Z. Wahrscheinlichkeitstheorie verw. Gebiete
65, 239–244 (1983)

Zeitschrift für
**Wahrscheinlichkeitstheorie**
und verwandte Gebiete


# A Note on Diffuse Random Measures

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**Summary.** Let $(\mu_x)$ be a random measure on a measure space $(\Omega, \Sigma, \mu)$, such that all $\mu_x$ are diffuse measures. Then there is a subalgebra $\Sigma_0 \subset \Sigma$ with $\mu_{|\Sigma_0}$ non-atomic such that $(\mu_{x|\Sigma_0})$ is absolutely $\mu$-continuous. This is applied to product measures and bounded linear operators on $L_1(\Omega, \nu)$.

## I. Random Measures

Let $(\Omega, \Sigma, \mu)$ and $(K, \mathscr{A}, \nu)$ always be countably generated, non-atomic probability measure spaces.

A family $(\mu_x)_{x \in K}$ of measures on $(\Omega, \Sigma)$ is called a *random measure* if the functions $x \in K \to \mu_x(A) \in \mathbb{R}$ are $\mathscr{A}$-measurable for all (fixed) $A \in \Sigma$.

A random measure $(\mu_x)_{x \in K}$ is *diffuse* if $\nu$-almost all $\mu_x$ are non-atomic.

**Theorem.** *Let $(\Omega, \Sigma, \mu)$ and $(K, \mathscr{A}, \nu)$ be as above. Then, for every diffuse random measure $(\mu_x)_{x \in K}$ on $(\Omega, \Sigma)$ with $\int_K \|\mu_x\| \, d\nu(x) < \infty$ there is a sub-$\sigma$-algebra $\Sigma' \subset \Sigma$ with $\mu_{|\Sigma'} \neq 0$ and non-atomic such that $\nu$-almost all $\mu_x$ are $\mu$-absolutely continuous on $\Sigma'$.*

A *disintegration* for a probability measure $P$ on a product space $(\Omega \times K, \Sigma \otimes \mathscr{A})$ is by definition a random measure $(\mu_x)_{x \in K}$ such that for all $M \in \Sigma \otimes \mathscr{A}$

$$P(M) = \int_K \int_\Omega \chi_M(t, x) \, d\mu_x(t) \, d\nu(x)$$

where $\nu$ is the marginal measure of $P$, i.e. $\nu(A) = P(\Omega \times A)$ for all $A \in \mathscr{A}$. If $\Omega$ is a topological space, $\Sigma$ the $\sigma$-Algebra of its Baire sets and the marginal measure $\mu(B) = P(B \times K)$, $B \in \Sigma$, is tight, then such a disintegration always exists (see [3]).

As an immediate consequence of the theorem we have

**Corollary.** *Let $P$ be a probability measure on the product space $(\Omega \times K, \Sigma \otimes \mathscr{A})$ with marginal measure $\nu$ on $(K, \mathscr{A})$ and a disintegration $(\mu_x)_{x \in K}$ consisting of diffuse measures. Then, for every non-atomic measure $\mu$ on $\Sigma$ there is a non-atomic sub-$\sigma$-algebra $\Sigma' \subset \Sigma$ such that $P$ is $\mu \otimes \nu$-absolutely continuous on $\Sigma' \otimes \mathscr{A}$.*

*Proof of Theorem.* Choose a system $A_i^n$, $i=1,\ldots,2^n, n\in\mathbb{N}$ of measurable subsets of $\Omega$ such that

i) $A_i^n$, $i=1,\ldots,2^n$, is a partition of $\Omega$ for all $n$.

ii) $A_i^n=A_{2i-1}^{n+1}\cup A_{2i}^{n+1}$, $\mu(A_i^n)=2^{-n}$.

iii) $\overline{\lim_n}\, A_{i_n}^n$ contains at most one atom for every sequence $i_n$, $i_n\in\{1,\ldots,2^n\}$.

Consider the $L_1(K,\nu)$-valued martingale

$$F_n(t)=\sum_{i=1}^{2^n} f_i^n\,\chi_{A_i^n}(t), \qquad f_i^n(x)=2^n\,\mu_x(A_i^n)$$

with respect to the algebras $\Sigma_n$ generated by $A_1^n,\ldots,A_{2^n}^n$. $(F_n,\Sigma_n)$ is $L_1$-bounded since

$$\int_\Omega \|F_n(t)\|\,d\mu(t)\leqq\int_K \|\mu_x\|\,d\nu(x)<\infty$$

and for every $x\in K$ with $\mu_x$ non-atomic we have

$$\Big|\int_{A_{i_n}^n} F_n\,d\mu\Big|(x)\leqq|\mu_x|(A_{i_n}^n)\xrightarrow[n\to\infty]{}0$$

for every sequence $i_n$, $i_n\in\{1,\ldots,2^n\}$.

Therefore

$$\max_{i=1}^{2^n}\Big|\int_{A_i^n} F_n\,d\mu\Big|\to 0 \qquad \text{a.e. on } K$$

and also in $L_1(K,\nu)$-norm by the dominated convergence theorem. By the following lemma there are algebras $\Sigma_n'\subset\Sigma_n$, generating a subalgebra $\Sigma'\subset\Sigma$ such that $\mathscr{E}_{\Sigma_n'}(F_n)$ converges to some $F\in L_1(\mu,L_1(K,\nu))$. Then $k(x,t)=F(t)(x)$ is $\Sigma\times\mathscr{A}$-measurable and for all $E\in\mathscr{A}$ and $A\in\bigcup\Sigma_n$ we have

$$\int_E\int_A k(x,t)\,d\mu(t)\,d\nu(x)=\int_E \mu_x(A)\,d\nu(x).$$

Since $\bigcup\Sigma_n$ is countable and generates $\Sigma'$ it follows that $k(x,\cdot)\,\mu_{|\Sigma'}=\mu_{x|\Sigma'}$ for $\nu$-almost all $x\in K$. □

Recall that a *dyadic martingale* is a martingale $(F_n,\Sigma_n)$ (for general reference see [1, 6]) such that each algebra $\Sigma_n$ is generated by $2^n$ atoms $A_i^n$ such that $\mu(A_i^n)=2^{-n}$ and $A_i^n=A_{2i-1}^{n+1}\cup A_{2i}^{n+1}$ for $i=1,\ldots,2^n$.

**Lemma.** *Let $X$ be a Banach lattice with order-continuous norm and assume that the $L_1(X)$-bounded $X$-valued dyadic martingale $(F_n,\Sigma_n)$ satisfies*

$$\sup_{i=1}^{2^n}\Big|\int_{A_i^n} F_n\,d\mu\Big|\to 0$$

*weakly.*

*Then there are algebras $\Sigma_n'\subset\Sigma_n$ with $\Sigma_{n+1}'\supset\Sigma_n'$ such that $\bigcup_n\Sigma_n'$ generates a non-atomic subalgebra of $(\Sigma,\mu)$ and $(\mathscr{E}_{\Sigma_n'}(F_n),\Sigma_n')$ converges in $L_1(X)$.*

*Proof.* Since $X$ is order-continuous and $(F_n)$ takes its values in a separable subspace of $X$ we may assume by a well known representation theorem (e.g. [5] Theorem 1.6.14) that $X$ is a Banach function lattice over a probability space $(K, \nu)$ such that $L_\infty(K,\nu) \subset X \subset L_1(K,\nu)$. It is enough to show that for some $\Sigma'_n$, $(\mathscr{E}_{\Sigma'_n}(F_n), \Sigma'_n)$ converges to some $F \in L_1(L_1(K,\nu))$ in the $L_1$-norm. Indeed, since $U_{X^{\times\times}}$ is closed in $L_1(K,\nu)$ ($X^{\times\times}$ has the Fatou-property, i.e.[1] $f_n \in U_{X^{\times\times}}$, $f_n \nearrow f$ implies $f \in U_{X^{\times\times}}$) it follows from the pointwise convergence theorem that $F \in L_1(X^{\times\times})$ and therefore $F \in L_1(X)$ because $\|f\|_X = \|f\|_{X^{\times\times}}$ for $f \in X$.

Hence we may restrict ourselves to $X = L_1(K,\nu)$ with

$$\sup_{i=1}^{2^n} \Big| \int_{A_i^n} F_n \, d\mu \Big| \to 0$$

in norm.

Assume for a moment that we already constructed a subsequence $n_m$ and a 'tree' $B_i^m$, $i=1,\dots,2^m$, such that

(1) $B_i^m \in \Sigma_{n_m}$, $i=1,\dots,2^m$, forms a partition of $\Omega$.

(2) $B_i^m = B_{2i-1}^{m+1} \cup B_{2i}^{m+1}$, $\mu(B_i^m) = 2^{-m}$.

(3) $\|\mu(B_i^m)^{-1} \cdot \int_{B_i^m} F_{n_m} \, d\mu - \mu(B_{2i+k}^{m+1})^{-1} \int_{B_{2i+k}^{m+1}} F_{n_{m+1}} \, d\mu\| \leqq \frac{1}{2^m}$ for $i=1,\dots,2^m$, $k=-1,0$.

For $n_m \leqq k < n_{m+1}$ let $\Sigma'_k$ be the algebra generated by $B_i^m$, $i=1,\dots,2^m$ and put $F'_k = \mathscr{E}_{\Sigma'_k}(F_k)$. Then for all $m,k \in \mathbb{N}$

$$\begin{aligned}\int_\Omega \|F'_{n_{m+k}}(t) - F'_{n_m}(t)\| \, d\mu(t) &\leqq \sum_{j=0}^{k-1} \sup_{t\in\Omega} \|F'_{n_{m+j+1}}(t) - F'_{n_{m+j}}(t)\| \\ &\leqq \sum_{j=0}^{k-1} \frac{1}{2^{m+j}} \leqq \frac{1}{2^{m-1}}\end{aligned}$$

and $(F'_k, \Sigma'_k)$ is Cauchy in $L_1(X)$.

We construct the tree $(B_i^m)$ by induction. Put $B^0 = \Omega$, $n_1 = 1$. If $B_i^m$, $i=1,\dots,2^m$, and $n_m$ are known already then apply the following procedure to all $L = B_i^m$, $i \in \{1,\dots,2^m\}$.

If $A_{i_1}^n, \dots, A_{i_r}^n$ is a partition of $L$ for some $n > n_m$ we write $E_1 = A_{i_1}^n, \dots, E_r = A_{i_r}^n$.

Put $\sigma_{2j-1} = r_j$, $\sigma_{2j} = -r_j$ for $j=1,\dots,\frac{r}{2}$ where $r_j$ is the $j^{\text{th}}$ Rademacherfunction on $[0,1]$. Define for $s \in [0,1]$:

$$g_s(t) = \sum_{j=1}^r \sigma_j(s) \chi_{E_j}(t), \qquad f_j = \int F_n(t) \chi_{E_j}(t) \, d\mu(t).$$

Then $\int g_s(t) \, d\mu(t) = 0$ for all $s \in [0,1]$ and by the Khintchine-inequality and the Cauchy-Schwarz-inequality we get:

$$\begin{aligned}&\min_{s\in[0,1]} \Big\| \int_\Omega F_n(t) g_s(t) \, d\mu(t) \Big\| \\ &\leqq \int_0^1 \int_K \Big| \sum_{j=1}^r \sigma_j(s) f_j(x) \Big| \, dx \, ds \\ &\leqq \int_K \int_0^1 \Big| \sum_{j=1}^r \sigma_j(s) f_j(x) \Big| \, ds \, dx \leqq C \int_K \Big( \sum_{j=1}^r |f_j(x)|^2 \Big)^{\frac{1}{2}} dx\end{aligned}$$

[1] $X^\times$ Köthe-Dual

$$\leqq C \int_K \left(\max_{j=1}^{r} |f_j(x)|\right)^{\frac{1}{2}} \left(\sum_{j=1}^{r} |f_j(x)|\right)^{\frac{1}{2}} dx$$

$$\leqq C \left\| \max_{i=1}^{2^n} \left| \int_{A_i^n} F_n \, d\mu \right| \right\|^{\frac{1}{2}} \cdot \left\| \int_\Omega |F_n|(t) \, d\mu(t) \right\|^{\frac{1}{2}}$$

By assumption, there is $n_{m+1} > n_m$ and $s_0 \in [0,1]$ such that

$$\left\| \int_\Omega F_{n_{m+1}}(t) \, g_{s_0}(t) \, d\mu \right\| \leqq 2^{-2m}.$$

Put $L_0 = \{g_{s_0} > 0\}$ and $L_1 = \{g_{s_0} < 0\}$. Since $\int_\Omega g_{s_0} \, d\mu = 0$ and $|g_{s_0}| \equiv \chi_L$ we have $\mu(L_0) = \mu(L_1) = \frac{1}{2}\mu(L)$.

Furthermore

$$\left\| 2^m \int_L F_{n_m} \, d\mu - 2^{m+1} \int_{L_0} F_{n_{m+1}} \, d\mu \right\| = 2^m \left\| \int_{L_1} F_{n_{m+1}} \, d\mu - \int_{L_0} F_{n_{m+1}} \, d\mu \right\|$$

$$= 2^m \left\| \int_L F_{n_{m+1}}(t) \, g_{s_0}(t) \, d\mu(t) \right\| \leqq 2^{-m}$$

and the same estimate holds if we replace $L_0$ by $L_1$. If we define $B_{2i-1}^{m+1} = L_0$, $B_{2i}^{m+1} = L_1$ then (1), (2), (3) follow.

*Remark.* The order continuity assumption cannot be dropped. If $X$ is $\sigma$-complete but not $\sigma$-order continuous, then $X$ contains $l_\infty$ as a sublattice (see [5] Proposition 1.a.7). Let $T$ be an isomorphic embedding $L_1(\Omega, \mu) \subset l_\infty \subset X$ and let $(F_n, \Sigma_n)$ be the $X$-valued martingale given by $T$:

$$F_n(t) = \sum_{i=1}^{2^n} \chi_{A_i^n}(t) \cdot T(2^n \chi_{A_i^n}).$$

Then

$$\left\| \sup_{i=1}^{2^n} \left| \int_{A_i^n} F_n \, d\mu \right| \right\|_X \leqq \sup_{i=1}^{2^n} \|T\chi_{A_i^n}\| \xrightarrow{n \to \infty} 0$$

but $T_{|L_1(\Sigma', \mu)}$ is an isomorphic embedding for all subalgebras $\Sigma'$ and not an representable operator (compare [1], Chap. III.2).

## II. Operators in $L_1(\Sigma, \mu)$

N.J. Kalton has shown that if $(\Omega, \Sigma, \mu)$ is a standard measure space with a finite measure $\mu$ then for every bounded linear operator $T$: $L_1(\Omega, \Sigma, \mu) \to L_1(K, \mathscr{A}, \nu)$ there is a random measure $(\nu_x)_{x \in K}$ such that

$$Tf(x) = \int f(t) \, d\nu_x(t) \qquad \nu\text{-a.e.}$$

for all $f \in L_1(\Omega, \mu)$. Decomposing $(\nu_x)$ into its diffuse and its atomic part gives

$$Tf(x) = \sum_{n=1}^{\infty} a_n(x) \, f(\sigma_n(x)) + \int f(t) \, d\mu_x(t)$$

where $(\mu_x)_{x \in K}$ is a diffuse random measure and the measurable maps $\sigma_n$: $K \to \Omega$, $a_n$: $K \to \mathbb{R}$ satisfy (see [4]):

i) $\sigma_m(x) \neq \sigma_n(x)$ for $m \neq n$, $x \in K$,

ii) $|a_n(x)| \geqq |a_{n+1}(x)|$ $\nu$-a.e., $x \in K$.

Say that $T$ has an *atomic representation* if $\mu_x = 0$ for almost all $x \in K$ and that $T$ has a *diffuse representation* if $a_n = 0$ for all $n$. If $\nu$ almost all $\nu_x$ are absolutely $\mu$-continuous, then $T$ is just an integral operator. It is well known (e.g. [1] Kap. III) that $T$ is an integral operator if and only if every $A \in \Sigma$, $\mu(A) > 0$, contains some $A' \in \Sigma$, $\mu(A') > 0$, such that $T_{|L_1(A', \mu|_{A'})}$ is compact.

**Theorem.** *Let* $T: L_1(\Omega, \Sigma, \mu) \to L_1(K, \mathscr{A}, \nu)$ *where* $(\Omega, \Sigma, \mu)$ *is a standard measure space.*

a) *$T$ has a diffuse representation if and only if for every $A \in \Sigma$, $\mu(A) > 0$, there is a non-atomic subalgebra $\Sigma'$ of measurable subsets of $A$ such that $T_{|L_1(A, \Sigma', \mu)}$ is an integral operator.*

b) *$T$ has an atomic representation if and only if for every $\varepsilon > 0$ there is a $A \in \Sigma$ with $\mu(\Omega - A) \leqq \varepsilon$ such that for every bounded sequence $f_n \in L_1(A, \mu|_A)$ which converges to zero in measure, $Tf_n$ also converges to zero in measure.*

*Proof.* a) "$\Rightarrow$" follows from Theorem I. On the other hand, if $T$ has non-zero atomic part, then (by [4] Theorem 5.5) there is an $A \in \Sigma$, $\mu(A) > 0$, such the $T_{|L_1(A, \mu|_A)}$ is an isomorphism. Hence $T_{|L_1(A, \Sigma', \mu)}$ cannot be an integral operator if $\mu_{|\Sigma'}$, $\Sigma' \subset \Sigma$, is non-atomic.

b) "$\Rightarrow$" We may assume that $T \geqq 0$ and that

$$Tf(x) = \sum_{n=1}^{\infty} a_n(x)\, f(\sigma_n(x))$$

with $a_n$ and $\sigma_n$ as above. Each operator $S_n f(x) = a_n(x)\, f(\sigma_n(x))$ maps a bounded $\mu$-convergent sequence into a $\nu$-convergent sequence (since $S_n(f \wedge g) = S_n f \wedge S_n g$ for all $f, g \in L_1(\mu)$) and therefore

$$T_m f(x) = \sum_{n=1}^{m} a_n(x)\, f(\sigma_n(x))$$

has the same property for all $m \in \mathbb{N}$.

Since $(T'_m 1)(t) \nearrow (T' 1)(t)$ for $\mu$-almost all $t \in \Omega$, there is an $A \in \Sigma$ with $\mu(\Omega - A) \leqq \varepsilon$ and

$$\begin{aligned} \|(T - T_m)\chi_A\|_{L_1} &= \|\chi_A (T' - T'_m)\|_{L_\infty} \\ &\leqq \sup_{t \in A} |(T' - T'_m)\, 1\, (t)| \to 0 \end{aligned}$$

for $m \to \infty$ by Egoroff's theorem. Therefore $T\chi_A$ has the required property.

"$\Leftarrow$" Write $T = T^a + T^d$ where $T^a$ is the atomic part of $T$ and $T^d$ is the diffuse part. By the first part of the proof there is a set $A \in \Sigma$, $\mu(A) > 0$, such that $Tf_n$ converges to zero in measure whenever the bounded sequence $f_n \in L_1(A, \mu|_A)$ does. If $T^d \neq 0$ we can find a non-atomic $\sigma$-Algebra $\Sigma'$ of measurable subsets of $A$ such that $T^d_{|L_1(A, \Sigma', \mu)}$ is a compact integral operator. Then there is always a sequence $A_n \in \Sigma'$ with $\mu(A_n) \to 0$ such that $T^d\left(\frac{1}{\mu(A_n)} \chi_{A_n}\right)$ norm-converges to some $f \neq 0$ (otherwise every martingale representing $T^d_{|L_1(A,\Sigma',\mu)}$ converges to zero in $L_1(L_1)$). It follows from our choice of $A$ that $T\left(\frac{1}{\mu(A_n)}\chi_{A_n}\right)$ does not converge to 0 in measure either. □

It was shown by Doss [2], that for a locally compact abelian group every multiplier from singular measures to singular measures is necessarily given by convolution with a discrete measure. Our theorem allows us to give a short, purely measure-theoretic proof also for the non-abelian case.

**Corollary.** *Let $G$ be a locally compact group and denote by $\lambda$ its right Haar measure. Then, for every diffuse measure $\mu$ on $G$ there is a $\lambda$-singular measure $\nu_0$ on $G$ such that $\nu_0 * \mu$ is $\lambda$-absolutely continuous.*

*Proof.* $T\nu = \nu * \mu$ defines an operator $T: L_1(G,\lambda) \to L_1(G,\lambda)$ which has the diffuse representation $\nu_x(A) = \nu(Ax^{-1})$. Choose a sub-$\sigma$-algebra $\Sigma'$ of the Borel sets of $G$, such that $T_{|L_1(\Sigma',\lambda_{|\Sigma'})}$ is compact and also a bounded sequence $f_n \in L_1(\Sigma', \lambda_{|\Sigma'})$ such that $f_n \cdot \lambda$ converges to some singular measure $\nu_0$ in the weak topology ($\sigma(M(G), C(G))$-topology). But then $T(f_n) = f_n * \mu$ norm converges and $\nu_0 * \mu = T(\nu_0)$ belongs to $L_1(G,\lambda)$. □

An extension of these results to larger classes of operators will be given in a forthcoming paper [8].

*Acknowledgement.* A special case of Theorem I was proved independently by M. Talagrand [7]. I also want to thank H.v. Weizsäcker for several stimulating discussions on this subject.

## References

1. Diestel, J., Uhl, J.: Vector measures, Mathematical surveys No. 15. Prov. R.I.: Amer. Math. Soc. 1977
2. Doss, R.: Convolution of singular measures. Studia Math. **45**, 111–117 (1973)
3. Edgar, G.: Disintegration of measures and the vector-valued Radon-Nikodym theorem. Duke Math. J. **42**, 447–450 (1975)
4. Kalton, N.J.: The Endomorphisms of $L_p$ ($0 \leq p \leq 1$). Indiana Univ. Math. J. **27**, 353–381 (1978)
5. Lindenstrauß, J., Tzafriri, L.: Classical Banach Spaces II. Berlin-Heidelberg-New York: Springer 1979
6. Neveu, J.: Martingales à temps discret. Paris: Masson 1972
7. Talagrand, M.: Un théorème de théorie de la mesure lié à deux théorèmes de Mokobodzki. [To appear in Ann. Scient. Éc. Norm. Sup.]
8. Weis, L.: On the Representation of Order continuous operators by Random Measures. To appear in Trans. Amer. Math. Soc.

Received March 4, 1982