quad\leqq T_-\,\phi\big(u\,r'(T_-)/\sqrt{\lambda_2}\big)+\lambda_4^{-\frac{1}{2}}\int_{u r'(t_0)}^{u r'(T_-)}\phi\big(x/\sqrt{\lambda_2}\big)\,dx\\
&\quad\leqq\frac{T_-}{\sqrt{2\pi}}\exp\big(-(2+\delta)\,\lambda_2\log T_-/2\lambda_2\big)+\sqrt{\lambda_2/\lambda_4}\,\Phi\big(u\,r'(T_-)/\sqrt{\lambda_2}\big)\to 0
\end{aligned}$$

by the definition (5.12) of T_-.

b) Separate $|y|\leqq y_-$, $>y_-$ $(=T)$. For $|y|\leqq y_-$ and $\varepsilon>0$ arbitrary, Lemma 5.2 says that $|m_{uy}(t)|/\sigma(t)\geqq(1-\varepsilon)\,|u\,r'(t)|/\sqrt{\lambda_2}$, $|\eta_{uy}(t)|\leqq(1+\varepsilon)\,u\,r''(t)/\sqrt{\lambda_4}$ if u and t are large. For $|y|>y_-$ we use Lemma 5.3. In total

$$\begin{aligned}
&\int_{y=-u/\beta}^{\infty}\int_{t=t_0}^{T_-}\phi\big(m_{uy}(t)/\sigma(t)\big)\,\Psi\big(\eta_{uy}(t)\big)\,q_u^*(y)\,dt\,dy\\
&\quad\leqq\int_{|y|\leqq y_-}\int_{t=t_0}^{T_-}\phi\left(\frac{(1-\varepsilon)\,u\,r'(t)}{\sqrt{\lambda_2}}\right)\Psi\left(\frac{(1+\varepsilon)\,u\,r''(t)}{\sqrt{\lambda_4}}\right)q_u^*(y)\,dt\,dy\\
&\qquad+\int_{|y|>y_-}K\,T_-\,q_u^*(y)\,dy.
\end{aligned}$$

The first integral can be estimated as in the proof of part a), if we take ε so small that $(1-\varepsilon)(1+\frac{1}{2}\delta)>1$. For the second integral, we use that

$$T_-\int_{|y|>y_-}q_u^*(y)\,dy\sim 2\,T_-\big(1-\Phi(y_-\sqrt{\lambda_2}\,\beta)\big)\sim K\,T_-\,\phi\big(T\sqrt{\lambda_2}\,\beta\big)/T\to 0\qquad\text{as } u\to\infty.$$

By Lemma 5.4 the only relevant region is $T_-\leqq t\leqq T+x$, and there we can sharpen Lemma 5.2, which will finally give us what we need to prove the theorem.

Lemma 5.5. *If* $T=T_\theta^0(u)$, $y_-=T$, *and* T_- *fulfills* (5.12) *then*

$$\begin{gathered}
\phi\big(m_{uy}(t)/\sigma(t)\big)/\phi\big(u\,r'(t)/\sqrt{\lambda_2}\big)\to 1\\
\Psi\big(\eta_{uy}(t)\big)\sim\Psi\big(u\,r''(t)/\sqrt{\lambda_4}\big)\to\Psi(0)=1/\sqrt{2\pi}
\end{gathered}$$

uniformly in $T_-\leqq t\leqq T$, $|y|\leqq y_-$, *as* $u\to\infty$.

Proof. ϕ) We must show that $m_{uy}^2(t)/\sigma^2(t) - u^2 r'(t)^2/\lambda_2 \to 0$ uniformly. The difference is

$$\sigma^{-2}(t)\{u^2 r'(t)^2(\lambda_2 - \sigma^2(t))/\lambda_2 + y^2(\lambda_2 r'(t) + r'''(t))^2 - 2uy r'(t)(\lambda_2 r'(t) + r'''(t))\}$$
$$= A + B - C, \quad \text{say}.$$

Since $\lambda_2 - \sigma^2(t) = O(r'(t)^2)$, $r'(t) = O(t^{-\gamma})$, $r'''(t) = O(r'(t))$, and, by (5.11), $T_- \leqq y_- = T \leqq M' u^{1/\gamma}$, we have

$$A \leqq K_1 u^2 r'(T_-)^4 \leqq K_2 T_-^{-2} \log T_- \to 0,$$
$$B \leqq K_3 y_-^2 r'(T_-)^2 \leqq K_4 (y_-/u)^2 \log T_- \leqq K_5 u^{-2(1-1/\gamma)} \log u \to 0,$$
$$|C| \leqq K_6 u y_- r'(T_-)^2 \leqq K_7 (y_-/u) \log T_- \to 0,$$

which shows the assertion.

$$\Psi)\ 0 \leqq u r''(t) \leqq u r''(T_-) = -u r'(T_-) \cdot r''(T_-)/(-r'(T_-)) = O(T_-^{-1} \sqrt{\log T_-}) \to 0$$

by condition C3.c and (5.12). Lemma 5.2 then shows that $\eta_{uy}(t) \to 0$ uniformly, so that $\Psi(\eta_{uy}(t)) \to \Psi(0)$ uniformly.

We now have everything that we need to show Theorem 5.1: By Lemma 5.4 it is enough to show that, (if $T = T_\theta^0(u)$),

$$\lim_{u\to\infty} \int_{y=-u/\beta}^{\infty} \left\{ \int_{t=T_-}^{T+x} \omega\, \phi(m_{uy}/\sigma)\, \Psi(\eta_{uy})\, dt \right\} q_u^*(y)\, dy = \lim_{u\to\infty} I = \theta$$

where

$$I = \int_{t=T_-}^{T} \sqrt{\frac{\lambda_4}{\lambda_2}}\, \phi(u r'(t)/\sqrt{\lambda_2})\, \Psi(u r''(t)/\sqrt{\lambda_4})\, dt.$$

Separate the double integral into three parts:

$$I_1 = \int_{|y| \leqq y_-} \int_{T_-}^{T}, \quad I_2 = \int_{|y| \leqq y_-} \int_{T}^{T+x}, \quad I_3 = \int_{|y| > y_-} \int_{T_-}^{T+x}.$$

By Lemma 5.5 and (5.5) every factor in I_1 is asymptotically equal to the corresponding factor in I, thus $I_1 \to I$. In I_2 the inner integral is over an interval of fixed length, in which the integrand is bounded by $K_1 \phi(K_2 u r'(t))\, q_u^*(y)$. Since $-u r'(t) \geqq -u r'(T+x) \sim -u r'(T) \to \infty$, the bound is $q_u^*(y) \cdot o(1)$ uniformly in $(T, T+x)$ so that $I_2 \to 0$. Finally, by Lemma 5.3, I_3 is bounded by

$$KT \int_{|y| > T} q_u^*(y)\, dy \sim 2KT(1 - \Phi(T\sqrt{\lambda_2}\beta)) \sim K' T \phi(T\sqrt{\lambda_2}\beta)/T \to 0 \quad \text{as } u \to \infty.$$

The proof of Theorem 5.1 is then complete.

We now turn to $P(\tau_u - T > x) = P(\nu_{T+x} = 0)$ when T is as in Theorem 5.1 (remember the definition (5.1)).

Theorem 5.2. *With the same conditions and notations as in Theorem* 5.1

$$P(\tau_u - T_\theta^0 \leqq x) \to 1 - e^{-\theta} \quad \textit{for all } x, \quad \textit{as } u \to \infty.$$

Remark. If $-r''(t)/r'(t) \to 0$ then the function $-ur'(t)$ behaves almost like a large constant for large u and t. Therefore we might expect that the stream of zero-crossings of $ur'(t)+\xi'(t)$ has the same asymptotic Poisson-character as has the stream of crossings of a very high level by a stationary process, derived by Cramér and others, cf. [3, Ch. 12], although we here have a high function, g_u, say. Actually, Theorem 5.2 contains the first term in the asymptotic Poisson-distribution of the number of local minima. In general, if $(g_u)_{u \in R}$ is a family of functions, and $T(u)$ is such that the expected number of zero-crossings by $\xi(t)-g_u(t)$ in $(0, T(u))$ is θ, then the number of zero-crossings in $(0, T(u))$ is asymptotically Poisson (θ), provided

$$\begin{gathered} \inf_{0 \leqq t \leqq T(u)} g_u(t) \to \infty, \qquad \sup_{0 \leqq t \leqq T(u)} |g_u'(t)| \to 0, \\ \sup_{0 \leqq s, t \leqq T(u)} |g_u^2(s) - g_u^2(t)| \quad \text{is bounded}, \end{gathered} \tag{5.13}$$

and r fulfills the usual regularity conditions. This can be shown by means of the standard proof, modified as in the proof of Theorem 5.2. That (5.13) cannot be substantially relaxed, follows from the results by Qualls [10] on multiple level crossings.

Proof. Let y be fixed, and define the event E_I, $I \subseteq R$, by

$$E_I: \quad \{m_{uy}(t)+\delta(t)<0 \ \text{ for all } t \in I\},$$

where $m_{uy}(t)=ur'(t)-y(\lambda_2 r'(t)+r'''(t))$. Suppose we can prove that

$$P(E_{(0,T]}) \to e^{-\theta} \qquad \text{as } u \to \infty, \tag{5.14}$$

$(T=T_\theta^0(u))$. Since the expected number of crossings in $(T, T+x]$ tends to zero, we would then have

$$\begin{aligned} \lim_{u\to\infty} P(\tau_u - T > x) &= \lim_{u\to\infty} \int P(E_{(0,T+x]})\, q_u^*(y)\, dy \\ &= \lim_{u\to\infty} \int P(E_{(0,T]})\, q_u^*(y)\, dy = \int e^{-\theta} \lim_{u\to\infty} q_u^*(y)\, dy = e^{-\theta} \end{aligned}$$

by dominated convergence, which is just the content of the theorem.

We thus only have to show (5.14) for any fixed y. Start along well-known lines and divide the interval $[T_-, T]$ into n subintervals of length $\Delta > 0$:

$$T_- = t_0 < t_1 = t_0 + \Delta < \cdots < t_k = t_0 + k\Delta < \cdots < t_n = T.$$

Let α be a small number, and split the subintervals into two parts:

$$I_k = [t_k, t_k + (1-\alpha)\Delta], \qquad J_k = (t_k + (1-\alpha)\Delta, t_{k+1}).$$

Also let $t_{k,j} = t_k + j(1-\alpha)\Delta/n_k$, $j = 0, 1, \ldots, n_k$ be a subdivision of the I_k-intervals, and write for short

$$E_k = E_{I_k}$$

$$F_k: \ \{m_{uy}(t_{k,j}) + \delta(t_{k,j}) < 0 \qquad \text{for } j = 0, 1, \ldots, n_k\}.$$

Then (5.14) is the direct combination of the following four relations, valid if $n_k = [\log t_k]$ and $\alpha \to 0$, $\Delta \to 0$ sufficiently slowly as $u \to \infty$.

$$\lim_{u\to\infty} P(E_{(0,T]}) = \lim_{u\to\infty} P\left(\bigcap_{k=1}^{n} E_k\right), \tag{5.15}$$

$$\lim_{u\to\infty} P\left(\bigcap_{k=1}^{n} E_k\right) - P\left(\bigcap_{k=1}^{n} F_k\right) = 0, \tag{5.16}$$

$$\lim_{u\to\infty} P\left(\bigcap_{k=1}^{n} F_k\right) - \prod_{k=1}^{n} P(F_k) = 0, \tag{5.17}$$

$$\lim_{u\to\infty} \prod_{k=1}^{n} P(F_k) = e^{-\theta}. \tag{5.18}$$

To prove these relations we need to know a few more details about $m_{uy}(t)$, or equivalently $-u\,r'(t)$, and the process $\delta(t)$.

Lemma 5.6. *For any constant $\delta > 0$, y fixed, and $T_- \leqq t \leqq T$, we have*

$$\sqrt{(2-\delta)\log t} \leqq \sqrt{(2-\delta)\log T} \leqq |m_{uy}(t)|/\sigma(t) \leqq \sqrt{(2+\delta)\log T_-} \leqq \sqrt{(2+\delta)\log t}$$

if u is large enough.

Proof. By Lemma 5.2, $|m_{uy}(t)|/\sigma(t) \sim -u\,r'(t)/\sqrt{\lambda_2}$ for $T_- \leqq t \leqq T$. Since

$$-u\,r'(t) \leqq -u\,r'(T_-) = \sqrt{(2+\delta)\,\lambda_2 \log T_-} \leqq \sqrt{(2+\delta)\,\lambda_2 \log t},$$

we have the right hand inequality. Since $-t\,r''(t)/r'(t)$ is bounded, the following inequality holds in the interval $(T - T/\log T, T)$: $-u\,r'(t) \leqq -u\,r'(T) + \sup(u\,r''(t)) \cdot T/\log T \leqq -u\,r'(T) + K/\sqrt{\log T}$. Therefore

$$\begin{aligned}\int_0^T \phi\left(\frac{u\,r'(t)}{\sqrt{\lambda_2}}\right) dt &\geqq \int_{T-T/\log T}^{T} \phi\left(\frac{u\,r'(t)}{\sqrt{\lambda_2}}\right) dt \\ &\geqq K' \frac{T}{\log T} \exp\left\{-\frac{1}{2\lambda_2}\left(-u\,r'(T) + K/\sqrt{\log T}\right)^2\right\} \\ &\geqq K' \frac{T}{\log T} \exp\left\{-\frac{1}{2\lambda_2} u^2 r'(T)^2 + K''\right\},\end{aligned}$$

from which it follows that $-u\,r'(T) \geqq \sqrt{(2-\delta)\,\lambda_2 \log T}$ (since the left hand integral has a finite limit). Thus $-u\,r'(t) \geqq -u\,r'(T) \geqq \sqrt{(2-\delta)\,\lambda_2 \log T} \geqq \sqrt{(2-\delta)\,\lambda_2 \log t}$ as asserted.

Lemma 5.7. *The covariance function $\bar{c}$ of the normalized process $\overline{\delta(t)} = \delta(t)/\sigma(t)$,*

$$\bar{c}(s,t) = c(s,t)/\sigma(s)\,\sigma(t), \tag{5.19}$$

fulfills

a)
$$\bar{c}(t, t+h) = 1 - (1 + g_h + g_h(t))\,\frac{\lambda_4}{\lambda_2} \cdot \frac{h^2}{2} \qquad (h \geqq 0),$$

where $g_h \to 0$ as $h \downarrow 0$, and $\sup_{h \leqq h_0} |g_h(t)| \to 0$ as $t \to \infty$.

b) *There are* $\gamma>1$ *and* M *such that*

$$|\bar{c}(t,t+h)|\leqq M\,h^{-\gamma} \quad \textit{for}\ t\geqq 0,\ h>0.$$

Proof. a) By definition,

$$-c(t,t)+2c(t,t+h)-c(t+h,t+h)=-2\big(\lambda_2+r''(h)\big)+h^2\,r'(t)^2\,F_h(t),$$

where $F_h(t)$ is bounded as $h\downarrow 0$, $t\to\infty$. Divide by

$$\sigma(t)\,\sigma(t+h)=\sqrt{c(t,t)\,c(t+h,t+h)},$$

and observe that

$$\big(\lambda_2+r''(h)\big)/\sqrt{c(t,t)\,c(t+h,t+h)}=\big(1+f_h+f_h(t)\big)\frac{\lambda_4}{\lambda_2}\cdot\frac{h^2}{2},$$

where $f_h\to 0$ as $h\downarrow 0$, and $\sup_{h\leqq h_0}|f_h(t)|\to 0$ as $t\to\infty$. Including $r'(t)^2\,F_h(t)$ in $f_h(t)$, and writing $a=\sqrt{c(t+h,t+h)/c(t,t)}-1$, we get

$$\bar{c}(t,t+h)=\frac{1}{2}\,\{(1+a)+(1+a)^{-1}\}-\big(1+f_h+f_h(t)\big)\frac{\lambda_4}{\lambda_2}\cdot\frac{h^2}{2}.$$

Now $a\to 0$, so that

$$a=\sqrt{c(t+h,t+h)/c(t,t)-1+1}-1\sim\tfrac{1}{2}\big(c(t+h,t+h)-c(t,t)\big)/c(t,t)=h\,r'(t)^2\,G_h(t),$$

where $G_h(t)$ is bounded as $h\downarrow 0$, $t\to\infty$. The observation that $\frac{1}{2}\{(1+a)+(1+a)^{-1}\}=1+a^2/2(1+a)$ finishes the proof.

b) Any γ less than the number γ that exists according to condition C 3.b will do.

Proof of (5.15). Write ν_J for the number of upcrossing zeros of the process $m_{uy}(t)+\delta(t)$ for $t\in J=\bigcup J_k$, and let $\nu_{T_-}(y)$ as before denote the number of upcrossing zeros in $(0, T_-]$. Then

$$0\leqq P\left(\bigcap_{k=1}^{n}E_k\right)-P(E_{(0,T]})\leqq P\big(\nu_{T_-}(y)\geqq 1\big)+P(\nu_J\geqq 1),$$

where $P\big(\nu_{T_-}(y)\geqq 1\big)\leqq E\big(\nu_{T_-}(y)\big)\to 0$ as $u\to\infty$. Furthermore, as $\alpha\to 0$,

$$P(\nu_J\geqq 1)\leqq E(\nu_J)=\int_J\omega\,\phi(m_{uy}/\sigma)\,\Psi(\eta_{uy})\,dt\leqq K\int_J\phi\left(\frac{u\,r'}{\sqrt{\lambda_2}}\right)dt\to 0.$$

Proof of (5.16). With

$\nu_k=$ the number of upcrossing zeros of $m_{uy}(t)+\delta(t)$ in I_k,
$\nu'_k=$ the number of upcrossing zeros of the sequence

$$m_{uy}(t_{k,j})+\delta(t_{k,j}) \quad \text{for} \quad j=0,1,\ldots,n_k,$$

we have

$$0\leqq P\left(\bigcap_{k=1}^{n}F_k\right)-P\left(\bigcap_{k=1}^{n}E_k\right)\leqq P\left(\bigcup_{k=1}^{n}E_k{}^*F_k\right)\leqq\sum_{k=1}^{n}P(E_k{}^*F_k)$$

$$\leqq\sum_{k=1}^{n}P(\nu_k\geqq 1,\nu'_k=0)\leqq\sum_{k=1}^{n}E(\nu_k-\nu'_k).$$

To make this difference small, we take a small $\varepsilon>0$ and choose u large enough to apply Lemma 5.5 and (5.5). Then

$$E(v_k)=\int_{I_k}\omega\,\phi(m_{uy}/\sigma)\,\Psi(\eta_{uy})\,dt\leqq(1+\varepsilon)\int_{I_k}\sqrt{\frac{\lambda_4}{\lambda_2}}\,\phi\left(\frac{ur'(t)}{\sqrt{\lambda_2}}\right)\Psi(0)\,dt,$$

$$\begin{aligned}E(v_k')&=\sum_{j=1}^{n_k}P\big(m_{uy}(t_{k,j-1})+\delta(t_{k,j-1})<0<m_{uy}(t_{k,j})+\delta(t_{k,j})\big)\\&=\sum_{j=1}^{n_k}Q_{kj}\cdot\sqrt{\frac{\lambda_4}{\lambda_2}}\,\phi\big(m_{uy}(t_{k,j})/\sigma(t_{k,j})\big)\,\Psi(0)\cdot(t_{k,j}-t_{k,j-1})\\&\geqq(1-\varepsilon)\sum_{j=1}^{n_k}Q_{kj}\cdot\sqrt{\frac{\lambda_4}{\lambda_2}}\,\phi\big(ur'(t_{k,j})/\sqrt{\lambda_2}\big)\,\Psi(0)\cdot(t_{k,j}-t_{k,j-1}).\end{aligned}\tag{5.20}$$

Below we will show that

$$Q_{kj}=\frac{P\big(m_{uy}(t_{k,j-1})+\delta(t_{k,j-1})<0<m_{uy}(t_{k,j})+\delta(t_{k,j})\big)}{\sqrt{\frac{\lambda_4}{\lambda_2}}\,\phi\big(m_{uy}(t_{k,j})/\sigma(t_{k,j})\big)\,\Psi(0)\cdot(t_{k,j}-t_{k,j-1})}$$

is uniformly greater than $1-\varepsilon$ in all the intervals I_k if u is large. If we then use the remainder of (5.20) as an approximating Riemann-sum, (remember that $ur'(t)$ is monotonous), we get

$$E(v_k')\geqq(1-\varepsilon)^3\int_{I_k}\sqrt{\frac{\lambda_4}{\lambda_2}}\,\phi\left(\frac{ur'(t)}{\sqrt{\lambda_2}}\right)\Psi(0)\,dt,$$

and

$$\sum_{k=1}^{n}E(v_k-v_k')\leqq\{(1+\varepsilon)-(1-\varepsilon)^3\}\sum_{k=1}^{n}\int_{I_k}\sqrt{\frac{\lambda_4}{\lambda_2}}\,\phi\left(\frac{ur'(t)}{\sqrt{\lambda_2}}\right)\Psi(0)\,dt\leqq4\varepsilon\,\theta,$$

from which (5.16) follows.

It remains to show that Q_{kj} is uniformly not less than $1-\varepsilon$. Suppress the index k and write

$$x_j=-m_{uy}(t_{k,j})/\sigma(t_{k,j}),\qquad \bar\delta_j=\delta(t_{k,j})/\sigma(t_{k,j}),$$

$$\Delta_k=t_{k,j}-t_{k,j-1}=(1-\alpha)\Delta/n_k,\qquad \zeta_j=(\bar\delta_j-\bar\delta_{j-1})/\Delta_k,$$

and let $f_j(\cdot,\cdot)$ be the density function of the r.v. $(\bar\delta_{j-1},\zeta_j)$. Then Δ_k^{-1} times the probability in Q_{kj} is

$$\begin{aligned}&\Delta_k^{-1}P\left(x_j-\Delta_k\cdot\frac{\bar\delta_j-\bar\delta_{j-1}}{\Delta_k}<\bar\delta_{j-1}<x_{j-1}\right)\\&\geqq\Delta_k^{-1}\int_{z=0}^{\infty}\int_{x=x_j-\Delta_k z}^{x_{j-1}}f_j(x,z)\,dx\,dz=\big(y=(x_j-x)/\Delta_k z\big)\\&=\int_{z=0}^{\infty}z\int_{y=(x_j-x_{j-1})/\Delta_k z}^{1}f_j(x_j-\Delta_k z\,y,z)\,dy\,dz\\&\geqq\int_{z=0}^{\infty}z\int_{y=0}^{1}f_j(x_j-\Delta_k z\,y,z)\,dy\,dz.\end{aligned}\tag{5.21}$$

Here we used that $x_j-x_{j-1}<0$ for large u.

Now $\bar{\delta}_{j-1}$ and ζ_j have a joint normal distribution with mean zero and the covariance matrix

$$D_j = \begin{pmatrix} 1 & d_{12} \\ d_{12} & d_{22} \end{pmatrix} = \begin{pmatrix} 1 & \Delta_k^{-1}(\bar{c}(t_{k,j-1}, t_{k,j}) - 1) \\ \dots & 2\Delta_k^{-2}(1 - \bar{c}(t_{k,j-1}, t_{k,j})) \end{pmatrix},$$

so that

$$f_j(x_j - \Delta_k z\, y, z) = \frac{1}{2\pi\sqrt{\det D_j}} \exp(-A/2 \det D_j),$$

where

$$A = d_{22}(x_j - \Delta_k z\, y)^2 - 2 d_{12}(x_j - \Delta_k z\, y)\, z + z^2.$$

Here $\max_{0 \leq y \leq 1} A = A_{y=0} = d_{22} x_j^2 - 2 d_{12} x_j z + z^2$, $(d_{22} = -2\Delta_k^{-1} d_{12}!)$, and we get the following lower bound for the integral (5.21):

$$\begin{aligned} &\int_{z=0}^{\infty} z \frac{1}{2\pi\sqrt{\det D_j}} \exp\left(-(d_{22} x_j^2 - 2 d_{12} x_j z + z^2)/2 \det D_j\right) dz \\ &= \frac{1}{2\pi\sqrt{\det D_j}} \exp\left(-\tfrac{1}{2} x_j^2\right) \int_{z=0}^{\infty} z \exp\left(-(z - d_{12} x_j)^2/2 \det D_j\right) dz. \end{aligned} \tag{5.22}$$

If $\bar{c} = \bar{c}(t_{k,j-1}, t_{k,j})$ then, by Lemma 5.7a, we have $\det D_j = \Delta_k^{-2}(1 - \bar{c}^2) \to \lambda_4/\lambda_2$ as $\Delta_k \to 0$, $t_{k,j-1} \to \infty$. Since furthermore, by Lemma 5.6 and 5.7a,

$$|d_{12} x_j| = \Delta_k^{-1}(1 - \bar{c}) \frac{|m_{uy}(t_{k,j})|}{\sigma(t_{k,j})} \leq K \Delta_k \sqrt{\log t_{k,j}} \leq K \Delta\, n_k^{-1} \sqrt{\log t_k} \to 0$$

if $n_k = [\log t_k]$, the integral in the right-most side in (5.22) tends to λ_4/λ_2. Thus the lower bound in (5.21) is at least $(1-\varepsilon)\sqrt{\lambda_4/\lambda_2}\, \phi(x_j)\, \Psi(0)$, i.e. $Q_{kj} \geq 1 - \varepsilon$. The proof of (5.16) is complete.

Proof of (5.17). We will estimate the difference $P(\cap F_k) - \prod P(F_k)$ by a method originally used by Berman [1] and improved by Cramér and Leadbetter [3], et al. The present version uses the following inequality, due to Qualls and Watanabe [11]:

$$\begin{aligned} \left| P\left(\bigcap_{k=1}^{n} F_k\right) - \prod_{k=1}^{n} P(F_k) \right| &\leq \sum\sum_{1 \leq k < l \leq n} \sum_{i=0}^{n_k} \sum_{j=0}^{n_l} |\bar{c}_{ij}| \\ &\quad \cdot \int_0^1 \phi(x_{ki}, x_{lj};\ \lambda \bar{c}_{ij})\, d\lambda, \end{aligned} \tag{5.23}$$

where, for $i = 0, 1, \dots, n_k$, $j = 0, 1, \dots, n_l$, ($\bar{c}$ beeing defined by (5.19))

$$\bar{c}_{ij} = \bar{c}(t_{k,i}, t_{l,j}), \qquad x_{ki} = -m_{uy}(t_{k,i})/\sigma(t_{k,i}),$$

and $\phi(\cdot,\cdot;\ \rho)$ is the standarized, bivariate normal density with correlation coefficient ρ:

$$\phi(x, y;\ \lambda \bar{c}) = \frac{1}{2\pi\sqrt{1 - \lambda^2 \bar{c}^2}} \exp\left(-\tfrac{1}{2} B(\lambda)\right),$$

$$B(\lambda) = (x^2 - 2\lambda \bar{c}\, x\, y + y^2)/(1 - \lambda^2 \bar{c}^2).$$

Since, by Lemma 5.6, $x_{k,i}$, $x_{l,j} \geqq \sqrt{(2-\delta)\log T}$, we have $B(\lambda) \geqq (2-\delta)\log T \cdot (2-2\lambda\bar{c})/(1-\lambda^2\bar{c}^2) \geqq 2(2-\delta)(1+|\bar{c}|)^{-1}\log T$, and so the integral in (5.23) is bounded by $(2\pi)^{-1}(1-\bar{c}^2)^{-\frac{1}{2}}\exp\{-(2-\delta)(1+|\bar{c}|)^{-1}\log T\}$. Now the time has come to choose the value of δ, used as an unspecified positive constant in (5.12), Lemma 5.6, and elsewhere. Since there is an $\varepsilon > 0$, such that $\min(1-\bar{c}^2) \geqq \varepsilon$, we can find a $\delta > 0$ so small that $(2-\delta)(1+|\bar{c}|)^{-1} \geqq 1+\delta' > 1$ for all $\bar{c}$. The integral in (5.23) is then bounded by

$$(2\pi)^{-1}\varepsilon^{-\frac{1}{2}}\exp\left(-(1+\delta')\log T\right). \tag{5.24}$$

To estimate $\bar{c}_{ij}$ in (5.23), we use Lemma 5.7 b: there is an M such that $|\bar{c}(t, t+h)| \leqq M h^{-\gamma}$. Since $|t_{k,i} - t_{l,j}| \geqq |t_k - t_l| - (1-\alpha)\Delta \geqq \Delta|k-l| - (1-\alpha)\Delta$, we therefore have

$$|\bar{c}_{ij}| \leqq M|t_{k,i} - t_{l,j}|^{-\gamma} \leqq M_\alpha \Delta^{-\gamma}|k-l|^{-\gamma}. \tag{5.25}$$

Using $n_k = [\log t_k] \leqq \log T$, and inserting (5.24) and (5.25) into (5.23), we get the following upper bound for (5.23):

$$\begin{aligned}
&\sum_{1\leqq k<l\leqq n}\sum n_k n_l M'_\alpha \Delta^{-\gamma}|k-l|^{-\gamma}\varepsilon^{-\frac{1}{2}}\exp\left(-(1+\delta')\log T\right)\\
&\quad\leqq M''_\alpha \Delta^{-\gamma}\sum_{1\leqq k<l\leqq n}\sum (\log T)^2 |k-l|^{-\gamma} T^{-1-\delta'}\\
&\quad\leqq M''_\alpha \Delta^{-\gamma}\sum_{k=1}^{n}(\log T)^2 T^{-1-\delta'}\sum_{v=1}^{\infty} v^{-\gamma}\\
&\quad\leqq M'''_\alpha \Delta^{-\gamma} T\Delta^{-1}(\log T)^2 T^{-1-\delta'} = M'''_\alpha \Delta^{-1-\gamma}(\log T)^2 T^{-\delta'}.
\end{aligned}$$

Thus the double sum in (5.23) is bounded by a function of T that tends to zero as $T \to \infty$. The bound may become large when Δ and α are small, but it can be kept small by regulating the rate with which they tend to zero. This completes the proof of (5.17).

Proof of (5.18). We have to prove that if

$$\bar{m}(t) = m_{uy}(t)/\sigma(t), \qquad \bar{\delta}(t) = \delta(t)/\sigma(t),$$

and the event F_k is $\{\bar{m}(t_{k,i}) + \bar{\delta}(t_{k,i}) < 0$ for $i = 0, 1, \ldots, n_k\}$ then

$$-\sum_{k=1}^{n}\log P(F_k) \to \theta.$$

The idea in the proof is to approximate the covariance function $\bar{c}$ of the non-stationary process $\bar{\delta}$ in each of the intervals I_k by two simple covariance functions:

$$c^+(h), \quad c^-(h) = \cos\left\{(1\pm\varepsilon)\sqrt{\frac{\lambda_4}{\lambda_2}}\cdot h\right\}.$$

By Lemma 5.7, for any $\varepsilon > 0$, there is $(1-\alpha)\Delta$ such that for all large t

$$c^+(h) \leqq \bar{c}(t, t+h) \leqq c^-(h), \qquad 0 \leqq h \leqq (1-\alpha)\Delta. \tag{5.26}$$

Now let $\{\delta^+(h), h \in R\}$ and $\{\delta^-(h), h \in R\}$ be separable, stationary, Gaussian processes with mean zero and the covariance functions c^+ and c^- respectively.

Hence they are simple cosine-processes! By a well-known theorem by Slepian [12, Lemma 1 and Theorem 1], the suprema of normalized Gaussian processes are stochastically ordered, in the sense that a process with a uniformly greater covariance function has a greater probability of staying below a certain boundary, see the relations (5.29) below.

To formulate Slepian's result in this context, define

$$\Delta m_k = \frac{\varepsilon}{u|r'(t_k)|}, \qquad m_k^+ = -\frac{u r'(t_{k+1})}{\sqrt{\lambda_2}} - \Delta m_k, \qquad m_k^- = -\frac{u r'(t_k)}{\sqrt{\lambda_2}} + \Delta m_k.$$

By Lemma 5.5, $\bar{m}^2(t) - u^2 r'(t)^2/\lambda_2 \to 0$ so that, for $t \in I_k$,

$$m_k^+ \leqq -\bar{m}(t) \leqq m_k^-. \tag{5.27}$$

Since $P(F_k) = P(\bar{\delta}(t_{k,i}) \leqq -\bar{m}(t_{k,i}), i=0,1,\ldots,n_k)$, (5.27) implies that

$$P(F_k) \begin{array}{l} \geqq P(\bar{\delta}(t_{k,i}) \leqq m_k^+, i=0,1,\ldots,n_k) = P_1^+ \\ \leqq P(\bar{\delta}(t_{k,i}) \leqq m_k^-, i=0,1,\ldots,n_k) = P_1^- \end{array} \text{ say.} \tag{5.28}$$

Let $\Delta_k = (1-\alpha)\Delta/n_k$. Then Slepian's result, combined with (5.26), implies

$$\begin{aligned} P_1^+ &\geqq P(\delta^+(i\Delta_k) \leqq m_k^+,\ i=0,1,\ldots,n_k) \\ &\geqq P(\delta^+(h) \leqq m_k^+, 0 \leqq h \leqq (1-\alpha)\Delta) = P_2^+, \text{ say.} \\ P_1^- &\leqq P(\delta^-(i\Delta_k) \leqq m_k^-, i=0,1,\ldots,n_k) \\ &\leqq P(\delta^-(h) \leqq m_k^-, 0 \leqq h \leqq (1-\alpha)\Delta) \\ &\quad + P\begin{pmatrix} \delta^- \text{ has at least two } m_k^-\text{-crossings in one of} \\ \text{the intervals } [i\Delta_k, (i+1)\Delta_k], i=0,1,\ldots,n_k-1 \end{pmatrix} = P_2^- + Q, \text{ say.} \end{aligned} \tag{5.29}$$

The probabilities P_2^+ and P_2^- can be computed, see Slepian [12]:

$$\begin{aligned} P_2^+ &= \Phi(m_k^+) - \frac{(1-\alpha)\Delta}{2\pi}(1+\varepsilon)\sqrt{\frac{\lambda_4}{\lambda_2}}\exp(-\tfrac{1}{2} m_k^{+2}) \\ &\geqq 1 - \frac{(1-\alpha)\Delta}{2\pi}(1+\varepsilon)^2\sqrt{\frac{\lambda_4}{\lambda_2}}\exp(-\tfrac{1}{2} m_k^{+2}), \qquad \text{since } m_k^+ \to \infty, \\ P_2^- &= \Phi(m_k^-) - \frac{(1-\alpha)\Delta}{2\pi}(1-\varepsilon)\sqrt{\frac{\lambda_4}{\lambda_2}}\exp(-\tfrac{1}{2} m_k^{-2}) \\ &\leqq 1 - \frac{(1-\alpha)\Delta}{2\pi}(1-\varepsilon)\sqrt{\frac{\lambda_4}{\lambda_2}}\exp(-\tfrac{1}{2} m_k^{-2}). \end{aligned}$$

From the definition of m_k^+ and m_k^- it is easily seen that the exponentials in these expressions are asymptotically equal to $\exp(-u^2 r'(t_k)^2/2\lambda_2)$, so that

$$\begin{aligned} P_2^+ &\geqq 1 - \frac{(1-\alpha)\Delta}{2\pi}(1+\varepsilon)^3\sqrt{\frac{\lambda_4}{\lambda_2}}\exp(-u^2 r'(t_k)^2/2\lambda_2) \\ P_2^- &\leqq 1 - \frac{(1-\alpha)\Delta}{2\pi}(1-\varepsilon)^2\sqrt{\frac{\lambda_4}{\lambda_2}}\exp(-u^2 r'(t_k)^2/2\lambda_2). \end{aligned} \tag{5.30}$$

Since δ^- is a simple cosine-process, the probability Q can be estimated by integration:

$$Q=P\begin{pmatrix}\text{there is a subinterval of length } (1-\alpha)\Delta/n_k \text{ in } I_k \text{ in} \\ \text{which } \delta^- \text{ has at least two crossings of the level } m_k^-\end{pmatrix}$$
$$\leqq \text{const}\cdot n_k\cdot P(\text{at least two crossings in } (0,\Delta/n_k))$$
$$\leqq \text{const}\cdot n_k(\Delta/n_k)^3\, m_k^-\, \exp\left(-\tfrac{1}{2}m_k^{-\,2}\right)=o(1)\exp\left(-u^2 r'(t_k)^2/2\lambda_2\right).$$

In total, we get

$$P_2^- + Q \leqq 1-\frac{(1-\alpha)\Delta}{2\pi}(1-\varepsilon)^3\sqrt{\frac{\lambda_4}{\lambda_2}}\exp\left(-u^2 r'(t_k)^2/2\lambda_2\right). \tag{5.31}$$

Summing the estimates (5.28)–(5.31), we obtain

$$P(F_k)\gtreqless 1-\frac{(1-\alpha)\Delta}{2\pi}(1\pm\varepsilon)^3\sqrt{\frac{\lambda_4}{\lambda_2}}\exp\left(-u^2 r'(t_k)^2/2\lambda_2\right).$$

Standard inequalities for $\log(1-x)$ finally yields

$$-\sum_{k=1}^{n}\log P(F_k)\leqq(1-\alpha)(1+\varepsilon)^3\sum_{k=1}^{n}\frac{\Delta}{2\pi}\sqrt{\frac{\lambda_4}{\lambda_2}}\exp\left(-u^2 r'(t_k)^2/2\lambda_2\right)+R_n$$
$$=(1-\alpha)(1+\varepsilon)^3\sum_{k=1}^{n}\Delta\sqrt{\frac{\lambda_4}{\lambda_2}}\,\phi\left(u\,r'(t_k)/\sqrt{\lambda_2}\right)\Psi(0)+R_n\leqq(1-\alpha)(1+\varepsilon)^3\,\theta+R_n,$$

where $R_n\leqq\text{const}\cdot\sum\Delta^2\exp\left(-\mathrm{u}^2\,\mathrm{r}'(\mathrm{t}_k)^2/\lambda_2\right)\to 0$ as $\Delta\to 0$ and $u\to\infty$. Similarly

$$-\sum_{k=1}^{n}\log P(F_k)\geqq(1-\alpha)(1-\varepsilon)^3\sum_{k=1}^{n}\frac{\Delta}{2\pi}\sqrt{\frac{\lambda_4}{\lambda_2}}\exp\left(-u^2 r'(t_k)^2/2\lambda_2\right)$$
$$\geqq(1-\alpha)(1-\varepsilon)^3\,\theta+o(1).$$

Since ε is arbitrary, and α is permitted to tend slowly to zero, we have proved (5.18).

This completes the proof of Theorem 5.2.

6. Decreasing Covariance Function

$$-r''(t)/r'(t)\to C,\qquad 0<C\leqq\infty,\quad \text{as}\quad t\to\infty$$

This section deals with the case when the constant $C=\lim -r^{(k)}(t)/r^{(k-1)}(t)$ is strictly positive or infinite. As in Section 5 we will define a function $T(=T^C(u), T^\infty(u))$ such that $E(v_T)$ has a finite limit as $u\to\infty$; v_t being the number of upcrossing zeros of $\xi_u'(s)$ is $(0,t]$.

Theorem 6.1. *If r fulfills condition* C3, *if* $-r''(t)/r'(t)\to C$ *as* $t\to\infty$, *with* $0<C\leqq\infty$, *and if* $T^C=T^C(u)$, $T^\infty=T^\infty(u)\to\infty$ *as* $u\to\infty$ *so that*

$$-u\,r'(T^C)=1 \qquad (0<C<\infty),$$
$$-u\,r'(T^\infty)=\sqrt{-r'(T^\infty)/r''(T^\infty)} \qquad (C=\infty)$$

then, as $u \to \infty$,

$$E(\nu_{T^C+x}) \to \int_{-\infty}^{x} \sqrt{\frac{\lambda_4}{\lambda_2}}\, \phi(e^{-Ct}/\sqrt{\lambda_2})\, \Psi(C e^{-Ct}/\sqrt{\lambda_4})\, dt \qquad (0<C<\infty), \qquad (6.1)$$

$$E(\nu_{T^\infty+x}) \to \begin{cases} 0 & \text{for } x<0 \\ \dfrac{1}{2}+\dfrac{x}{2\pi}\sqrt{\dfrac{\lambda_4}{\lambda_2}} & \text{for } x \geqq 0 \end{cases} \qquad (C=\infty). \qquad (6.2)$$

Remark. The right hand limits in (6.1) and (6.2) are actually $\lim E(\mu_{T+x})$, where μ_t is the number of upcrossing zeros of $u\,r'(s)+\xi'(s)$ in $(0, t]$, so that Theorem 6.1 is a more explicit one than Theorem 5.1.

Proof. The main ideas from the proof of Theorem 5.1 works, but since more can be said about $-u\,r'$ (see Lemma 6.2 below), the proof will be simpler. This is also reflected in the more explicit statement of Theorem 6.1.

Of course Lemma 5.1 and 5.3 hold unchanged, since their proofs do not depend of the value of C. Lemma 5.2 must be replaced by

Lemma 6.1. *If* $y_- = T = T^C, T^\infty$ *then*

$$\frac{m_{uy}(t)}{\sigma(t)} = \frac{u\,r'(t)}{\sqrt{\lambda_2}}\,(1+o(1)),$$

$$\eta_{uy}(t) = \frac{u\,r''(t)}{\sqrt{\lambda_4}}\,(1+o(1)),$$

where $o(1) \to 0$ *uniformly in* $|y| \leqq y_-$ *as* $u \to \infty$, $t \to \infty$ (*for* $0<C<\infty$), *and uniformly in* $|y| \leqq y_-$, $t_0 \leqq t \leqq T^\infty$ *as* $u \to \infty$, $t_0 \to \infty$, $t_0 < T^\infty$, (*for* $C=\infty$).

Proof of Lemma 6.1. It is only for $C=\infty$ that we need to modify the proof of Lemma 5.2. Since $\mu(t) = O(r^{\mathrm{IV}}(t))$ if $C=\infty$, the residuals R_{uy} and S_{uy} in (5.10) are uniformly small if

a) $r'''(t)\,r^{\mathrm{IV}}(t)/r''(t) \to 0$; this again follows from the convexity of $(-1)^k r^{(k)}(t)$.

b) y_-/u, $y_-\,r'''(T)/u\,r'(T)$, $y_-\,r^{\mathrm{IV}}(T)/u\,r''(T) \to 0$ as $u \to \infty$; by C3.c and the definition of T^∞ we have $M u T^{-2} \geqq u\,r''(T) \to \infty$ so that $y_-/u = T/u \to 0$. Also

$$[T r'''(T)/u\,r'(T)]^2 = -T^2 r'''(T) \cdot [r''(T)\, r'''(T)/r'(T)] \to 0,$$

and

$$[T r^{\mathrm{IV}}(T)/u\,r''(T)]^2 = T^2 r^{\mathrm{IV}}(T) \cdot [-r'(T)\, r^{\mathrm{IV}}(T)/r''(T)] \to 0.$$

The choice of the splitting point T_-, which had to be made carefully in Section 5 (definition (5.12)), is here simpleminded. All we need to do is to let $T_- \to \infty$ so that $E(\nu_{T_-}) \to 0$; actually it suffices that $-u\,r'(T_-) \to \infty$. Lemma 5.4 is therefore superfluous.

The following lemma motivates both the name "exponential case" for $0<C<\infty$, and the choice of T^∞ for $C=\infty$.

Lemma 6.2. *As* $u \to \infty$ *then, for any* $t_0 > t_0' > 0$,

$$-u\,r'(T^C+t)/\exp(-C\,t) \to 1 \qquad \textit{uniformly in } |t| \leqq t_0 \qquad (0<C<\infty)$$

$$-u\,r'(T^{\infty}+t)\to\begin{cases}\infty & \textit{uniformly in } -t_0\leqq t\leqq -t_0' \\ 0 & \textit{uniformly in } t\geqq 0\end{cases}$$

$$u\,r''(T^{\infty}+t)\to\begin{cases}\infty & \textit{uniformly in } -t_0\leqq t\leqq 0 \\ 0 & \textit{uniformly in } t>t_0'.\end{cases} \qquad (C=\infty)$$

Proof of Lemma 6.2. If $-r''(t)/r'(t)\to C<\infty,\ >0$ then for every $\varepsilon>0$ there is a t^* such that $-(C-\varepsilon)\,r'(t)<r''(t)<-(C+\varepsilon)\,r'(t)$ for $t\geqq t^*$. By continuity arguments, this can be shown to imply that $-r'(s)\exp\big(-(C+\varepsilon)(t-s)\big)\leqq -r'(t)\leqq -r'(s)\exp\big(-(C-\varepsilon)(t-s)\big)$ for $t\geqq s\geqq t^*$, which gives the lemma in this case.

If $-r''(t)/r'(t)\to\infty$ the result follows if one considers the definition of T^{∞}.

Of course the uniform convergence in Lemma 6.2 takes place even if $t_0\to\infty$, $t_0'\to 0$ sufficiently slowly. Therefore we can find $T_-=t_0\to\infty$, such that $T_- - T\to-\infty$, $-u\,r'(T_-)\to\infty$, $E(v_{T_-})\to 0$, and such that the conclusions in Lemma 6.2 hold.

Now separate $C<\infty$ and $C=\infty$, and let $y_-=T=T^C,\ T^{\infty}$. $0<C<\infty$: If

$$\Lambda(s)=\sqrt{\lambda_4/\lambda_2}\,\phi\big(e^{-Cs}/\sqrt{\lambda_2}\big)\,\Psi\big(C\,e^{-Cs}/\sqrt{\lambda_4}\big)$$

then we shall prove that $E(v_{T+x})$ has the limit

$$\lim\int_y\int_{t=0}^{T+x}\omega\,\phi(m_{uy}/\sigma)\,\Psi(\eta_{uy})\,q_u^*(y)\,dt\,dy=\lim\int_{|y|\leqq y_-}\int_{t=T_-}^{T+x}$$

$$=\lim\int_{s=T_--T}^{x}\Lambda(s)\,ds=\int_{-\infty}^{x}\Lambda(s)\,ds.$$

But this is easy; we only need to note that

$$\omega(T+s)\,\phi\big(m_{uy}(T+s)/\sigma(T+s)\big)\,\Psi\big(\eta_{uy}(T+s)\big)/\Lambda(s)\to 1$$

uniformly in $T_- - T\leqq s\leqq x$, $|y|\leqq y_-$ as $u\to\infty$. $C=\infty$: First take $x\leqq 0$ and let $\varepsilon>0$ be arbitrary. Since $\eta_{uy}(t)\sim u\,r''(t)/\sqrt{\lambda_4}\to\infty$ we have

$$|m_{uy}(t)/\sigma(t)|\geqq-(1-\varepsilon)\,u\,r'(t)/\sqrt{\lambda_2}\quad\text{and}\quad\Psi\big(\eta_{uy}(t)\big)\leqq(1+\varepsilon)\,u\,r''(t)/\sqrt{\lambda_4}$$

uniformly in $T_-\leqq t\leqq T+x$, $|y|\leqq y_-$ for large u. Thus

$$\int_{T_-}^{T+x}\omega\,\phi(m_{uy}/\sigma)\,\Psi(\eta_{uy})\,dt\leqq(1+\varepsilon)^2\sqrt{\frac{\lambda_4}{\lambda_2}}\int_{T_-}^{T+x}\phi\left((1-\varepsilon)\frac{u\,r'(t)}{\sqrt{\lambda_2}}\right)\frac{u\,r''(t)}{\sqrt{\lambda_4}}\,dt$$

$$=\frac{(1+\varepsilon)^2}{(1-\varepsilon)}\left\{\Phi\left((1-\varepsilon)\frac{u\,r'(T+x)}{\sqrt{\lambda_2}}\right)-\Phi\left((1-\varepsilon)\frac{u\,r'(T_-)}{\sqrt{\lambda_2}}\right)\right\}\to\begin{cases}0 & \text{if } x<0\\ \dfrac{(1+\varepsilon)^2}{2(1-\varepsilon)} & \text{if } x=0.\end{cases}$$

This in turn gives that $\lim E(v_{T+x})=0$ if $x<0$, and that $\limsup E(v_T)\leqq\frac{1}{2}$. A reverse inequality is obtained from

$$E\big(v_T(y)\big)\geqq P\big(v_T(y)\geqq 1\big)\geqq P\big(m_{uy}(T)+\delta(T)>0\big)=\Phi\big(m_{uy}(T)\big),$$

(cf. the definition (5.3)). Then $m_{uy}(T)\to 0$ implies $\liminf E\big(v_T(y)\big)\geqq\frac{1}{2}$, $\liminf E(v_T)\geqq\frac{1}{2}$, and subsequently the theorem if $x=0$.

It remains to show that if $x>0$ then

$$\int_{|y|\leq y_-} \left\{ \int_T^{T+x} \omega\, \phi(m_{uy}/\sigma)\, \Psi(\eta_{uy})\, dt \right\} q_u^*(y)\, dy \to \frac{x}{2\pi} \sqrt{\frac{\lambda_4}{\lambda_2}}. \tag{6.3}$$

But for any t in $(T, T+x)$ both $m_{uy}(t)$ and $\eta_{uy}(t)$ tend to zero, so that the inner integrand in (6.3) tends to $\sqrt{\lambda_4/\lambda_2}\,\phi(0)\,\Psi(0)=\sqrt{\lambda_4/\lambda_2}/2\pi$. Simple calculations show that the convergence is dominated, and thus (6.3) follows.

This completes the proof of Theorem 6.1.

In Theorem 6.1 we used that $m_{uy}(T+t)$ is approximately known for moderate t-values. We will now more explicitly use that $\delta(T+t)$ is asymptotically equivalent to $\xi'(t)$ for large u, and compare the zeros of $m_{uy}(T+t)+\delta(T+t)$ with certain function crossings of $\xi'(t)$, and so get the asymptotic distribution of the wavelength τ_u defined by (2.6).

Theorem 6.2. *With the same notations and conditions as in Theorem* 6.1

$$P(\tau_u - T^C \leq x) \to 1 - P\big(\sup_{t\leq x} \xi'(t) - e^{-Ct} < 0\big) \qquad (0<C<\infty),$$

$$P(\tau_u - T^\infty \leq x) \to \begin{cases} 0 & \text{for } x<0 \\ 1-P\big(\sup_{0\leq t\leq x} \xi'(t)<0\big) & \text{for } x\geq 0 \end{cases} \qquad (C=\infty).$$

Proof. The crucial point in the proof is the observation that the translated process

$${}^T\delta(t) = \delta(T+t)$$

converges weakly to $\xi'(t)$. More precisely, let $\mathscr{P}_T$ and $\mathscr{P}$ be probability measures for the processes $\{{}^T\delta(t), |t|\leq t_0\}$ and $\{\xi'(t), |t|\leq t_0\}$ respectively on the space $\{C, \mathscr{C}\}$ of continuous functions with the topology for uniform convergence.

Lemma 6.3. $\mathscr{P}_T \Rightarrow \mathscr{P}$ *as* $T\to\infty$, *i.e.* $\mathscr{P}_T$ *converges weakly to* $\mathscr{P}$.

Proof of Lemma 6.3. The process ${}^T\delta$ has the covariance function $c(T+s, T+t)$ which tends to $-r''(s-t)$ as $T\to\infty$, so that all finite-dimensional distributions of $\mathscr{P}_T$ tend to those of $\mathscr{P}$. To establish that $\mathscr{P}_T\Rightarrow\mathscr{P}$, we have to prove that $\{{}^T\delta\}$ is tight, see e.g. Billingsley [2, Ch. 2]. A sufficient condition for this is

a) $\sup_T V({}^T\delta(0)) < \infty$,

b) there are T_0, h_0, K such that, for $T\geq T_0$, $|h|\leq h_0$

$$V({}^T\delta(t+h) - {}^T\delta(t)) < Kh^2.$$

Here a) is obviously fulfilled, since $V({}^T\delta(0)) = V(\delta(T)) = c(T, T)$, which tends to $\lambda_2<\infty$ as $T\to\infty$. For b), we recall from the proof of Lemma 5.7 that

$$\begin{aligned} V({}^T\delta(t+h) - {}^T\delta(t)) &= c(T+t+h, T+t+h) + c(T+t, T+t) - 2c(T+t+h, T+t) \\ &\leq 2(\lambda_2 + r''(h)) + K_1 h^2 \leq K h^2 \end{aligned}$$

for small h and all T.

Now let $\varepsilon > 0$ be small and define

$$m_\varepsilon^+(t) = \begin{cases} (1+\varepsilon)\exp(-C\,t) & (0<C<\infty) \\ \infty \quad \text{for } t<0 & \\ \varepsilon \quad \text{for } t \geqq 0 & (C=\infty), \end{cases}$$

$$m_\varepsilon^-(t) = \begin{cases} (1-\varepsilon)\exp(-C\,t) & (0<C<\infty) \\ -t/\varepsilon \quad \text{for } t<0 & \\ 0 \quad \text{for } t \geqq 0 & (C=\infty). \end{cases}$$

Also write, for fixed y, ${}^T m_u(t) = m_{uy}(T+t)$. Then, Lemma 6.2 implies that, for any t_0,

$$m_\varepsilon^-(t) \leqq -{}^T m_u(t) \leqq m_\varepsilon^+(t) \qquad \text{for } |t| \leqq t_0 \tag{6.4}$$

holds for all sufficiently large u. It is also possible to find a t_0 such that

$$1 - P\big(\xi'(t) < m_\varepsilon^+(t) \text{ for } t < -t_0\big) \leqq \varepsilon,$$

and such that, for all large T,

$$1 - P\big({}^T\delta(t) < -{}^T m_u(t) \text{ for } -T < t < -t_0\big) \leqq \varepsilon.$$

Then, by (6.4),

$$\begin{aligned} P\big(m_{uy}(t) + \delta(t) < 0 \text{ for } 0 < t < T+x\big) \\ = P\big({}^T m_u(t) + {}^T\delta(t) < 0 \text{ for } -T < t < x\big) \\ \leqq P\big({}^T\delta(t) < m_\varepsilon^+(t) \text{ for } -t_0 < t < x\big), \end{aligned}$$

and the weak convergence in Lemma 6.3 implies that this expression tends to

$$\begin{aligned} P\big(\xi'(t) < m_\varepsilon^+(t) \text{ for } -t_0 < t < x\big) \\ \leqq P\big(\xi'(t) < m_\varepsilon^+(t) \text{ for } t < x\big) + \varepsilon. \end{aligned}$$

Therefore

$$\limsup_{u\to\infty} P(\tau_u > T+x) \leqq P\big(\xi'(t) < m_\varepsilon^+(t) \text{ for } t < x\big) + \varepsilon. \tag{6.5}$$

A lower bound for $P\big(m_{uy}(t) + \delta(t) < 0 \text{ for } 0 < t < T+x\big)$ is

$$\begin{aligned} P\big({}^T\delta(t) < -{}^T m_u(t) \text{ for } -t_0 < t < x\big) \\ -\big(1 - P\big({}^T\delta(t) < -{}^T m_u(t) \text{ for } -T < t < -t_0\big)\big) \\ \geqq P\big({}^T\delta(t) < m_\varepsilon^-(t) \text{ for } -t_0 < t < x\big) - \varepsilon, \end{aligned}$$

which has the limit

$$\begin{aligned} P\big(\xi'(t) < m_\varepsilon^-(t) \text{ for } -t_0 < t < x\big) - \varepsilon \\ \geqq P\big(\xi'(t) < m_\varepsilon^-(t) \text{ for } t < x\big) - \varepsilon, \end{aligned}$$

and so

$$\liminf_{u\to\infty} P(\tau_u > T+x) \geqq P\big(\xi'(t) < m_\varepsilon^-(t) \text{ for } t < x\big) - \varepsilon. \tag{6.6}$$

Since the right hand sides in (6.5) and (6.6) can be made arbitrarily close to the required probability by taking ε small, we have proved the asserted convergence in Theorem 6.2.

7. Distribution of Amplitude, Decreasing Covariance Function

The amplitude, or crest-to-trough wave-height, $\delta_u = u - \xi_u(\tau_u)$, defined by (2.6) tends to be of the order u after a high maximum. In the exponential and over-exponential case ($0 < C \leqq \infty$ in condition C3.c) even more can be said about its distribution. Theorems 7.1 and 7.2 give details.

Theorem 7.1. *If r fulfills condition* C3 *then the amplitude δ_u is of the order u as $u \to \infty$, i.e.*

$$\delta_u/u \xrightarrow{\mathscr{P}} 1 \quad as \quad u \to \infty.$$

Proof. By definition (2.4)

$$\delta_u/u = (u - \xi_u(\tau_u))/u = 1 - r(\tau_u) + u^{-1}\eta_u(\lambda_2 r(\tau_u) + r''(\tau_u)) - u^{-1}\Delta(\tau_u).$$

Since, by Lemma 5.1, τ_u tends to infinity in probability both $r(\tau_u)$ and $u^{-1}\eta_u(\lambda_2 r(\tau_u) + r''(\tau_u))$ tend to zero, and we have only to estimate $u^{-1}\Delta(\tau_u)$. Since $\Delta(0) = 0$ (a.s.), we have

$$|u^{-1}\Delta(\tau_u)| = u^{-1}\left|\int_0^{\tau_u} \delta(t)\,dt\right| \leqq u^{-1}\tau_u \sup_{0<t<\tau_u} |\delta(t)|.$$

Thus, for all $t_u, x_u > 0$,

$$\begin{aligned} P(|u^{-1}\Delta(\tau_u)| \leqq u^{-1} t_u x_u) &\geqq P(\tau_u \leqq t_u, \sup_{0<t<t_u} |\delta(t)| \leqq x_u) \\ &\geqq 1 - P(\tau_u > t_u) - P(\sup_{0<t<t_u} |\delta(t)| > x_u). \end{aligned} \tag{7.1}$$

By condition C3.b, there is a $\gamma > 1$ and a constant K such that $|r'(t)| \leqq K t^{-\gamma}$ for large t. Take $\delta > 0$ and $1 < \gamma' < \gamma$, and define

$$t_u = u^{1/\gamma'}, \qquad x_u = \sqrt{(2+\delta)\lambda_2 \log u}.$$

Then it is an easy consequence of Theorems 5.1 and 6.1 that

$$P(\tau_u > t_u) \to 0 \quad \text{as} \quad u \to \infty. \tag{7.2}$$

Also, if $t_0 \to \infty$ slowly enough, then

$$\begin{aligned} &\lim_{u\to\infty} P(\sup_{0<t<t_u} |\delta(t)| > x_u) = \lim_{u\to\infty} P(\sup_{t_0<t<t_u} |\delta(t)| > x_u) \\ &\leqq \lim_{u\to\infty} P\left\{\begin{matrix}\delta(t) \text{ crosses one of the levels } \pm x_u \\ \text{at least once in } (t_0, t_u)\end{matrix}\right\} \\ &\leqq \lim_{u\to\infty} E(\text{the number of crossings of } \pm x_u \text{ by } \delta(t) \text{ in } (t_0, t_u)). \end{aligned}$$

This expectation can be calculated by a similar formula as (5.3), and is less than or equal to

$$K\int_{t_0}^{t_u}\phi(x_u/\sigma)\,\Psi\left(\frac{\mu}{\sqrt{1-\mu^2}}\cdot\frac{x_u}{\sigma}\right)dt \leqq K'\,t_u\,x_u\exp(-x_u^2/(2+\delta)\,\lambda_2)$$
$$= K''\,u^{1/\gamma'}\cdot\sqrt{\log u}\cdot\exp(-\log u)\to 0.$$

Thus

$$P\big(\sup_{0<t<t_u}|\delta(t)|>x_u\big)\to 0. \tag{7.3}$$

Since, furthermore $u^{-1}\,t_u\,x_u\to 0$, we can insert (7.2) and (7.3) into (7.1) and conclude that $u^{-1}\Delta(\tau_u)\to 0$ in probability. This proves the theorem.

Theorem 7.2. *If the conditions of Theorem* 6.1 *are fulfilled, and if the r.v.* τ *is defined by*

$$\tau=\begin{matrix}\inf\{t;\ \xi'(t)\geqq e^{-Ct}\} & (0<C<\infty)\\ \inf\{t\geqq 0;\ \xi'(t)\geqq 0\} & (C=\infty),\end{matrix}$$

then, as $u\to\infty$,

$$(\tau_u-T^C,\delta_u-u)\xrightarrow{\mathscr{L}}(\tau,\ -C^{-1}e^{-C\tau}-\xi(\tau)) \qquad (0<C<\infty)$$
$$(\tau_u-T^\infty,\delta_u-u)\xrightarrow{\mathscr{L}}(\tau,\ -\xi(\tau)) \qquad (C=\infty).$$

Remark. It is easily shown that the r.v. τ is finite (a.s.).

Proof. We already know that $\bar\tau=\tau_u-T\xrightarrow{\mathscr{L}}\tau$, $(T=T^C,T^\infty)$. Also

$$(\tau_u-T,\delta_u-u)=(\bar\tau,\ -u\,r(T+\bar\tau)+\eta_u(\lambda_2\,r(T+\bar\tau)+r''(T+\bar\tau))-\Delta(T+\bar\tau))$$
$$=(\bar\tau,\ -u\,r(T+\bar\tau)+o_p(1)-{}^T\!\Delta(\bar\tau)),$$

where $o_p(1)\xrightarrow{\mathscr{P}}0$, $u\,r(T+\bar\tau)\sim C^{-1}\exp(-C\bar\tau)$ or 0 according to if $0<C<\infty$ or $C=\infty$, and

$${}^T\!\Delta(t)=\Delta(T+t)={}^T\!\Delta(0)+\int_0^t {}^T\delta(s)\,ds, \qquad {}^T\delta(t)=\delta(T+t).$$

If we use the tightness criterion in the proof of Lemma 6.3, now on the conditional process $({}^T\delta(\cdot)\,|\,{}^T\!\Delta(0)=x)$, we can conclude that ${}^T\!\Delta(\bar\tau)$ behaves like

$$\xi(0)+\int_0^\tau \xi'(s)\,ds=\xi(\tau),$$

and get the theorem.

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