Z. Wahrscheinlichkeitstheorie verw. Gebiete
65, 245–267 (1983)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Flows of Stochastic Dynamical Systems: The Functional Analytic Approach

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1. Introduction

1.1. Aims

Many results about the flows of stochastic dynamical systems on compact manifolds M may be proved easily by considering the infinite dimensional manifold $\mathscr{D}^s(M)$ of H^s diffeomorphisms of M. The technique was introduced for ordinary differential equations by Ebin and Marsden [10], and utilized for stochastic differential equations by Elworthy [11]. Recently a similar approach has been used by Ustunel [34]. There is a detailed description in [13].

The procedure is to induce from the dynamical system on M a dynamical system on $\mathscr{D}^s(M)$, which we shall call the "lift", and whose solution is the flow of the original system. Results proved for compact manifolds can often be extended to non-compact (but σ-compact) manifolds, in particular to $\mathbb{R}^n$. In this way many of the long technical details of the proofs of the basic theorems about flows of stochastic dynamical systems (in particular the inductive steps) are subsumed once and for all in the known results about Sobolev spaces.

Our aim here is to present the lift and some of its applications in a unified form and in a simplified way: rather than deal with manifolds directly we first take our stochastic dynamical system to be defined on $\mathbb{R}^n$ and have support in some bounded domain B. We lift it to the space of H^s diffeomorphisms of $\mathbb{R}^n$ which restrict to the identity on the complement of B. This is an open set in an affine subspace of a Hilbert space, and so infinite dimensional manifold theory is not required. Results for systems on finite dimensional manifolds M will follow from these results by embedding M in $\mathbb{R}^P$, for some P; see § 7.

The Sobolev spaces H^s are used as a tool: here we are really interested in the C^∞ case and not primarily in the precise H^s regularity etc obtainable. A method of deducing some C^r results from C^∞ results by approximation is described in [13]. We will prove the basic results on regularity of the flows, convergence of piecewise linear approximations, the generalised Itô formula of Bismut (Theorem 5.3), [2, 3], some of Kunita's results [21, 22] on backward stochastic differential equations, and his criterion for the flow to be a diffeomorphism. They are mostly first proved for stochastic dynamical systems with compact support, and the version for general systems on $\mathbb{R}^n$ is deduced later.

1.2. Stochstic Dynamical Systems: Their Solutions and Flows

All stochastic differential equations will be Stratonovitch equations with an m-dimensional Brownian motion B_t as the driving noise. However, the methods will work more generally.

A stochastic dynamical system on a separable Hilbert space H will therefore be a pair (X, z) where

$$X: H \to \mathbb{L}(\mathbb{R}^{m+1}; H)$$

is a map into the space of linear maps of $\mathbb{R}^{m+1}$ to H and $z_t: \Omega \to \mathbb{R}^{m+1}$ is given by

$$z_t = (B_t^1, \ldots, B_t^m, t)$$

for $(\Omega, \{\mathscr{F}_t\}_{t \geqq 0}, P)$ the probability space, with filtration, of B_t. A process $\{\xi_{t,x}: 0 \leqq t \leqq T\}$ (always adapted and with continuous sample paths) is a *solution to* (X, z) *starting from* $x \in H$ if

$$\xi_{t,x} = x + \int_0^t X(\xi_{s,x}) \circ dz_s \qquad 0 \leqq t \leqq T$$

(where "$\circ$" denotes Stratonovich integral).

By a *flow* for (X, z) on H, in the time interval $[0, T]$ we mean a family of continuous maps $\xi_t(\omega): H \to H$, for $(t, \omega) \in [0, T] \times \Omega$ such that $\{\xi_t(\cdot)x: 0 \leqq t \leqq T\}$ is a solution to (X, z) starting from x, for each x in H.

When X has bounded first and second derivatives and $\dim H < \infty$ it follows easily [1, 4, 19, 27], from Totoki's generalisation of Kolmogorov's theorem [33] that a flow exists for all time, moreover it can be chosen to be jointly continuous in (t, x).

Strictly speaking we should define the flow (or solution starting at x) to be the equivalence class under the relation of almost sure equality. By $\{\xi_t\}_t$ we mean a "version" of the flow, and one of our aims will be to obtain especially nice versions.

Note 1.2. For our systems the Markov property implies that if (X, z) has a flow for the time interval $[0, T]$ then it has a flow for all time.

2. Preliminary Results

2.1. Behaviour of Solutions Under Transformations

Proposition 2.1. *Let U and V be open in Hilbert spaces H and K. Consider C^2 stochastic dynamical systems (X, z) on U and (Y, z) on V for which there is a C^2*

map θ: $U \to V$, not necessarily bijective, such that $D\theta(u)(X(u)e) = Y(\theta(u))e$ for all (u, e) in $U \times \mathbb{R}^{n+1}$. Then θ maps solutions of (X, z) to solutions of (Y, z).

Proof. Suppose

$$\xi_t = u + \int_0^t X(\xi_s) \circ dz_s.$$

Then by the Itô formula

$$\theta \xi_t = \theta u + \int_0^t D\theta(\xi_s) X(\xi_s) \circ dz_s = \theta u + \int_0^t Y(\theta \xi_s) \circ dz_s.$$

In case the solution ξ is only defined up to a stopping time τ the above equations still hold on the set $\{t < \tau\}$. //

2.2. *Uniform Covers: a Non-explosion Criterion*

Let (X, z) be a system on the open set U of H. A *uniform cover* for X, (radius $r > 0$, bound k), is a family $\{\phi_i\}_i$ of diffeomorphisms ϕ_i: $U_i \to V_i$ of open subsets of U onto open sets of H such that

1. $B_{3r} \subset \phi_i(U_i)$ each i (B_α denotes the open ball about 0, radius α).
2. $\{\phi_i^{-1}(B_r)\}_i$ covers U.
3. If $(\phi_i)_*(X)$: $V_i \to \mathbb{L}(\mathbb{R}^{m+1}; H)$ is defined by

$$(\phi_i)_*(X)(v)\, e = D\phi_i(\phi_i^{-1} v)\, X(\phi_i^{-1} v)\, e$$

then $(\phi_i)_*(X)$ and its first derivative are bounded by k on B_{2r}.

Theorem 2.2 (Itô [18]). *If X admits a uniform cover then (X, z) has solutions going on for all (positive) time.*

Proof. See Itô [18], Clarke [7], or Elworthy [13]. //

When solutions exist for all (positive) time we will say (X, z) is *complete*, sometimes the terms *non-explosive* or *conservative* are used.

2.3. *Piecewise Linear Approximations*

By the piecewise linear approximation to the system (X, z) in the time interval $[0, T]$, with respect to the partition Π given by $0 = t_0 < t_1 < \ldots < t_m = T$ we mean the system (X, z_π) where z_π is the piecewise linear approximation to z, given by

$$z_\pi(t) = (t_j - t_{j-1})^{-1} [(t - t_{j-1})\, z(t_j) + (t_j - t)\, z(t_{j-1})] \qquad t_{j-1} \leqq t \leqq t_j.$$

Note that (X, z_π) can be considered as a family of ordinary dynamical systems parametrized by $\omega \in \Omega$ since the sample paths of z_π are all piecewise C^1.

Theorem 2.3 (McShane [25], Elworthy [11, 13], see also [3, 9, 16, 17]). *Suppose (X, z) on the open set U of H is C^2 and complete and that the approximations (X, z_π) are also complete. Then as mesh $\Pi \to 0$ the solutions to (X, z_π) converge to the corresponding solutions of (X, z) in measure uniformly in $t \in [0, T]$ for the given time interval $[0, T]$, i.e. for $\delta > 0$*

$$P\{\sup_{0 \leq t \leq T} \|\xi^\pi_{t,x} - \xi_{t,x}\|_H > \delta\} \to 0$$

as mesh $\pi \to 0$, where $\xi^\pi_{t,x}$ refers to the solution to (X, z_π) starting from x in U. //

3. The Lift to the Diffeomorphism Group

3.1. Sobolev Spaces of H^s Maps

Let $C^r_0(\mathbb{R}^n; \mathbb{R}^p)$ denote the space of C^r functions f: $\mathbb{R}^n \to \mathbb{R}^p$ with compact support. For $s = 0, 1, 2, \ldots$ and f, g in $C^\infty_0(\mathbb{R}^n; \mathbb{R}^p)$ set

$$\langle f, g\rangle_s = \sum_{|\alpha| \leq s} \int_{\mathbb{R}^n} \langle D^\alpha f(x), D^\alpha g(x)\rangle_{\mathbb{R}^p}\, dx$$

where the sum is over multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $D^\alpha = \dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$ with $|\alpha| = \alpha_1 + \ldots + \alpha_n$. Then, for $s = 0, 1, 2, \ldots$ the Sobolev space $H^s(\mathbb{R}^n; \mathbb{R}^p)$ can be considered as the Hilbert space completion of $C^\infty_0(\mathbb{R}^n; \mathbb{R}^p)$ under the inner product $\langle\ ,\ \rangle_s$. There is a natural isomorphism

$$H^0(\mathbb{R}^n; \mathbb{R}^p) \cong L^2(\mathbb{R}^n; \mathbb{R}^p)$$

which we will take as an identification, and natural dense continuous inclusions

$$\ldots \subset H^{s+1} \subset H^s \subset \ldots \subset H^0 = L^2.$$

The following references may be useful for Sobolev space theory: [5, 30, 31, 32].

For U open in $\mathbb{R}^n$ a map f: $U \to \mathbb{R}^p$ is in H^s_{loc}, or $H^s_{\text{loc}}(U; \mathbb{R}^p)$, if ϕf is in $H^s(\mathbb{R}^n; \mathbb{R}^p)$ for all $\phi \in C^\infty_0(\mathbb{R}^n; \mathbb{R})$. Then $H^s_{\text{loc}}(U; \mathbb{R}^p)$ is given the topology determined by the seminorms $\{f \mapsto \|\phi f\|_s$: $\phi \in C^\infty_0\}$, i.e. the smallest topology such that $f \mapsto \phi f$ is continuous into H^s for each ϕ in C^∞_0. Give the space $C^r(\mathbb{R}^n; \mathbb{R}^p)$ of C^r maps the topology of uniform convergence on compacta of the first r derivatives. By the Sobolev embedding theorem we then have a continuous inclusion

$$H^s_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^p) \subset C^r(\mathbb{R}^n; \mathbb{R}^p) \qquad \text{for } s > \frac{n}{2} + r,$$

as well as the obvious one

$$C^r(\mathbb{R}^n; \mathbb{R}^p) \subset H^r_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^p).$$

From these we obtain

$$C^\infty(\mathbb{R}^n;\mathbb{R}^p)=\bigcap_{s=0}^{\infty} H^s_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^p)$$

both as a set and as a topological space.

For X a function of compact support on $\mathbb{R}^n$ and $s>\frac{n}{2}$ set

$$H^s_X(\mathbb{R}^n;\mathbb{R}^p)=\{f\in H^s(\mathbb{R}^n;\mathbb{R}^p) \text{ with } \operatorname{supp} f\subset \operatorname{supp} X\}.$$

Since $s>\frac{n}{2}$ the evaluation maps $f\mapsto f(x)$ are continuous on H^s so H^s_X is a well defined closed subspace of H^s, or equivalently of H^s_{loc}.

For $s>\frac{n}{2}+1$ set

$$\mathscr{D}^s_X=\{f\colon \mathbb{R}^n\to\mathbb{R}^n\colon f \text{ is a } C^1 \text{ diffeomorphism, } f\in H^s_{\mathrm{loc}}, \text{ and } f|\mathbb{R}^n-\operatorname{supp} X\equiv \mathrm{Id}\}$$

where Id denotes the identity map. Since diffeomorphisms are open in the C^1 topology (see e.g. [28] or [15]), $\mathscr{D}^s_X$ is an open subset of the affine subspace Id $+H^s_X(\mathbb{R}^n;\mathbb{R}^n)$ of H^s_{loc}. In particular it can be identified with an open subset of the Hilbert space H^s_X. This will enable us to talk of differentiable functions on $\mathscr{D}^s_X$ and stochastic dynamical systems on $\mathscr{D}^s_X$.

From now on take $s>\frac{n}{2}+1$. Since $\mathscr{D}^s_X$ can be identified with closed submanifolds of the diffeomorphism groups $\mathscr{D}^s(S^n)$, [10], or $\mathscr{D}^s(B)$ for suitable domains B of $\mathbb{R}^n$, [5], the following can be obtained from [5, 10, 29]: (However they are comparatively easy to prove directly.)

1. $\mathscr{D}^s_X$ is a topological group under composition
2. For all h in $\mathscr{D}^s_X$ right multiplication

$$R_h\colon \mathscr{D}^s_X\to\mathscr{D}^s_X \qquad R_h(f)=f\circ h$$

is C^∞.

3. For $k=0,1,2,\dots$ the composition map

$$\phi_k\colon \mathscr{D}^{s+k}_X\times\mathscr{D}^s_X\to\mathscr{D}^s_X \qquad \phi_k(f,h)=f\circ h$$

is C^k. [Indeed it is C^k as a map $H^{s+k}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^p)\times\mathscr{D}^s_X\to H^s_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^p)$].

4. For $k=1,2,\dots$, inversion, considered as a map

$$\mathscr{I}_k\colon \mathscr{D}^{s+k}_X\to\mathscr{D}^s_X \qquad h\mapsto h^{-1}$$

is C^k with derivative given by

$$D\mathscr{I}_k(h)(f)(x)=-Dh(h^{-1}(x))^{-1}(f\circ h^{-1}(x))=-Dh^{-1}(x)(f\circ h^{-1}(x))$$
$$h\in\mathscr{D}^{s+k}_X,\ f\in H^{s+k}_X,\ x\in\mathbb{R}^n.$$

3.2. The Lift $(\tilde{X}, z)$ on $\mathscr{D}^s$

Suppose now that $X\colon \mathbb{R}^n \to \mathbb{L}(\mathbb{R}^{m+1}; \mathbb{R}^n)$ has compact support and lies in $H^{s+2}(\mathbb{R}^n; \mathbb{L}(\mathbb{R}^{m+1}; \mathbb{R}^n))$ where, as always, $s > \frac{n}{2} + 1$. Then $X(-)\,e \in H_X^{s+2}$ for all e in $\mathbb{R}^{m+1}$. Define $\tilde{X}\colon \mathscr{D}_X^s \to \mathbb{L}(\mathbb{R}^{m+1}; H_X^s)$ by

$$\tilde{X}(h)(e)\,x = X(h(x))(e) \qquad x \in B.$$

Then

$$\tilde{X}(h)(e) = \phi_2(X(-)\,e, h)$$

so that $\tilde{X}$ is C^2 by 3 above. Equivalently we could define $\tilde{X}$ by

$$\tilde{X}(\mathrm{Id})(e) = X(-)(e) \text{ and right invariance:}$$

$$\tilde{X}(h)(e) = DR_h(\mathrm{Id})(X(-)\,e) \qquad h \in \mathscr{D}_X^s.$$

Theorem 3.2. *The stochastic dynamical system $(\tilde{X}, z)$ on $\mathscr{D}_X^s$ is complete. If ξ_t: $\Omega \to \mathscr{D}_X^s$ is the solution with $\xi_0 = \mathrm{Id}$ then for each $x \in \mathbb{R}^n$ the $\mathbb{R}^n$-valued process $\xi_t(-)(x)$ is a version of the solution to (X, z) starting from x. Thus ξ is a flow for (X, z).*

Proof. Since $\tilde{X}$ is C^2, solutions of $(\tilde{X}, z)$ exist for some positive time and are unique. To show completeness take an open neighbourhood U of Id in $\mathscr{D}_X^s$ and define

$$\theta\colon U \to H_X^s$$

by $\theta(h) = h - \mathrm{Id}$.

For h in $\mathscr{D}_X^s$ set

$$U_h = R_h(U) \subset \mathscr{D}_X^s \quad \text{and} \quad \theta_h = \theta \circ R_{h^{-1}}\colon U_h \to H_X^s.$$

Since $R_{h^{-1}}$ is C^∞ so is θ_h. By taking U sufficiently small we see that $\{(U_h, \theta_h)\}_{h \in \mathscr{D}_X^s}$ is a uniform cover for $\tilde{X}$ and we can apply Theorem 2.2.

To show that $\xi_t(-)\,x$ is a solution to (X, z) consider the evaluation map

$$\mathrm{ev}_x\colon \mathscr{D}_X^s \to \mathbb{R}^n$$

given by $\mathrm{ev}_x(h) = h(x)$.

This is the restriction of a bounded linear map and is therefore C^∞ with

$$D(\mathrm{ev}_x)(h)\,[\tilde{X}(h)\,e] = X(h(x))\,e \qquad h \in \mathscr{D}_X^s.$$

Corollary 3.2. *If $X\colon \mathbb{R}^n \to \mathbb{L}(\mathbb{R}^{m+1}; \mathbb{R}^n)$ has compact support and is in H_{loc}^{s+2} for some $s > \frac{n}{2} + 1$ (or in particular is C^∞) then (X, z) has a flow defined for all positive time. The flow is an H_{loc}^s (respectively C^∞) diffeomorphism of $\mathbb{R}^n$ which is the identity outside of the support of X. As a function of time t it is continuous into the H^s (or C^∞) topology.*

Proof. This is simply a rephrasing of the theorem. The C^∞ case follows by the remarks in § 3.1. //

4. Properties of the Flow When X Has Compact Support

The results of this section are preliminary to those given later for general X. Here we assume throughout that X has compact support and is in H^{s+2}_{loc} for some $s>n/2+1$. Corresponding C^∞ results follow automatically.

4.1. Uniformity of Piecewise Linear Approximation

Proposition 4.1. *For each partition Π of a fixed time interval $[0, T]$ and each $\omega\in\Omega$ let $\xi^\pi_t(\omega)$: $\mathbb{R}^n\to\mathbb{R}^n$, $0\leqq t\leqq T$, be the flow of the ordinary differential equation*

$$\dot{x}_t=X(x_t)\frac{dz_\pi}{dt}(\omega).$$

Then, as mesh $\Pi\to 0$ so ξ^π_t converges in measure in the H^s topology uniformly in $t\in[0, T]$, with limit the flow ξ_t of (X, z). i.e. if $\delta>0$ then

$$P\{\sup_{0\leqq t\leqq T}\|\xi^\pi_t-\xi_t\|_s>\delta\}\to 0 \qquad \text{as mesh } \Pi\to 0$$

where $\|\ \|_s$ denotes the H^s norm.

In particular $\xi^\pi_t(-)x$ converges in measure uniformly in $x\in\mathbb{R}^n$ to $\xi_{t,x}$.

Proof. Since ξ^π can be identified with the solution to $(\tilde{X}, z_\pi)$ starting from Id in $\mathscr{D}^s_X$ as in Theorem 3.2, the result follows by Theorem 2.3. //

4.2. The Generalized Itô Formula

The results given here are essentially special cases of those of Bismut [2, 3], see also Kunita [20] and Ustunel [34].

Proposition 4.2. *Suppose $s>\frac{n}{2}+2$. Let ρ_t be a continuous semi-martingale with values in $\mathbb{R}^n$. Suppose X has compact support and is in H^{s+2}_{loc}. If ξ is the flow of (X, z) then the process*

$$\eta_t=\xi_t\,\rho_t$$

i.e.

$$\eta_t(\omega)=\xi_t(\omega)\,(\rho_t(\omega))$$

is an $\mathbb{R}^n$-valued semi-martingale with

$$d\eta_t=D\,\xi_t(\rho_t)\circ d\rho_t+X(\eta_t)\circ dz_t. \tag{1}$$

Remarks 4.2. (a) By (1) we mean

$$\begin{aligned}\eta_t=\rho_0&+\int_0^t D\,\xi_t(\rho_t)\,d\rho_t+\tfrac{1}{2}\int_0^t DX(\eta_t)\,(D\,\xi_t(\rho_t)\,d\rho_t)\,dz_t\\&+\tfrac{1}{2}\int_0^t D^2\,\xi_t(\rho_t)\,(d\rho_t, d\rho_t)+\int_0^t X(\eta_t)\,dz_t\\&+\tfrac{1}{2}\int_0^t DX(\eta_t)\,(X(\eta_t)\,dz_t)\,dz_t \quad \text{a.s.}\end{aligned} \tag{2}$$

see (b) below. Taking this to be the *definition* of equation (1) means that there is no need to spend time discussing the existence of the Stratonovich integral $\int_0^t D\xi_t(\rho_t)\circ d\rho_t$. (In fact given one extra degree of differentiability our argument below would apply with $(\xi_t(\rho_t), D\xi_t(\rho_t))$ replacing $\xi_t(\rho_t)$ to show that $D\xi_t(\rho_t)$ is a semi-martingale; more generally we could use Corollary 1D Chap. VIII of [13]).

Remark 4.2. (b) Since differentiation gives a map

$$D\colon \mathscr{D}_X^s \to H_X^{s-1}(\mathbb{R}^n; \mathbb{L}(\mathbb{R}^n; \mathbb{R}^n)) \qquad s>\frac{n}{2}+1$$

which is continuous linear, and hence C^2, Itô's formula shows that $D\xi_t$ considered as a process with values in

$$H_X^{s-1}(\mathbb{R}^n; \mathbb{L}(\mathbb{R}^n; \mathbb{R}^n))$$

satisfies

$$d(D\xi_t) = DX(\xi_t(-))\,(D\xi_t(-)\,(-))\circ dz_t.$$

Proof. Since $s>\frac{n}{2}+2$ the map

$$E\colon \mathscr{D}_X^s \times \mathbb{R}^n \to \mathbb{R}^n \qquad E(h,x)=h(x)$$

is C^2 with derivatives

$$DE(h,x)\,(f,v)=f(x)+Dh(x)\,v$$

and

$$D^2E(h,x)\,((f',v'),(f,v)) = DF(x)\,v' + Df'(x)\,v + D^2h(x)\,(v',v),$$

for $(h,x)\in\mathscr{D}_X^s\times\mathbb{R}^n$, and (f,v) and (f',v') in $H_X^s\times\mathbb{R}^n$.

The result follows from the Itô formula, using Remark 4.2(b) above, since $\eta_t = E(\xi_t, \rho_t)$. However the most easily available version of the Itô formula in this generality, [26], Remarks 3.9(2), Chap. 2, requires D^2E to be uniformly continuous on bounded subsets of $\mathscr{D}_X^s\times\mathbb{R}^n$ (more precisely we shall work on the linear space $H_X^s\times\mathbb{R}^n$). In fact this is so since it holds for the first two terms in the expression for D^2E, by local compactness of $\mathbb{R}^n$, and the last term

$$(h,x)\mapsto D^2h(x) \qquad H_x^s\times\mathbb{R}^n\to\mathbb{L}(\mathbb{R}^n,\mathbb{R}^n;\mathbb{R}^n)$$

mapping into the space of bilinear maps, factorizes

$$H_X^s\times\mathbb{R}^n\to C^2(\mathbb{R}^n;\mathbb{R}^n)\times\mathbb{R}^n\to\mathbb{L}(\mathbb{R}^n,\mathbb{R}^n;\mathbb{R}^n)$$

where the first map comes from the natural inclusion $H_X^s\subset C^2$ and the second is the map $(h,x)\mapsto D^2h(x)$, which is continuous. However by the Rellich theorem the inclusion $H_X^s\to C^2$ is compact i.e. sends bounded sets to sets with compact closure [30, 32]. //

4.3. The Inverse of the Flow

Proposition 4.3 (c.f. Kunita [21]). *Suppose that* $s>\frac{n}{2}+2$. *Then the inverse* ξ_t^{-1} *of the flow* ξ_t *of* (X,z) *satisfies*

$$d(\xi_t^{-1}(x))=-D\,\xi_t^{-1}(x)\,X(x)\circ dz_t.$$

Proof. The map $\mathscr{I}_2:\mathscr{D}_X^s\to\mathscr{D}_X^{s-2}$ given by $\mathscr{I}_2(h)=h^{-1}$ is C^2 with

$$D\mathscr{I}_2(h)(f)(x)=-Dh^{-1}(x)(f\circ h^{-1}(x)) \qquad h\in\mathscr{D}_X^s,\ f\in H_X^s,\ x\in\mathbb{R}^n.$$

Since $s>\frac{n}{2}+2$ the evaluation map

$$\mathrm{ev}_x:\ \mathscr{D}_X^{s-2}\to\mathbb{R}^n$$

is continuous affine. The result follows by applying Itô's formula to $\mathrm{ev}_x\circ\mathscr{I}_2$. //

5. The General Case

5.1. The Partial Flow

When X is merely in H_{loc}^{s+2} the system may not have a flow defined for all time, for example the solutions may explode. However there is a partially defined flow as we shall show. First we recall a well known local uniqueness result.

Lemma 5.1. *Consider two stochastic dynamical systems* (X_1,z) *and* (X_2,z) *on* $\mathbb{R}^n$. *Suppose that both* X_1 *and* X_2 *are* C^2 *and agree on a bounded open subset* U *of* $\mathbb{R}^n$. *Let* $\xi_{t,x}^i$ *denote the solution from* x *of* (X_i,z) *and let* τ_x^i *be its first exit time from* U. *Then if* $x\in\mathbb{R}^n$

$$\tau_x^1=\tau_x^2 \qquad \text{a.s.}$$

and

$$\xi_{t,x}^1=\xi_{t,x}^2 \qquad \text{a.s. } \textit{for } t<\tau_x^1.$$

Proof. See for example [14] or [13]. //

The following result (in the C^r case) is due to Kunita [19]. The method we use is essentially that used in [13] for the manifold case (where slightly more refined results are proved). See also [27].

Theorem 5.1. *Consider the stochastic dynamical system* (X,z) *on* $\mathbb{R}^n$ *where* X *is in* H_{loc}^{s+2} *for some* $s>\frac{n}{2}+1$. *Then there is an explosion time map* τ: $\mathbb{R}^n\times\Omega\to[0,\infty]$ *and a partially defined flow* ξ *for* (X,z) *such that if*

$$M(t)(\omega)=\{x\in\mathbb{R}^n:\ t<\tau(x,\omega)\} \qquad \omega\in\Omega$$

then for all ω *in* Ω:

(i) $M(t)(\omega)$ *is open in* $\mathbb{R}^n$ i.e. $\tau(-,\omega)$ *is lower semi-continuous.*

(ii) $\xi_t(\omega)x$ *is defined for* $x \in M(t)(\omega)$ *and*

$$\xi_t(\omega)\colon\ M(t)(\omega) \to \mathbb{R}^n$$

is in H^s_{loc} *and is a diffeomorphism onto an open subset of* $\mathbb{R}^n$ *with inverse in* H^s_{loc}.

(iii) *For each* x *in* $\mathbb{R}^n$ *the random variable* $\tau(x)$ *is a stopping time and* $\{\xi_t(x)\}_{0 \leqq t < \tau(x)}$ *is a maximal solution to* (X, z) *from* x. *Moreover for any compact set* K *in* $\mathbb{R}^n$ *set*

$$\tau(K)(\omega) = \inf\{\tau(x)(\omega)\colon\ x \in K\}.$$

Then on $\{\tau(K) < \infty\}$ *we have almost surely*

$$\sup\{|\xi_t(x)(\omega)|\colon\ x \in K\} \to \infty \qquad \text{as } t \to \tau(K)(\omega).$$

(iv) *The map* $\alpha \mapsto \xi_\alpha(\omega)$ *of* $[0, t]$ *into* $H^s_{\text{loc}}(M(t)\,\omega; \mathbb{R}^n)$ *is continuous for each* t *and* ω.

(v) *When* X *is* C^∞ *then* $\xi_t(\omega)$ *can be chosen to be* C^∞ *on* $M(t)(\omega)$ *and so that in* (iv) *the map is continuous into the* C^∞ *topology.*

Proof. Take C^∞ maps $\lambda_r\colon \mathbb{R}^n \to [0, 1]$ with

$$\operatorname{supp} \lambda_r \subset B_{r+1}$$

and

$$\lambda_r | B_r \equiv 1 \qquad r = 1, 2, \ldots.$$

Set

$$X_r = \lambda_r X.$$

By Corollary 3.2 the systems (X_r, z) have flows ξ^r in $\mathscr{D}^s_r$ (where we write $\mathscr{D}_r$ for $\mathscr{D}_{X_r}$). Let $\tau^r(x)$ be the first exit time of $\xi^r(x)$ from B_r and set

$$M^r_t(\omega) = \{x \in \mathbb{R}^n\colon\ t < \tau^r(x)(\omega)\} \qquad \omega \in \Omega.$$

By local uniqueness, Lemma 5.1, for each r and each x in $\mathbb{R}^n$ we have

$$\tau^r(x) \leqq \tau^{r+1}(x) \qquad \text{a.s.}$$

and

$$\xi^r_t(x) = \xi^{r+1}_t(x) \qquad \text{for } 0 \leqq t \leqq \tau^r(x) \quad \text{a.s.}$$

The separability of $\mathbb{R}^n$ and the continuity of the flows ensures that these inequalities hold for a set of full measure Ω', say, in Ω which is independent of $x \in \mathbb{R}^n$, as well as independent of the natural number r. On Ω' set

$$\tau(x) = \sup_r \tau^r(x) \leqq \infty,$$

and

$$\xi_t(x) = \lim_{r \to \infty} \xi^r_t(x) \qquad 0 \leqq t < \tau(x).$$

Since $\xi^r_t(x)$ is independent of r for r sufficiently large when $t < \tau(x)$, this limit exists on Ω'. Outside Ω' set $\xi_t(x) = x$ and $\tau(x) = \infty$: clearly all our assertions hold on this set so from now on we restrict ourselves to Ω'.

Clearly $\xi_t(x)$ is a solution of (X, z) and since $\overline{\lim}\, |\xi_t(x)| = \infty$ as $t \to \tau(x)$ whenever $\tau(x) < \infty$ it is a maximal solution, giving the first part of assertion (iii) after noting that the supremum of a sequence of stopping times is a stopping time.

For the second part observe that $\tau_K(\omega) = \tau(x, \omega)$ for some x in K, by the lower semi-continuity of τ and compactness of K. But since it is a maximal solution $\xi_t(x) \to \infty$ as $t \to \tau(x)$ almost surely on $\{\tau(x) < \infty\}$, as is well known (e.g. Corollary 6.2, Chap. VII of [13]).

Since $M(t)(\omega) = \bigcup_{r=1}^{\infty} M_t^r(\omega)$ assertion (i) holds. Also from this we see that for every compact subset K of $M(t)(\omega)$ there is an r with $\xi_s(-)(\omega)|K \equiv \xi_s^r(-)(\omega)|K$ for $0 \leqq s \leqq t$. From this we have assertion (iv) and the fact that $\xi_t(\omega)$ is a local diffeomorphism on $M(t)(\omega)$ in H_{loc}^s. Moreover it is seen to be injective by taking K to consist of any 2 given points. Thus (ii) is true.

Finally (v) follows since the C^∞ topology is the limit of the H_{loc}^s topology as $s \to \infty$. //

Remarks 5.1 (a). The second part of assertion (iii) of the theorem is a maximality condition on τ. It is enough to ensure uniqueness of the pair (ξ, τ). In fact suppose (ξ', τ') also satisfy assertions (i) to (iv). By separability of $\mathbb{R}^n$ the flows agree up to the time $\inf\{\tau(x), \tau'(x)\}$, after discarding a negligible set independent of x. To show that $\tau = \tau'$ a.s. let $\omega \in \Omega$ be fixed and set $N(\omega) = \{x\colon \tau'(x) < \tau(x)(\omega)\}$. Then if $B_r \cap N(\omega) \neq \emptyset$ we have

$$\tau'(B_r)(\omega) < \tau(B_r)(\omega).$$

It follows from (iii) that ω lies in some null set, Ω_r say. Then

$$\{\omega\colon N(\omega) \neq \emptyset\} \subset \bigcup_{r=1}^{\infty} \Omega_r$$

and so has probability zero. Thus (ξ, τ) and (ξ', τ') agree outside of one set of measure zero.

(b) Note the "almost surely" in (iii), (thanks to the referee). The "proof" in [13] only yields the rather obvious sure statement: $\overline{\lim}\, |\xi_t(\omega)(x)| = \infty$ as $t \to \tau(x, \omega)$ whenever $\tau(x, \omega) < \infty$.

Theorem 5.1 raises the following questions:

A. When can the explosion time map τ be chosen to have $\tau(x) \equiv \infty$ all $x \in \mathbb{R}^n$? If so we will say that (X, z) is *strongly complete*; the term *strictly conservative* has also been used.

B. Given strong completeness when is ξ_t almost surely surjective for all t?

On $\mathbb{R}$ completeness implies strong completeness, as observed by Kunita [21]. For higher dimensions this is not so: the simplest example [11, 13] is of the system restricted to $\mathbb{R}^2 - \{0\}$ given by (X, B) where B is a 2-dimensional Brownian motion and $X\colon \mathbb{R}^2 - \{0\} \to \mathbb{L}(\mathbb{R}^2; \mathbb{R}^2)$ is simply $X(z)e = e$ for all z in $\mathbb{R}^2 - \{0\}$. To get an example on all of $\mathbb{R}^2$ we only need to apply the inversion $z \mapsto \frac{1}{z}$, in complex notation. The resulting system $(\tilde{X}, B)$ on $\mathbb{R}^2$ with

$$\tilde{X}(x, y)(e_1, e_2) = (-(x^2 - y^2)e_1 + 2xye_2,\ -(x^2 - y^2)e_2 - 2xye_1)$$

is complete but not strongly so. In [6] Carverhill constructs a stochastic differential equation on $\mathbb{R}^2$ with the same associated infinitesimal generator as $(\tilde{X}, B)$ but which is strongly complete.

Bessel processes furnish good examples of one dimensional processes which are strongly complete but do not have the surjective property. These are the radial components of n-dimensional Brownian motions where $n \geqq 2$, and are defined on $(0, \infty)$ rather than $\mathbb{R}$. However their logarithms give a process on $\mathbb{R}$. The lack of surjectivity follows from the results of [21] (or [12]).

The strong completeness of Itô equations given global Lipschitz conditions was shown by Blagovescenskii and Freidlin in 1961 [4], using an extension of Kolmogorov's criterion for the existence of sample continuous versions of stochastic processes. This method was also used by Baxendale [1], see also [19] and [11], [13]. Surjectivity also holds in this case and the fact that there is a flow of homeomorphisms is proved, most beautifully, by Kunita in [19], using ideas of Varadhan. Malliavin gave a completely different approach in discussing the existence of flows consisting of diffeomorphisms see [23] or [24], and recent interest in the problem was mainly stimulated by his work. The treatments in the books of Ikeda and Watanabe [17] and Bismut [3] are close to those of Malliavin. A careful analysis when the coefficients have bounded derivatives was given by Bismut [2]. See also Meyer [27].

Warning. The statement of Theorem 2.4, Chap. V of [17] seems to be incorrect: see the examples in [13] Chap. VIII.

5.2 Uniform Convergence of the Piecewise Linear Approximations

Theorem 5.2. (Elworthy [13], see also Bismut [3], Ikeda and Watanabe [17], Baxendale [0]).

Fix a time interval $[0, T]$. Consider (X, z) on $\mathbb{R}^n$ with $X \in H^{s+2}_{\text{loc}}$ and let ξ and τ be its partial flow and explosion time map as in Theorem 5.1. For partitions Π of $[0, T]$ let ξ^π be the partial flow of (X, z_π).

Then ξ^π_t converges to ξ_t in H^s_{loc} uniformly in each subinterval $[0, S]$ of $[0, T]$, in measure, in the sense that if U is open in $\mathbb{R}^n$ and $\delta > 0$ then

$$P\{\bar{U} \subset M(S) \,\&\, \sup_{0 \leqq t \leqq S} \|\xi^\pi_t | U - \xi_t | U\|' > \delta\} \to 0 \qquad \text{as mesh } \Pi \to 0$$

where $\| \ \|'$ is any continuous seminorm on $H^s_{\text{loc}}(U)$ and $M(S)(\omega)$ is as in 5.1, and the norm is taken to be infinite when ξ^π_t is not defined on U.

In particular there is convergence of $\xi^\pi_t(x)$ to $\xi_t(x)$ uniformly in $t \in [0, S]$ and on compacta in $\mathbb{R}^n$: if K is compact and $\delta > 0$ then

$$P\{K \subset M(S) \,\&\, \sup\{\|\xi^\pi_t(x) - \xi_t(x)\|_{\mathbb{R}^n} \colon 0 \leqq t \leqq S, x \in K\} > \delta\} \to 0$$
$$\text{as mesh } \Pi \to 0,$$

with the same convention if $\xi^\pi_t(x)$ is not defined.

When X is C^∞ the convergence in x is that of convergence uniformly on compacta of derivatives of all orders.

Proof. The measurability of the events considered in the statement of the theorem, and below, follows from the general theory in [8] Theorem No. 44. It is also discussed in [13] where there is an alternative proof of the result we are proving.

We shall use the notation of the proof of Theorem 5.1. By definition of the topology of H^s_{loc} we can assume that $\bar{U}$ is compact. We will ignore the negligible event $\Omega - \Omega'$. Now

$$\{\omega\colon \bar{U} \subset M(S)(\omega)\} = \bigcup_{r=1}^{\infty} \{\omega\colon \bar{U} \subset M^r_S(\omega)\}$$

and the right hand side is an increasing union. The convergence in H^s_{loc} will therefore be assured if we can show that

$$P\{\bar{U} \subset M^{r-1}_S \,\&\, \sup_{0 \leqq t \leqq S} \|\xi^\pi_t | U - \xi^r_t | U\|' > \delta\} \to 0 \quad \text{as mesh } \Pi \to 0 \tag{*}$$

for all sufficiently large r, since $\xi^r_t | U = \xi_t | U$ for $0 \leqq t \leqq S$ if $U \subset M^{r-1}_S$.

Set

$$\Theta(r, \Pi) = \{\sup_{0 \leqq t \leqq S} \|\xi^{r,\pi}_t - \xi^r_t\|_{C^0} \leqq 1\},$$

where $\xi^{r,\pi}$ is the flow of (X_r, z_π). On $\Theta(r, \pi) \cap \{\bar{U} \subset M^{r-1}_S\}$ we have

$$\xi^{r,\pi}_t(\bar{U}) \subset B_r \qquad 0 \leqq t \leqq S$$

and therefore

$$\xi^\pi_t | \bar{U} = \xi^{r,\pi}_t | \bar{U} \qquad 0 \leqq t \leqq S.$$

Theorem 4.1 then yields

$$P(\{\bar{U} \subset M^{r-1}_S\} \cap \Theta(r, \Pi) \cap \{\sup_{0 \leqq t \leqq S} \|\xi^\pi_t | U - \xi^r_t | U\|' > \delta\}) \to 0 \quad \text{as mesh } \Pi \to 0$$

giving (*) since $P(\Theta(r, \Pi)) \to 1$ as mesh $\Pi \to 0$ by Theorem 4.1. The other assertions follow as usual from the Sobolev theorems. //

For Malliavin's approximations to the flow by flows of (X, z_ε) where z_ε is the 'regularization'

$$z_\varepsilon(t) = \frac{1}{\varepsilon} \int_0^\varepsilon z(t+s)\, u\left(\frac{s}{\varepsilon}\right) ds$$

for a suitable bump function u see Malliavin [23], [24]. For approximation by flows of generalized Ornstein-Uhlenbeck processes see Dowell [9].

5.3. Generalized Itô Formula

Continuing with the same notation as in Theorem 5.1 and its proof set

$$M(\omega) = \{(x, t) \in \mathbb{R}^n \times [0, \infty); \quad t < \tau(x)(\omega)\}$$

and

$$M^r(\omega) = \{(x, t) \in \mathbb{R}^n \times [0, \infty); t < \tau^r(x)(\omega)\} \qquad r = 1, 2, \ldots.$$

Then each $M^r(\omega)$ is open in $\mathbb{R}^n \times [0,\infty)$ and so therefore is $M(\omega)$ since $M(\omega) = \bigcup_{r=1}^{\infty} M^r(\omega)$.

If ρ is a sample continuous adapted process with values in $\mathbb{R}^n$ let $\tau_\rho(\omega)$ and $\tau^r_\rho(\omega)$ be the first exit times of $\{(\rho_t(\omega), t)\colon t \geqq 0\}$ from $M(\omega)$ and $M^r(\omega)$ respectively. Then τ^r_ρ is the first exit time of $\xi^r \cdot \rho$ from B_r and $\tau_\rho = \sup_r \tau^r_\rho$. Therefore τ_ρ is a stopping time.

We can now extend the generalized Itô formula of Bismut [2], [3] and of Kunita [20]. As we saw in § 4.2 it is essentially an integration by parts formula. See also Ustunel [34].

Theorem 5.3. *Let ρ_t be a continuous semi-martingale with values in $\mathbb{R}^n$. Suppose X is in H^{s+2}_{loc} where $s > \frac{n}{2} + 2$. If ξ is he partial flow of (X, z) then the process*

$$\eta_t = \xi_t \cdot \rho_t \qquad t < \tau_\rho$$

is an $\mathbb{R}^n$-valued semi-martingale with

$$d\eta_t = D\xi_t(\rho_t) \circ d\rho_t + X(\eta_t) \circ dz_t.$$

Proof. Set $\eta^r_t = \xi^r_t \cdot \rho_t$. Then $\eta_t = \eta^r_t$ for $t < \tau^r_\rho$ and $\tau^r_\rho \uparrow \tau_\rho$ as $r \to \infty$. The result follows by applying Proposition 4.2 to η^r_t, and observing that $D\xi^r_t(\rho_t) = D\xi_t(\rho_t)$ for $t < \tau^r_\rho$. //

5.4. The Backward Equation for the Inverse: Kunita's Surjectivity Criterion

For $0 \leqq s \leqq t < \infty$ let $\mathscr{F}_{st}$ be the completion of the σ-algebra generated by $z(t') - z(s')$ for $s \leqq s' \leqq t' \leqq t$. For fixed $T > 0$ define the $\mathbb{R}^{m+1}$ valued process $z^\vee$ by

$$z^\vee(t) = z(T-t) - z(T) \qquad 0 \leqq t \leqq T.$$

Then, given X as before, the system $(X, z^\vee)$ is of the same type as we have been considering, apart from the fact that our time interval is restricted to $[0, T]$, and that the filtration is $\{\mathscr{F}_{T-t,T}\colon 0 \leqq t \leqq T\}$.

In our preliminary lemma we follow essentially the same method as used by Malliavin when he discussed the diffeomorphism property of these flows [23, 24]; and it is a special case of what he obtained.

Lemma 5.4. *Suppose X is in H^{s+2}_{loc} and has compact support. Let $\{\xi^\vee_t\colon 0 \leqq t \leqq T\}$ denote the flow of $(X, z^\vee)$ and $\{\xi_t\colon 0 \leqq t < \infty\}$ that of (X, z). Then, with probability one,*

$$\xi^\vee_t \cdot \xi_T = \xi_{T-t} \qquad 0 \leqq t \leqq T$$

and in particular

$$\xi^\vee_T = \xi_T^{-1}.$$

Proof. For partitions Π of $[0, T]$ let $\xi^{\vee\pi}$ be the flow of $(X, (z_\pi)^\vee)$ where

$$(z_\pi)^\vee(t) = z_\pi(T-t) - z(T),$$

and let $\Pi^\vee$ be the partition $0 \leqq T - t_m \leqq \ldots \leqq T - t_1$ if Π is given by $0 \leqq t_1 \leqq \ldots \leqq t_m \leqq T$.

Observe that $(z^\vee)_{\pi^\vee} = (z_\pi)^\vee$ and mesh $\Pi^\vee \to 0$ as mesh $\Pi \to 0$. Therefore $\xi^{\vee\pi}$ converges to $\xi^\vee$ in $\mathscr{D}^s_x$ uniformly in $[0, T]$ as mesh $\Pi \to 0$ as in Proposition 4.1. Now

$$\xi_t^{\vee\pi} \cdot \xi_T^\pi = \xi_{T-t}^\pi \qquad 0 \leqq t \leqq T$$

since

$$\frac{d}{dt} \xi_{T-t}^\pi = -\tilde{X}(\xi_{T-t}^\pi)\, \dot{z}_\pi(T-t) = \tilde{X}(\xi_{T-t}^\pi) \frac{d}{dt}(z_\pi)^\vee$$

using the lifted system $\tilde{X}$ on $\mathscr{D}^s_X$. Also composition

$$\mathscr{D}^s_X \times \mathscr{D}^s_X \to \mathscr{D}^s_X$$

is continuous and so taking limits as mesh $\Pi \to 0$ by Proposition 4.1 we obtain, with probability one,

$$\xi_t^\vee \cdot \xi_T = \xi_{T-t} \qquad \text{all } 0 \leqq t \leqq T$$

as required. //

The following extension of Kunita [21], Theorem 2, is due to Carverhill, it was proved by Kunita under the assumption that both (X, z) and $(X, z^\vee)$ are strongly complete, and with X of class C^5 and extended to include X of class C^3 in [22]. See Malliavin [23, 24] and Bismut [2, 3] for earlier results. As with all the main results we give it holds equally well when $\mathbb{R}^n$ is replaced by an arbitrary smooth σ-compact manifold, with essentially the same proof.

Theorem 5.4. *Suppose that X is in H^{s+2}_{loc}. Let $\{\xi_t\colon 0 \leqq t < \infty\}$ and $\{\xi_t^\vee\colon 0 \leqq t \leqq T\}$ denote the partial flows of (X, z) and $(X, z^\vee)$ respectively, where $T > 0$ is fixed. Set*

$$M(t)(\omega) = \{x \in \mathbb{R}^n\colon t < \tau(x)\,\omega\}$$

and

$$M^\vee(t)(\omega) = \{x \in \mathbb{R}^n\colon t < \tau^\vee(x)\,\omega\} \quad \omega \in \Omega, \qquad 0 \leqq t \leqq T$$

where τ and $\tau^\vee$ are the explosion time maps of ξ and $\xi^\vee$. Then with probability one, $\xi_T(\omega)$ maps the open set $M(T)(\omega)$ as an H^s diffeomorphism onto the open set $M^\vee(T)(\omega)$ and its inverse is $\xi_T^\vee(\omega)$.

Furthermore, with probability one,

$$\xi_t^\vee \cdot \xi_T = \xi_{T-t} | M(T) \qquad \textit{for all } 0 \leqq t \leqq T,$$

In particular, for every T, ξ_T is almost surely a diffeomorphism of $\mathbb{R}^n$ onto $\mathbb{R}^n$ if and only if both (X, z) and $(X, z^\vee)$ are strongly complete.

Proof. Let ξ^r and $\xi^{\vee r}$ be the flows of (X_r, z) and $(X_r, z^\vee)$ for X_r as in the proof of Theorem 5.1. As in that proof let τ^r and $\tau^{\vee r}$ be their exit time maps from B_r

and set

$$M_T^r = \{x \in \mathbb{R}^m \colon \tau^r > T\}$$

and

$$M_T^{\vee r} = \{x \in \mathbb{R}^m \colon \tau^{\vee r} > T\}.$$

We can throw away one negligible set to assume that

$$\xi_t | M_T^r = \xi_t^r \qquad 0 \leqq t \leqq T$$

$$\xi_t^\vee | M_T^{\vee r} = M_T^{\vee r} \qquad 0 \leqq t \leqq T$$

$$M(T)(\omega) = \bigcup_r M_T^r(\omega), \qquad M^\vee(T)(\omega) = \bigcup_r M_T^{\vee r}(\omega)$$

and, by Lemma 5.4, $\xi_T^{\vee r} \cdot \xi_T^r = \xi_{T-t}^r$ $0 \leqq t \leqq T$, $r = 1, 2, 3, \ldots$.

For $x \in M(T)(\omega)$ we can therefore choose r with $x \in M^r(T)(\omega)$. Set $y = \xi_T(\omega)(x)$. Then, for $0 \leqq t \leqq T$,

$$\xi_t^{\vee r}(\omega)(y) = \xi_t^{\vee r}(\omega) \circ \xi_T^r(\omega)(x) = \xi_{T-t}^r(\omega)(x) \in B_r.$$

It follows that y is in $M_T^{\vee r}(\omega)$ and therefore

$$\xi_r^{\vee r}(\omega)(y) = \xi_t^\vee(\omega)(y) \qquad 0 \leqq t \leqq T.$$

Consequently $y \in M^\vee(T)(\omega)$ and

$$\xi_t^\vee(\omega)(y) = \xi_{T-t}(\omega)(x) \qquad 0 \leqq t \leqq T$$

as required. //

Corollary 5.4 (Kunita [21], Theorem 3, in the C^5 case, see also [22] for the C^2 case). *Consider a stochastic differential equation on* $\mathbb{R}^n$

$$dx_t = X_0(x_t) \circ dB_t + A(x_t)\, dt$$

where $A \colon \mathbb{R}^n \to \mathbb{R}^n$ *and* $X_0 \colon \mathbb{R}^n \to \mathbb{L}(\mathbb{R}^m; \mathbb{R}^n)$ *are in* H_{loc}^{s+2} *and* B_t *is a Brownian motion on* $\mathbb{R}^m$.

If this system is strongly complete then it has a flow which for each $T > 0$ *is surjective with probability one if and only if the adjoint system*

$$dy_t = X_0(y_t) \circ dB_t - A(y_t)\, dt$$

is strongly complete.

Proof. This follows from the theorem since:

(a) $t^\vee = -t$ and $B^\vee$ is again a Brownian motion up to time t_0, so the adoint system corresponds to our system $(X, z^\vee)$ with $z^\vee$ replaced by a process with the same distributions.

(b) if z and $\tilde{z}$ have the same distributions then the strong completeness of (X, z) implies that of $(X, \tilde{z})$. This is discussed in detail in Elworthy [13]. //

Remark 5.4. Even if ξ_T is only assumed to be almost surely a diffeomorphism of $\mathbb{R}^n$ onto itself for some given $T > 0$ then the arguments above show that the adjoint equation has a flow defined at least up to time T. By the time homogeneity in law of Brownian increments it follows, as in (b) immediately above, that the adjoint system also has a flow of solutions starting from time T and continuing to time $2T$. Composition of these gives a flow from time 0 to time $2T$, and iteration of this shows that the adjoint equation must again be strongly complete.

Note that the negligible set in Theorem 5.4 and its Corollary may depend on T. More work is needed to get surjectivity of ξ_t for all $t \geqq 0$, with probability one. This was done by Kunita, and we discuss it in the next section (Corollary 6.2). For the moment let us simply observe that if we set

$$\eta_{s,T} = \xi^{\vee}_{T-s} \qquad 0 \leqq s \leqq T$$

and allow T to vary then it will be enough to prove that there is a version of $\{\eta_{0,t}: t \geqq 0\}$ (consisting of continuous flows) which is continuous in t.

6. Backward and Forward Equations

All of this section was inspired by Kunita [21]. For $0 \leqq s \leqq t < \infty$ let $\mathscr{F}_{st}$ be the σ-algebra defined in § 5.6. We shall consider processes indexed by pairs (s, t) and will insist that at the index (s, t) they are $\mathscr{F}_{st}$-measurable.

6.1. Backward and Forward Continuity

Let $\{\xi_{ut}: 0 \leqq u \leqq t < \infty\}$ be the flow for (X, z) starting at times $u \geqq 0$. We want to find a version which is continuous in u and t, and for fixed t, $t = T$ say, to find a stochastic integral equation satisfied by $(\xi_{uT}(x): 0 \leqq u \leqq T\}$ for $x \in \mathbb{R}^n$. (The relevant filtration will be $\{\mathscr{F}_{uT}: 0 \leqq u \leqq T\}$).

Lemma 6.1. *Suppose X has compact support and is in H^{s+2}_{loc}. Let $\xi_{0,t} \equiv \xi_t$ be its flow, $0 \leqq t < \infty$. For each $u \geqq 0$ set*

$$\xi_{ut} = \xi_t \cdot \xi_u^{-1} \qquad u \leqq t < \infty.$$

Then $\{\xi_{ut}: u \leqq t < \infty\}$ is a version of the flow starting at time u, and for all $\omega \in \Omega$

(i) *$\xi_{ut}(\omega)$ is in $\mathscr{D}^s_X$ for all $0 \leqq u \leqq t < \infty$ and*

(ii) *it is jointly continuous in u and t as a map into $\mathscr{D}^s_X$.*

Proof. By the uniqueness of flows we must have

$$\tilde{\xi}_{ut} \cdot \xi_u = \xi_t \quad \text{a.s.}$$

for any version $\tilde{\xi}_{ut}$ of the flow starting at time u. Therefore ξ_{ut} is a version of that flow. Since $\mathscr{D}^s_X$ is a topological group under composition both (i) and (ii) hold. //

The following construction of nice versions of ξ_{ut} is due to Carverhill:

Theorem 6.1. *Suppose X is in H^{s+2}_{loc}. Take X_r for $r=1,2,\dots$ as in the proof of Theorem 5.1 with corresponding flows $\{\xi^r_t\colon 0\leqq t<\infty\}$. Set*

$$\xi^r_{ut}=\xi^r_t\cdot(\xi^r_u)^{-1}\qquad 0\leqq u\leqq t<\infty$$

For each $u\geqq 0$ construct a version of the partial flow $\{\xi_{ut}\colon u\leqq t<\infty\}$ with the explosion time map denoted by τ_u using $\{\xi^r_{ut}\colon u\leqq t<\infty\}$ as in the proof of Theorem 5.1. Set

$$M_{uv}(\omega)=\{x\in\mathbb{R}^m\colon \tau_u(\omega)>v\}\qquad \omega\in\Omega,\ 0\leqq u\leqq v<\infty.$$

Then with probability one (independent of u and v):

(i) *$M_{uv}(\omega)$ is open in $\mathbb{R}^n$.*

(ii) *$M_{uv}(\omega)\subset M_{ut}(\omega)$ if $u\leqq t\leqq v$.*

(iii) *$\xi_{su}(\omega)\,M_{sv}(\omega)\subset M_{uv}(\omega)$ if $s\leqq u\leqq v$.*

(iv) *$\xi_{uv}(\omega)\colon M_{uv}(\omega)\to\mathbb{R}^n$ is a diffeomorphism onto an open subset in $\mathbb{R}^n$ and is in H^s_{loc}, for $0\leqq u\leqq v<\infty$.*

(v) *$\xi_{tv}(\omega)\cdot\xi_{ut}(\omega)|M_{uv}(\omega)=\xi_{uv}(\omega)\quad u\leqq t\leqq v$.*

(vi) *the map $t\mapsto\xi_{ut}(\omega)$ is continuous into $H^s_{\text{loc}}(M_{uv}(\omega))$ on the interval $[u,v]$.*

(vii) *Set $\tilde M_v(\omega)=\{(x,t)\in\mathbb{R}^n\times[0,v]\colon x\in M_{tv}(\omega)\}$.*

Then each $\tilde M_v(\omega)$ is open in $\mathbb{R}^m\times[0,v]$. For any open set in $\tilde M_v(\omega)$ of the form $U\times(a,b)$ the map $(t,u)\mapsto\xi_{tu}(\omega)$ is continuous into $H^s_{\text{loc}}(U)$ on the set $\{(t,u)\in(a,b)\times[0,v]\colon t\leqq u\}$.

Proof. We will use the notation of the proof of Theorem 5.1. We shall ignore the negligible event $\Omega-\Omega'$ of that proof.

For $u\geqq 0$, $x\in\mathbb{R}^n$ and $r=1,2,\dots$ set

$$\tau^r_u(x)(\omega)=\tau^r(\xi^r_u(\omega)^{-1}x)(\omega)\qquad \omega\in\Omega.$$

Then $\tau^r_u(x)$ is the first exit time of $\{\xi^r_{ut}\colon t\geqq u\}$ from B_r. Set

$$M^r_{ut}(\omega)=\{x\in\mathbb{R}^n\colon \tau^r_u(x)(\omega)>t\}.$$

By definition

$$\tau_u(\omega)(x)=\lim_{r\to\infty}\tau^r_u(\omega)(x)$$

and

$$M_{ut}(\omega)=\bigcup_r M^r_{ut}(\omega).$$

Assertions (i) and (ii) are immediate from the corresponding facts about M^r_{uv}. Also if $x\in M_{sv}(\omega)$ then $x\in M^r_{sv}(\omega)$ for some r; but then $x\in M^r_{su}(\omega)$ if $s\leqq u\leqq v$, and so

$$\xi_{su}(\omega)\,x=\xi^r_{su}(\omega)\,x\in M^r_{uv}(\omega)$$

since

$$\xi^r_{uv}(\omega)\cdot\xi^r_{su}(\omega)=\xi^r_{sv}(\omega).$$

Thus (iii) holds, and a similar argument proves (v).

Assertions (iv) and (vi) come as in Theorem 5.1. For (vii) set $\tilde M^r_v(\omega)=\{(x,t)\in\mathbb{R}^n\times[0,v]\colon x\in M^r_{tv}(\omega)\}$. Then

$$\tilde M_v(\omega)=\bigcup_r\tilde M^r_v(\omega).$$

Now each $\tilde{M}_v^r(\omega)$ is open in $\mathbb{R}^n \times [0, v]$ by the continuity of ξ_{tv}^r in (t, v) shown in Lemma 6.1. Thus $\tilde{M}_v(\omega)$ is open.

For the continuity of $\xi_{tu}(\omega)$ in (vii) we can assume without loss of generality that $\bar{U} \times [a, b]$ is contained in $\tilde{M}_v(\omega)$ and therefore in $\tilde{M}_v^r(\omega)$ for some r. But then $\xi_{tu}(\omega) = \xi_{tu}^r(\omega)$ on the relevant domain and the result follows from Lemma 6.1. //

From now on $\{\xi_{ut}: 0 \leqq u \leqq t < \infty\}$ *will always refer to a version constructed as in Theorem* 6.1.

6.2. *A Version of the Inverse*

For a fixed $T > 0$ let $z^\vee$ be as in §5.4. Replacing z by $z^\vee$ in Theorem 6.1 construct a version of the flow $\xi_{ut}^\vee$ of $(X, z^\vee)$ just as in the proof of that theorem: it will be defined for $0 \leqq u \leqq t \leqq T$ on subsets $M_{ut}^\vee(\omega)$ of $\mathbb{R}^n$.

Theorem 6.2. *Except for a set of probability zero, depending only on* T, *for all* $0 \leqq u \leqq s \leqq t \leqq T$, $\xi_{ut}(\omega)$ *maps* $M_{ut}(\omega)$ *as an* H^s *diffeomorphism onto* $M_{T-t,T-u}^\vee(\omega)$ *and has inverse* $\xi_{T-t,T-u}^\vee(\omega)$. *Furthermore,*

$$\xi_{T-t,T-s}^\vee \cdot \xi_{ut} = \xi_{us} | M_{ut}. \tag{1}$$

Proof. With the notation of the proof of Theorem 6.1, if $x \in M_{ut}(\omega)$ then $x \in M_{ut}^r(\omega)$ for some r. For this r we have

$$\xi_{ut}(\omega)\, x = \xi_{ut}^r(\omega)\, x = \xi_t^r(\omega) \circ \xi_u^r(\omega)^{-1} x.$$

By Lemma 5.6 we have $\xi_t^r = \xi_{T-t}^{\vee r} \cdot \xi_T^r$ giving

$$\begin{aligned} \xi_{T-t,T-s}^{\vee r}(\omega) \circ \xi_{ut}(\omega)\, x &= \xi_{T-s}^{\vee r}(\omega) \circ \xi_T^r(\omega) \circ \xi_u^r(\omega)^{-1} x \\ &= \xi_s^r(\omega) \circ \xi_u^r(\omega)^{-1} x = \xi_{us}^r(\omega)\, x \in B_r \end{aligned}$$

since $M_{ut}^r(\omega) \subset M_{us}^r(\omega)$. This shows that $\xi_{ut}(\omega) x$ is in $M_{T-t,T-s}^{\vee r}(\omega)$ since the equation is true for all s in $[u, t]$. However on that set $\xi_{T-t,T-s}^{\vee r}(\omega)$ agrees with $\xi_{T-t,T-s}^\vee(\omega)$. Therefore

$$\xi_{T-t,T-s}^\vee(\omega) \circ \xi_{ut}(\omega)\, x = \xi_{us}(\omega)\, x$$

proving (1).

Now take $s = u$. The proof so far shows that $\xi_{ut}(\omega)$ maps $M_{ut}(\omega)$ into $M_{T-t,T-u}^\vee(\omega)$ with the required inverse. However we can equally well interchange the roles of z and $z^\vee$; this will show that $\xi_{T-t,T-u}^\vee(\omega)$ maps $M_{T-t,T-u}^\vee(\omega)$ into $M_{ut}(\omega)$ with inverse $\xi_{ut}(\omega)$, and so complete the proof. //

Corollary 6.2. *If for each* $t \geqq 0$ *there is probability one that* $\xi_t(\omega)$ *is an* H^s *diffeomorphism of* $\mathbb{R}^n$ *onto itself then with probability one* $\xi_{ut}(\omega)$ *is such a diffeomorphism for all* $0 \leqq u \leqq t < \infty$.

Proof. Under the given hypotheses it will be enough to prove that for $T = 1, 2, \ldots$ with probability one $M_{ut}(\omega) = \mathbb{R}^n$ and $\xi_{ut}(\omega)$ is surjective, for all $0 \leqq u \leqq t \leqq T$.

To show $M_{ut}(\omega)=\mathbb{R}^n$ let $Z_t=\{\omega\colon \xi_t(\omega)$ not onto$\}$. Then $P(Z_t)=0$ for each $0\leqq t<\infty$ by hypothesis. From the definition

$$\tau_t|\mathbb{R}^n\times(\Omega-Z_t)\equiv\infty \qquad 0\leqq t<\infty. \tag{1}$$

Let A be the event that the conclusions of Theorem 6.1 hold. Then $P(A)=1$ and if $\omega\in A$

$$\xi_{ut}(\omega)\cdot\xi_u(\omega)=\xi_t(\omega) \qquad 0\leqq u\leqq t<\infty.$$

Therefore if $Z'_t=Z_t\cup(\Omega-A)$ for $\omega\notin Z'_t$ we have surjectivity for ξ_{ut}, $0\leqq u\leqq t$. Also for $\omega\in A$, by 6.1 (v),

$$\xi_{tT}(\omega)\cdot\xi_{ut}(\omega)|M_{uT}(\omega)=\xi_{uT}(\omega) \qquad 0\leqq u\leqq t\leqq T.$$

Therefore

$$\xi_{ut}(\omega)\,[M_{uT}(\omega)]=M_{tT}(\omega) \qquad 0\leqq u\leqq t\leqq T$$

whenever $\omega\in A$ and $\xi_{uT}(\omega)$ is surjective. In particular, if $\omega\notin Z'_T\cup Z'_t$, with $0\leqq t\leqq T$, so that $M_{tT}(\omega)=\mathbb{R}^n$ by (1), then

$$M_{uT}(\omega)=M_{ut}(\omega) \qquad 0\leqq u\leqq t,$$

Set

$$Z=\bigcup\{Z'_t\colon t\in\mathbb{Q}\}.$$

Then if $\omega\neq Z$ and $0\leqq u<t\leqq T$, choosing $t'\in\mathbb{Q}\cap(u,t)$,

$$M_{ut}(\omega)\subset M_{ut'}(\omega)=M_{uT}(\omega)\subset M_{ut}(\omega).$$

Thus

$$M_{ut}(\omega)=M_{uT}(\omega) \qquad 0\leqq u<t\leqq T,\ \omega\notin Z.$$

However for all $u\geqq 0$ we have

$$\tau_u(x,\omega)>u \qquad x\in\mathbb{R}^n,\ \omega\in\Omega.$$

Therefore

$$\mathbb{R}^n=\bigcup_{u<t\leqq T}M_{ut}(\omega) \qquad 0\leqq u<T$$

and so

$$M_{ut}(\omega)=M_{uT}(\omega)=\mathbb{R}^n \qquad \text{all } 0\leqq u\leqq t\leqq T$$

if $\omega\neq Z$, as required.

For surjectivity of $\xi_{ut}(\omega)$ we apply the above argument to $(X,z^\vee)$ using Theorem 5.4. This shows that $M^\vee_{T-t,T-u}(\omega)=\mathbb{R}^n$ for all $0\leqq u\leqq t\leqq T$, almost surely. The surjectivity then follows from the theorem. //

6.3. *The Backward Equation*

We continue with the same notation and hypotheses on X. For fixed $T>0$ let $\rho\colon M\times\Omega\to[0,T]$ be given by

$$\rho(x,\omega)=\sup\{u\in[0,T]\colon x\notin M_{uT}(\omega)\}.$$

It is the "backwards explosion time map". Note that $\rho(x)<T$ for all x in $\mathbb{R}^n$, almost surely, by Theorem 6.1, and that

$$\rho(x)=\inf\{\rho^r(x)\colon r=1,2,\ldots\}$$

where

$$\rho^r(x,\omega)=\sup\{u\in[0,T]\colon x\notin M^r_{uT}(\omega)\}.$$

Let us say that a map $\alpha\colon \Omega\to[0,T]$ is a *starting time* if for each $t\in[0,T]$ we have $\{\alpha<t\}\in\mathscr{F}_{tT}$.

Lemma 6.3. *For each x in $\mathbb{R}^n$, $\rho(x)$ is a starting time.*

Proof. It suffices to show that each $\rho^r(x)$ is a starting time. However $T-\rho^r(x)$ is the first exit time of the process $u\mapsto\xi^r_{T-u,T}(x)$ from B_r and this process is adapted to $\{\mathscr{F}_{T-u,T}\colon 0\leqq u\leqq T\}$. This means that

$$\{\rho^r(x)<t\}=\{T-\rho^r(x)>T-t\}\in\mathscr{F}_{tT}$$

as required. //

The following was given by Kunita [21], [22] under C^5, respectively C^2, conditions on X, and assuming that (X,z) is strongly complete.

Theorem 6.3. *For fixed $T>0$ set $\zeta_u=\xi_{uT}$, $0\leqq u\leqq T$. Then for each x in $\mathbb{R}^n$ if $f\colon\mathbb{R}^n\to\mathbb{R}$ is C^2 and τ is any starting time strictly greater than $\rho(x)$* we have

$$f(\zeta_{u\vee\tau}(x))=f(x)+\int_{u\vee\tau}^{T} X(f\cdot\zeta_s)(x)\circ dz_s \qquad 0\leqq u\leqq T, \quad \text{a.s.}$$

Proof. Since $\rho(x)=\inf\rho^r(x)$ we can assume that $\tau>\rho^r(x)$ for some fixed r, thereby only being wrong on a set of arbitrarily small measure. But then

$$\zeta_u(\omega)(x)=\xi^r_{uT}(\omega)(x)=\xi^{\vee r}_{T-u}(\omega)^{-1}(x) \qquad \tau(\omega)\leqq u\leqq T. \tag{1}$$

Set

$$x_t=\xi^{\vee r}_t(\omega)^{-1}x \qquad 0\leqq t<\infty.$$

By Proposition 4.3

$$dx_t=-D(\xi^{\vee r}_t)^{-1}(x)\,X_r(x)\circ dz^\vee_t$$

i.e.

$$\begin{aligned} f(x_t)&=f(x)-\int_0^t Df(x_s)\,D(\xi^{\vee r}_s)^{-1}(x)\,X_r(x)\circ dz^\vee_s\\ &=f(x)+\int_{T-t}^{T} Df(x_{T-s})\,D(\xi^{\vee r}_{T-s})^{-1}(x)\,X_r(x)\circ dz_s. \end{aligned}$$

Therefore if $y_t=x_{T-t}$, since $X_r(x)=X(x)$ for r sufficiently large

$$f(y_{u\vee\tau})=f(x)+\int_{u\vee\tau}^{T} Df(y_s)\,D(\xi^{\vee r}_{T-s})^{-1}(x)\,X(x)\circ dz_s.$$

Therefore, by (1)

$$\begin{aligned} f(\zeta_{u\vee\tau}(x)) &= f(x) + \int_{u\vee\tau}^{T} Df(\zeta_s(x))\, D\zeta_s(x)\, X(x)\circ dz_s \\ &= f(x) + \int_{u\vee\tau}^{T} D(f\cdot\zeta_s)(x)\, X(x)\circ dz_s \\ &= f(x) + \int_{u\vee\tau}^{T} X(f\cdot\zeta_s)(x)\circ dz_s \end{aligned}$$

as required. //

§ 7. Equations on Manifolds

Suppose now that our equation $dx = X(x)\circ dz$ is on a separable, finite dimensional, C^∞ manifold M. For simplicity we will suppose that X is C^∞.

If X has compact support, rather than work with the diffeomorphism groups as in [11], we can embed M as a closed C^∞ submanifold of $\mathbb{R}^p$ for some p, and take a C^∞ extension $\bar{X}$ of X with compact support, to obtain the equation $dx = \bar{X}\circ dz$ on $\mathbb{R}^p$.

Using Corollary 3.2 we obtain a flow $\bar{\xi}$ for $(\bar{X}, z)$. By separability of M, we can modify $\bar{\xi}$ on some negligible set so that it restricts to give a flow for (X, z) on M. Thus we obtain the analogue of Corollary 3.2 in the manifold case, at least for X of class C^∞.

To get a partial flow for X not of compact support we simply do exactly the same as in Theorem 5.1 but with $\{B_r\}_{r=1}^{\infty}$ now a family of open subsets of M, with compact closures, covering M and having $\bar{B}_r \subset B_{r+1}$ for each r. Theorem 5.1 then goes through in exactly the same way with M replacing $\mathbb{R}^n$, as do the other results with the embedding technique used to get the necessary preliminary version for the "approximations" X_r.

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Received June 15, 1982; in final form May 4, 1983