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Discontinuous Additive Functionals of Dual Processes*

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1. Introduction

Let X be a standard Markov process with state space E and let ξ be an excessive reference measure for X. In recent work ([6], [7]) Revuz associates with certain additive functionals of X measures on E which determine those additive functionals in case they are natural and X is in duality, relative to ξ, with another standard process $\hat{X}$. In this paper we use an analogous method to associate with every finite additive functional A of X a measure ν_A on $E \times E$ which turns out to be σ-finite and whose projection upon the second co-ordinate is Revuz's measure on E. Under the hypotheses of duality, but with no other assumptions as, for example, on the left-continuity of the fields or Feller properties of the resolvents, we give a formula for the "bipotential" of a finite additive functional A in terms of ν_A and we construct a canonical measure ν on $E \times E$ for the process X which reflects the behavior of the jumps of X. Using this canonical measure, we can prove that if A is a finite purely discontinuous quasi-left-continuous additive functional of X then A is of the form $A_t = \sum_{s \leqq t} F(X_{s-}, X_s)$, a result due to Motoo (see Watanabe [8]) in the case where X is a special standard process. We also use the canonical measure ν to prove that a Lévy system (n, H) exists for X, thus taking into a different context work of Watanabe [8] whose method for special standard processes involves the heavy machinary of stochastic integrals relative to square-integrable martingales.

All terminology and notation which is not specifically explained here will be that of Blumenthal and Getoor [1]. The basic object is a standard Markov process $X = (\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, P^x)$ whose state space E is $LCCB$ and which has transition semigroup $\{P_t; t \geqq 0\}$, resolvent $\{U^\alpha; \alpha \geqq 0\}$ and lifetime ζ. The σ-fields $\mathscr{E}$, $\mathscr{E}^n$ and $\mathscr{E}^*$ are respectively the Borel sets, nearly Borel sets and universally measurable sets in E. The object of our attention here is an additive functional (AF) of X. We call A a finite AF of X if $A_t < \infty$ on $[0, \zeta)$ a.s., and we denote by $\mathscr{A}$ the class of finite AF's of X. The restriction to finite AF's makes it possible to avoid a number of tricky points dealt with by Revuz [6].

We assume throughout that there is a σ-finite measure ξ on E which is an excessive reference measure for X. For the main results of this paper, when X is assumed to be in duality with a standard process $\hat{X}$ relative to the σ-finite measure ξ, then ξ automatically possesses all the above named properties. All the regularity properties discussed in Chapter V of [1] may be assumed. One particularly useful result is that if f and g are α-excessive and $f \leqq g$ a.e. (ξ), then $f \leqq g$ everywhere.

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2. The Bipotential Operator

For A an AF of X, we define the α-bipotential $\mathscr{U}_A^\alpha F$ of $F \in (\mathscr{E} \times \mathscr{E})_+^*$ relative to A by

$$\mathscr{U}_A^\alpha F(x) = E^x \int_0^\infty e^{-\alpha t} F(X_{t-}, X_t)\, dA_t. \tag{2.1}$$

A standard argument shows that $\mathscr{U}_A^\alpha F \in \mathscr{E}_+^*$. For $f \in \mathscr{E}_+^*$, define

$$U_A^\alpha f(x) = E^x \int_0^\infty e^{-\alpha t} f(X_t)\, dA_t \tag{2.2}$$

and

$$W_A^\alpha f(x) = E^x \int_0^\infty e^{-\alpha t} f(X_{t-})\, dA_t. \tag{2.3}$$

Then $U_A^\alpha f$ and $W_A^\alpha f \in \mathscr{E}_+^*$, and we call them respectively the right and left α-potentials of f with respect to A. Of course, $U_A^\alpha f$ is the usual α-potential of f with respect to A. One has the obvious relations

$$\mathscr{U}_A^\alpha F(x) = U_A^\alpha f(x) \quad \text{if } F(y,z) = f(z),$$

and

$$\mathscr{U}_A^\alpha F(x) = W_A^\alpha g(x) \quad \text{if } F(y,z) = g(y).$$

If A is natural (i.e. almost surely, A and X have no common discontinuity) then $U_A^\alpha f \equiv W_A^\alpha f$ and $\mathscr{U}_A^\alpha F$ reduces to $U_A^\alpha f$, where $f(x) = F(x,x)$. The usefulness of $\mathscr{U}_A^\alpha$ and W_A^α is confined therefore to their use in the study of non-natural AF's.

In the study of $\mathscr{U}_A^\alpha$, an important tool is the analogue of the resolvent equation to the effect that under certain finiteness assumptions,

$$U_A^\alpha f - U_A^\beta f = (\beta - \alpha)\, U^\alpha U_A^\beta f = (\beta - \alpha)\, U^\beta U_A^\alpha f.$$

In exactly the same way, we obtain

Lemma 2.1. *Let $A \in \mathscr{A}$, $\alpha \geqq 0$, $\beta \geqq 0$ and $F \in (\mathscr{E} \times \mathscr{E})_+^*$. If $\mathscr{U}_A^\alpha F(x)$ and $\mathscr{U}_A^\beta F(x)$ are finite, then*

$$\mathscr{U}_A^\alpha F(x) - \mathscr{U}_A^\beta F(x) = (\beta - \alpha)\, U^\alpha \mathscr{U}_A^\beta F(x) = (\beta - \alpha)\, U^\beta \mathscr{U}_A^\alpha F(x).$$

If one drops the finiteness assumptions, one has

$$\mathscr{U}_A^\alpha F \leqq \mathscr{U}_A^\beta F + (\beta - \alpha)\, U^\alpha \mathscr{U}_A^\beta F \qquad \text{if } \alpha \leqq \beta.$$

From this lemma follows the equation for U_A^α cited above as well as the analogous formula for W_A^α.

It should be remarked that in order to conform to the notational conventions about operators and kernels stated in [1], p. 253, one should regard $F(y,z)$ as a function of the single vector (y,z), and the kernel corresponding to the operator $\mathscr{U}_A^\alpha$ as $\mathscr{U}_A^\alpha(x, d(y,z))$.

Lemma 2.2. *If $F \in (\mathscr{E} \times \mathscr{E})_+^*$, then $\mathscr{U}_A^\alpha F \in \mathscr{S}^\alpha$ if $A \in \mathscr{A}$, and consequently $U_A^\alpha f$ and $W_A^\alpha f \in \mathscr{S}^\alpha$ if $f \in \mathscr{E}_+^*$, $A \in \mathscr{A}$.*

Proof. A routine computation of $P_t^\alpha \mathscr{U}_A^\alpha F$. □

Notice that if $A \in \mathscr{A}$ and $f \in b\mathscr{E}_+^*$, then

$$B_t = \int_{(0,t]} f(X_s)\, dA_s \quad \text{and} \quad B_t' = \int_{(0,t]} f(X_{s-})\, dA_s$$

are in $\mathscr{A}$, and if $F \in b(\mathscr{E} \times \mathscr{E})_+^*$, then $C_t = \int_{(0,t]} F(X_{s-}, X_s)\, dA_s$ is in $\mathscr{A}$.

A result of Meyer states (see [1] p. 157) that if A and B are AF's such that for some fixed $\alpha \geqq 0$, $u_A^\alpha (= U_A^\alpha 1)$ is everywhere finite and if $U_A^\alpha f = U_B^\alpha f$ for all $f \in C_K^+$, then A and B are equivalent. We want something a bit different from this.

Proposition 2.3. *Let A, $B \in \mathscr{A}$ and suppose that for some fixed $\alpha \geqq 0$, $u_A^\alpha < \infty$ a.e. (ξ) and that $U_A^\alpha f = U_\beta^\alpha f$ for all $f \in b\mathscr{E}_+$ having compact support. Then A and B are equivalent.*

Proof. The following nice proof was supplied to the author by R.K. Getoor.

Notice firstly that $\{u_A^\alpha = \infty\} = P$ is polar. If one examines the proofs of Proposition 2.8, 2.11 and 2.12 of ([1], Ch. IV), one sees that if $x \notin P$, then (2.9) holds with $M_t = 1_{[0,\zeta)}(t)$, and that (2.11) holds with $M_t = 1_{[0,\zeta)}(t)$ if one assumes $E^x\{f(X_t) A_t\} = E^x\{f(X_t) B_t\}$ for all $f \in b\mathscr{E}_+^*$ $t \geqq 0$, $x \notin P$. Finally, for A and B as in the statement of the present proposition, we obtain, as in the proof of (2.12) that for $\beta > \alpha$, $U_A^\beta f = U_A^\beta f$ off P, and consequently $U_A^\beta U^\alpha f = U_B^\beta U^\beta f$ off P. This implies that $A = B$ a.s. P^x, $x \notin P$. Consider now $T = \inf\{t\colon A_t \neq B_t\}$. It is easy to see that T is an exact terminal time, and $P^x(T = \infty) = 1$ if $x \notin P$. But $\varphi(x) = E^x\{e^{-T}\}$ is 1-excessive and equals 0 when $x \notin P$, so $\varphi(x)$ is identically 0, which tells us that A and B are equivalent. ▯

Thus, if $A \in \mathscr{A}$ and $u_A^\alpha < \infty$ a.e. (ξ), the operators U_A^α and $\mathscr{U}_A^\alpha$ determine A uniquely. This is not the case with the left α-potential operator W_A^α, unless A is assumed to be natural.

Proposition 2.4. *If A and B are AF's such that $E^x A_t = E^x B_t < \infty$ for every $x \in E$ and $t \geqq 0$, and if A has a finite α-potential for some $\alpha > 0$, then $W_A^\alpha = W_B^\alpha$ as operators on $b\mathscr{E}_+$. Conversely, if for some $\alpha > 0$, $W_A^\alpha = W_B^\alpha$ as operators on $b\mathscr{E}_+$, and if u_A^α is bounded then $E^x A_t = E^x B_t$ for all $x \in E$ and $t \geqq 0$.*

Proof. If $E^x A_t = E^x B_t$ for all $x \in E$ and $t \geqq 0$, then $A, B \in \mathscr{A}$, and it suffices to prove that $W_A^\alpha f = W_B^\alpha f$ for every $f \in C_K^+$, $x \in E$. Since $t \to f(X_{t-})$ is left continuous a.s. and $A_t - B_t$ is a martingale relative to each P^x, the result follows by T.17 of ([4], Ch. VII). Conversely, if $u_A^\alpha(x)$ is bounded and $W_A^\alpha = W_B^\alpha$, then

$$E^x A_t \leqq e^{\alpha t} E^x \int_{(0,t]} e^{-\alpha s}\, dA_s \leqq e^{\alpha t} u_A^\alpha(x) < \infty, \quad \text{and} \quad u_B^\alpha(x) = u_A^\alpha(x)$$

so $E^x B_t < \infty$ also, for all $t \geqq 0$, $x \in E$. From the equations in Lemma 2.1, we obtain that $W_A^\beta = W_A^\beta$ for all $\beta \geqq \alpha$, and hence $u_A^\beta = u_B^\beta$ is bounded for all $\beta \geqq \alpha$. Since $E^x e^{-\alpha t} A_t \leqq u_A^\alpha(x) \leqq C = \sup_x u_A^\alpha(x) < \infty$, then if $\beta > \alpha$, $E^x e^{-\beta t} A_t \leqq C e^{-(\beta-\alpha)t} \to 0$ as $t \to \infty$, and the same holds for B_t. Then we obtain

$$\begin{aligned} u_A^\beta(x) &= \lim_{t\to\infty} E^x \int_{(0,t]} e^{-\beta s}\, dA_s \\ &= \lim_{t\to\infty} E^x \left[e^{-\beta t} A_t + \beta \int_0^t e^{-\beta s} A_s\, ds \right] \\ &= \beta \int_0^\infty e^{-\beta s} E^x A_s\, ds \qquad \text{if } \beta > \alpha, \end{aligned}$$

and similarly

$$u_B^\beta(x) = \beta \int_0^\infty e^{-\beta s} E^x B_s\, ds.$$

Thus $E^x A_s = E^x B_s$ for a.a.s (Lebesgue) by uniqueness of Laplace transforms, and hence for all $s \geq 0$, $x \in E$, using right-continuity of these functions in s. $\square$

3. The Measures Associated with an AF

Inspired by Revuz [6], we define, for any AF A of X and $F \in b(\mathscr{E} \times \mathscr{E})_+^*$

$$\nu_A(F) = \sup_{t>0} t^{-1} E^\xi \int_{(0,t]} F(X_{s-}, X_s)\, dA_s.$$

For $f \in b\mathscr{E}_+^*$, define

$$\nu_A^1(f) = \sup_{t>0} t^{-1} E^\xi \int_{(0,t]} f(X_{s-})\, dA_s$$

$$\nu_A^2(f) = \sup_{t>0} t^{-1} E^\xi \int_{(0,t]} f(X_s)\, dA_s.$$

The $\nu_A(f)$ of Revuz is $\nu_A^2(f)$ in our notation. If $F(x, y) = f(x)$, $\nu_A(F) = \nu_A^1(f)$, and if $F(x, y) = g(y)$, then $\nu_A(F) = \nu_A^2(g)$.

We call A *integrable* if $\nu_A(1) < \infty$. (This is not the same as $E^x A_t < \infty$ for all $t \geq 0$ and $x \in E$.) If there is a decomposition of $E \times E$ into a countable union of sets F_i such that $\nu_A(1_{F_i}) < \infty$ for each i, we call A *σ-integrable*.

We remark that A is integrable in our sense if and only if it is integrable in Revuz's sense, but our definition for σ-integrability is more general than that of Revuz which allows only decompositions of the form $E \times E_i$.

With only a trivial modification of Revuz's proof of the analogous proposition, we obtain

Proposition 3.1. *Let A be an* AF *of X and $F \in (\mathscr{E} \times \mathscr{E})_+^*$.*

(a)
$$\begin{aligned}\nu_A(F) &= \lim_{t \to 0} t^{-1} E^\xi \int_{(0,t]} F(X_{s-}, X_s)\, dA_s \\ &= \lim_{\alpha \to \infty} \alpha \langle \xi, \mathscr{U}_A^\alpha F \rangle\end{aligned}$$

and the latter limit is increasing.

(b) *If A is σ-integrable, the mapping $F \to \nu_A(F)$ is a positive σ-finite measure which is a finite measure iff A is integrable. Denote the measure also by ν_A.*

Obviously, if A is σ-integrable, ν_A^1 and ν_A^2 are (possibly non-σ-finite) measures on E such that $\nu_A(\Gamma \times E) = \nu_A^1(\Gamma)$ and $\nu_A(E \times \Lambda) = \nu_A^2(\Lambda)$.

It is clear that

$$\nu_A^1(f) = \lim_{t \to 0} t^{-1} E^\xi \int_{(0,t]} f(X_{s-})\, dA_s = \lim_{\alpha \to \infty} \alpha \langle \xi, W_A^\alpha f \rangle$$

and

$$\nu_A^2(f) = \lim_{t \to 0} t^{-1} E^\xi \int_{(0,t]} f(X_s)\, dA_s = \lim_{\alpha \to \infty} \alpha \langle \xi, U_A^\alpha f \rangle$$

if $f \in b\mathscr{E}_+^*$.

Using the fact that if K is polar in E, then for all $x \in E$, $P^x\{X_t$ or $X_{t-} \in K$ for some $t > 0\} = 0$, one sees that $\nu_A^1(K) = \nu_A^2(K) = 0$ for all A. One sees in the same sort of way that if A is continuous and L is semipolar in E, then $\nu_A^1(L) = \nu_A^2(L) = 0$.

It is immediate that if $A \in \mathscr{A}$ and $F \in b(\mathscr{E} \times \mathscr{E})^*_+$, then

$$B_t = \int_{(0,t]} F(X_{s-}, X_s)\, dA_s \text{ is in } \mathscr{A} \text{ and } \nu_B(G) = \nu_A(FG), \quad G \in b(\mathscr{E} \times \mathscr{E})^*_+.$$

Thus, in particular, if $A \in \mathscr{A}$ is σ-integrable, B is σ-integrable and $d\nu_B = F\, d\nu_A$. Let $\mathscr{I}$ denote the class of integrable AF's of X, and let $\sigma\mathscr{I}$ denote the class of σ-integrable AF's of X. It is shown in [6] that $\mathscr{I} \subset \mathscr{A}$ and that every AF A of X whose jumps are a.s. bounded away from ∞ is in $\sigma\mathscr{I}$. We shall see later, under duality hypotheses, that $\mathscr{A} \subset \sigma\mathscr{I}$. For the moment, we content ourselves with a much simpler result.

Proposition 3.2. *If $A \in \mathscr{A}$, then A can be expressed as $\sum_{n=1}^{\infty} A^n$ where each A^n is integrable, $A^n \in \mathscr{A}$.*

Proof. Let $B_t = \sum_{s \leq t} \Delta A_s$, $C_t = A_t - B_t$, so that B and $C \in \mathscr{A}$, and C is continuous. Then C is σ-integrable by ([7], I.3), and

$$C = \sum_n C^n, \qquad C^n_t = \int_{(0,t]} 1_{E_n}(X_s)\, dA_s$$

defining an integrable AF. Write

$$B_t = \sum_{n=1}^{\infty} B^n_t \quad \text{where} \quad B^n_t = \sum_{s \leq t} \Delta A_s \cdot 1_{\{\Delta A_s \in [n-1, n)\}}.$$

Then each B^n is σ-integrable, by ([7], I.3), and so can be expressed as a sum of integrable AF's. ▯

This means that for $F \in (\mathscr{E} \times \mathscr{E})^*_+$ and $A \in \mathscr{A}$, $\nu_A(F) = \sum_{n=1}^{\infty} \nu_{A^n}(F)$, using the monotonicity in the limit which defines $\nu_A(F)$. Thus if $A \in \mathscr{A}$, ν_A is a countable sum of finite measures.

Proposition 3.3. *Let $A \in \sigma\mathscr{I}$. Then A is natural if and only if ν_A is carried by the diagonal D in $E \times E$, and in this case $\nu^1_A = \nu^2_A$. The same result holds if one assumes $A \in \mathscr{A}$.*

Proof. If $A \in \sigma\mathscr{I}$ is natural, and $F \in b(\mathscr{E} \times \mathscr{E})_+$ vanishes on D, then clearly

$$\mathscr{U}^\alpha_A F(x) = E^x \int_0^\infty e^{-\alpha t} F(X_{t-}, X_t)\, dA_t$$

is identically zero, so $\nu_A(F) = 0$, hence ν_A is carried by D. On the other hand, if $A \in \sigma\mathscr{I}$ and ν_A is carried by D, then for any pair of disjoint Borel sets K, L, $\nu_A(K \times L) = 0$, so for all $\alpha \geq 0$, $U^\alpha_A 1_{K \times L} = 0$ a.e. (ξ), and since $\mathscr{U}^\alpha_A 1_{K \times L} \in \mathscr{S}^\alpha$, $U^\alpha_A 1_{K \times L} = 0$ everywhere. Hence we have a.s. $\Delta A_t = 0$ for all t such that $X_{t-} \in K$ and $X_t \in L$. We can however, find a countable collection of such pairs K_n, L_n, $n \geq 1$, such that $E \times E \setminus D = \bigcup_n K_n \times L_n$ and then a.s. $\Delta A_t = 0$ for all t such that $X_{t-} \neq X_t$, $t < \zeta$.

The modification to the case where $A \in \mathscr{A}$ is simple. ▯

It is apparent from Proposition 3.2 that nothing new is being introduced when A is natural. A non-decreasing right-continuous function Φ on $[0, \infty)$ is called purely discontinuous if for every $t>0$, $\Phi(t)=\sum_{s\leq t} \Delta\Phi(s)$. An AF A of X is called purely discontinuous if the sample paths $t\to A_t$ are a.s. purely discontinuous. We call an AF A of X quasi-left-continuous if a.s., all jumps of the sample paths $t\to A_t$ occur at jump times of the sample paths $t\to X_t$. It is well known that A is quasi-left-continuous (q.l.c.) if and only if $A_{T_n}\to A_T$ whenever $\{T_n\}$ is a sequence of stopping times which increases to T, in case X is special standard.

Proposition 3.4. *If $A\in\sigma\mathscr{I}$, then A is purely discontinuous and q.l.c. if and only if $\nu_A(D)=0$. The same result holds if $A\in\mathscr{A}$.*

Proof. If A is purely discontinuous and q.l.c., then for every

$$\alpha>0, \qquad U_A^\alpha 1_D(x)=E^x\int_0^\infty e^{-\alpha t}\, 1_D(X_{t-}, X_t)\, dA_t=0 \qquad \text{for all } x\in E,$$

so $\nu_A(D)=0$. Conversely, if $\nu_A(D)=0$, then $U_A^\alpha 1_D=0$ a.e. (ξ) and hence $U_A^\alpha 1_D\equiv 0$, for each $\alpha>0$. This means that the measure on $[0,\zeta)$ determined by $t\to A_t(\omega)$ is carried by the countable set $\{t\colon X_{t-}(\omega)\neq X_t(\omega)\}$ a.s. and this obviously implies that A is purely discontinuous and q.l.c. $\square$

Once again, the case $A\in\mathscr{A}$ is settled by trivial modification.

For any $A\in\sigma\mathscr{I}$, the part of ν_A which will and be of interest to us is the off-diagonal part, and we shall see in the next section that under duality hypotheses, we can determine the most general purely discontinuous q.l.c. AF of X.

4. The Representation of the Bipotential

In this section, we assume that the standard process X is in duality with a standard process $\hat{X}$ relative to the σ-finite measure ξ. The reader is referred to [1], Chapter VI, for details, but briefly, it is assumed that there exist functions $u^\alpha(x,y)$ on $E\times E$, $\alpha>0$, such that the resolvents $\{U^\alpha\}$ and $\{\hat{U}^\alpha\}$ of X and $\hat{X}$ respectively satisfy, for all $\alpha>0$

$$\begin{array}{ll} \text{(i)} & U^\alpha(x,dy)=u^\alpha(x,y)\,\xi(dy), \qquad \hat{U}^\alpha(dx,y)=\xi(dx)\,u^\alpha(x,y) \\ \text{(ii)} & x\to u^\alpha(x,y) \text{ is } \alpha\text{-excessive for } X \\ \text{(iii)} & y\to u^\alpha(x,y) \text{ is } \alpha\text{-excessive for } \hat{X} \text{ (i.e. } \alpha\text{-coexcessive).} \end{array} \tag{4.1}$$

No regularity assumptions are made on the resolvents, nor is it assumed that either process is special standard. We write $\xi(dx)=dx$ and $\langle f,g\rangle=\int f(x)\,g(x)\,dx$ if $f, g\in\mathscr{E}_+^*$.

Our first result, though having very special hypotheses, contains the core of the results on representing the bipotential operator, and follows by trivial modification of the very nice argument of Revuz [6] in the natural case.

Proposition 4.1. *Let $A\in\mathscr{I}$. Then $u_A^\alpha<\infty$ a.e. (ξ) if $\alpha>0$, and*

$$u_A^\alpha(x)=U^\alpha\,\nu_A^1(x)=\int u^\alpha(x,y)\,\nu_A^1(dy).$$

Proof. The fact that $u_A^\alpha < \infty$ a.e. (ξ) follows from $\langle 1, \alpha u_A^\alpha \rangle \leqq \nu_A(1) < \infty$. Fix $\alpha > 0$, and for $\phi \in b\mathscr{E}_+ \cap \mathscr{L}^1(d\xi)$ consider

$$\nu_A^1(\phi\, \hat{U}^\alpha) = \lim_{\beta \to \infty} \beta E^\xi \int_0^\infty e^{-\beta s} \phi\, \hat{U}^\alpha(X_{s-})\, dA_s.$$

The remainder of the proof is exactly that of Revuz ([6], V.1). We provide a brief sketch.

By a theorem of Weil [9], the mapping $s \to \phi\, \hat{U}^\alpha(X_{s-})$ is left-continuous, so

$$\sum_{k=0}^\infty e^{-\beta k/2^n} \phi\, \hat{U}^\alpha(X_{k/2^n})\, 1_{(k/2^n, k+1/2^n]}(s) \to e^{-\beta s} \phi\, \hat{U}^\alpha(X_{s-})$$

as $n \to \infty$, and this implies that

$$\begin{aligned}
\nu_A^1(\phi\, \hat{U}^\alpha) &= \lim_{\beta \to \infty} \lim_{n \to \infty} \beta E^\xi \sum_{k=0}^\infty e^{-\beta k/2^n} \phi\, \hat{U}^\alpha(X_{k/2^n -})\,[A_{k+1/2^n} - A_{k/2^n}] \\
&\leqq \lim_{\beta \to \infty} \lim_{n \to \infty} \left(\beta \sum_{k=0}^\infty e^{-\beta k/2^n}\right) \int \phi\, \hat{U}^\alpha(y)\, E^y A_{1/2^n}\, \xi(dy) \\
&= \lim_{n \to \infty} \int 2^n \phi(y)\, U^\alpha(E^\bullet A_{1/2^n})(y)\, \xi(dy) \\
&= \alpha \int_0^\infty e^{-\alpha s} E^{\phi\xi}(A_s)\, ds = \langle \phi, u_A^\alpha \rangle.
\end{aligned}$$

Thus $\langle \phi, U^\alpha \nu_A^1 \rangle \leqq \langle \phi, u_A^\alpha \rangle$ for all $\phi \in b\mathscr{E}_+ \cap \mathscr{L}^1(d\xi)$, so $U^\alpha \nu_A^1 \leqq u_A^\alpha$ a.e. (ξ), hence everywhere, since both functions are in $\mathscr{S}^\alpha$.

On the other hand, the resolvent equations and Eq. (2.2) give

$$u_A^\alpha - U^\alpha \nu_A^1 = \lim_{\beta \to \infty} U^\alpha(\beta u_A^{\beta+\alpha} - \beta U^{\beta+\alpha} \nu_A^1),$$

so

$$\int dx\,[u_A^\alpha(x) - U^\alpha \nu_A^1(x)] \leqq \liminf_{\beta \to \infty} \alpha^{-1} \int dx\,[\beta u_A^{\beta+\alpha}(x) - \beta U^{\beta+\alpha} \nu_A^1(x)] = 0$$

so $u_A^\alpha = U^\alpha \nu_A^1$. □

Theorem 4.2. *If $A \in \mathscr{A}$ and $F \in (\mathscr{E} \times \mathscr{E})_+^*$, then*

$$\mathscr{U}_A^\alpha F(x) = \iint_{E \times E} u^\alpha(x, y)\, F(y, z)\, \nu_A(dy, dz).$$

In particular, if $f \in \mathscr{E}_+^*$,

$$W_A^\alpha f(x) = \int u^\alpha(x, y)\, f(y)\, \nu_A^1(dy)$$

and

$$U_A^\alpha f(x) = \iint u^\alpha(x, y)\, f(z)\, \nu_A(dy, dz).$$

Proof. One need only observe that $A = \sum A^n$ where each A^n is integrable and for which the corresponding formula holds, because if F is bounded and we define

$$B_t^n = \int_{(0,t]} F(X_{s-}, X_s)\, dA_s^n, \qquad \text{then } d\nu_{B^n} = F\, d\nu_{A^n}.$$

Upon passage through a monotone limit, the general result follows. □

Proposition 4.3. *If $A \in \mathscr{A}$ and $u_A^\alpha < \infty$ a.e. (ξ) then v_A determines A uniquely.*

Proof. Immediate, using Proposition 2.3 and Theorem 4.2. □

Proposition 4.4. *If $A \in \mathscr{A}$ is σ-integrable, then v_A determines A uniquely.*

Proof. Let $A, B \in \mathscr{A}$ and suppose $v_A = v_B$ is σ-finite.

Let $E \times E = \bigcup_{n=1}^{\infty} \Gamma_n$ (disjoint), where $v_A(\Gamma_n) < \infty$ for $n = 1, 2, \ldots$. Let

$$A_t^n = \int_{(0,t]} 1_{\Gamma_n}(X_{s-}, X_s)\, dA_s \quad \text{and} \quad B_t^n = \int_{(0,t]} 1_{\Gamma_n}(X_{s-}, X)\, dA_s.$$

We have $dv_{A^n} = dv_{B^n} = 1_{\Gamma_n} dv_A$, so by Proposition 4.3, A^n and B^n are equivalent, so $A = \sum A^n$ and $B = \sum B^n$ are equivalent. □

For later use, we record here the following characterization of associated AF's with bounded α-potentials. Recall that two AF's A, B of X are said to be associated if for all $x \in E$ and all $t \geqq 0$, $E^x A_t = E^x B_t < \infty$.

Proposition 4.5. *Let $A \in \mathscr{A}$ have bounded α-potential for some $\alpha > 0$ and let $B \in \mathscr{A}$. Then A and B are associated if and only if $v_A^1 = v_B^1$.*

Proof. If A and B are associated, then $W_A^\alpha = W_B^\alpha$ by Proposition 2.4, so $u_A^\alpha = u_B^\alpha$, and this proves that $v_A^1 = v_B^1$ because of Proposition 4.1 and Proposition 1.15 of ([1], Ch. VI).

Conversely, $v_A^1 = v_B^1$ implies $W_A^\alpha = W_B^\alpha$ because of Theorem 4.2, and so by Proposition 2.4, A and B are associated. □

We are now going to associate with the standard process X (in duality with $\hat{X}$ relative to ξ) a canonical measure v on $E \times E$ which will reflect the jumping behavior of X. In the sequel, we shall find ourselves frequently in the following situation: let K and L be Borel subsets of E which have disjoint compact closures in E, or more generally, let Γ be a Borel subset of $E \times E$ whose closure in $E \times E$ is compact and disjoint from the diagonal. Define in the first case, $J_{K,L} = \inf\{t > 0\colon X_{t-} \in K, X_t \in L\} \wedge \zeta$ and in the second case $J_\Gamma = \inf\{t > 0\colon (X_{t-}, X_t) \in T\} \wedge \zeta$. It is clear that $J = J_\Gamma$ is a thin terminal time, and we denote by J^n the n-th iterate of J, namely, $J^1 = J$ and inductively, $J^{n+1} = J^n + J \circ \theta_{J^n}$. Because X possesses left limits on $[0, \zeta)$ a.s., $\lim_{n\to\infty} J^n = \zeta$ a.s. We call $J_{K,L}$ the time of first jump from K to L, and J_Γ the time of first jump within Γ.

If Γ is as above, we consider the AF ${}^\Gamma A$ defined by

$$\begin{aligned} {}^\Gamma A_t &= \sum_{s \leqq t} 1_\Gamma(X_{s-}, X_s) \\ &= \sum_{n=1}^{\infty} 1_{\{J_\Gamma^n \leqq t\}} \end{aligned}$$

so that ${}^\Gamma A$ counts the number of jumps within Γ. Since $\lim_{n\to\infty} J_\Gamma^n = \zeta$, ${}^\Gamma A \in \mathscr{A}$. Since ${}^\Gamma A$ has bounded jumps, it is σ-integrable because of Theorem I.3 of Revuz [7]. The measure ${}^\Gamma v$ for ${}^\Gamma A$ clearly is carried by Γ, and if Λ is another such Borel subset of $E \times E$, it is easy to see that the σ-finite measures ${}^\Gamma v$ and ${}^\Lambda v$ agree on $\Gamma \cap \Lambda$. There exists therefore, a σ-finite measure v on $E \times E$ which assigns zero mass to D and such that if $\Gamma\in\mathscr{E}\times\mathscr{E}$ has compact closure disjoint from D, then ${}^{\Gamma}v$ is the restriction of v to Γ. Thus, for such Γ,

$$E^x \sum_{t\geqq 0} e^{-\alpha t}\, 1_\Gamma(X_{t-}, X_t) = \iint_{E\times E} u^\alpha(x,y)\, 1_\Gamma(y,z)\, v(dy,dz). \tag{4.2}$$

But (4.2) uniquely determines v, for the following reason. Let v_1 and v_2 be two σ-finite measures satisfying (4.2) for all $\Gamma\in\mathscr{E}\times\mathscr{E}$ having compact closure disjoint from D, with $\alpha=1$, say. Fix such a Γ and let $A={}^{\Gamma}A$. By Theorem I.3 of Revuz [7], E is the union of an increasing sequence E_n of nearly Borel sets in E such that for each $n\geqq 1$, $U_A^1(x,E_n)$ is bounded and integrable in x. Let $\Gamma_n=\Gamma\cap(E\times E_n)$, $A^n={}^{\Gamma_n}A$, $d\mu_1=1_{\Gamma_n}dv_1$, $d\mu_2=1_{\Gamma_n}dv_2$. We have $U_A^1(x,E_n)=u_{A^n}^1(x)$ which is equal, by (4.2), to $U^1\mu_1^1$ and $U^1\mu_2^1$. Thus $\mu_1^1=u_2^1$ by Proposition 1.15 of ([1], Ch. VI) and this implies that $v_1(\Gamma_n)=v_2(\Gamma_n)$, so that we obtain, finally, $v_1(\Gamma)=v_2(\Gamma)$, and thus $v_1=v_2$.

What has been done therefore is that a unique σ-finite measure v on $E\times E$ such that $v(D)=0$ has been associated with X in such a way that (4.2) holds.

Remark. The intuitive notion that $\hat{X}$ is X run backwards in time suggests that the canonical measure $\hat{v}$ for $\hat{X}$ ought to be obtained from v by reversing the coordinates. That this is so follows from recent work of Getoor [3].

Proposition 4.6. *The jumping measures v and $\hat{v}$ for the dual processes X and $\hat{X}$ satisfy $v(K\times L)=\hat{v}(L\times K)$ for any K, $L\in\mathscr{E}$.*

Proof. It suffices to prove that $v(K\times L)=\hat{v}(L\times K)$ in case K and L are open sets in E having disjoint compact closures. By Proposition 2.5 of [3], $T=J_{K,L}$ and $\hat{T}=\hat{J}_{L,K}$ are dual exact terminal times in the sense that $P_T^\alpha u^\alpha=u^\alpha\hat{P}_T^\alpha$. It follows inductively that $(P_T^\alpha)^n u^\alpha=u^\alpha(\hat{P}_T^\alpha)^n$ for $n=1,2,\ldots$, so $P_{T^n}^\alpha u^\alpha=u^\alpha P_{T^n}^\alpha$. The equations $\langle 1, U^\alpha 1\rangle=\langle 1\,\hat{U}^\alpha, 1\rangle$ and $\langle 1, P_{T^n}^\alpha U^\alpha 1\rangle=\langle 1\,\hat{U}^\alpha \hat{P}_{T^n}^\alpha, 1\rangle$ together imply, using Dynkin's lemma, that

$$E^\xi\int_0^{T^n} e^{-\alpha t}\,dt=\hat{E}^\xi\int_0^{T^n} e^{-\alpha t}\,dt,$$

so that $E^\xi e^{-\alpha T^n}=\hat{E}^\xi e^{-\alpha T^n}$ for $n=1,2,\ldots$. Thus, if

$$A_t=\sum_{n=1}^\infty 1_{\{T^n\leqq t\}} \quad\text{and}\quad \hat{A}_t=\sum_{n=1}^\infty 1_{\{\hat{T}^n\leqq t\}},$$

$$\hat{v}(L\times K)=\hat{v}_A(1)=\lim_{\alpha\to\infty}\alpha\hat{E}^\xi\sum_{n=1}^\infty e^{-\alpha T^n}=\lim_{\alpha\to\infty}\alpha E^\xi\sum_{n=1}^\infty e^{-\alpha T^n}=v_A(1)=v(K\times L). \quad\square$$

5. The Representation of a Purely Discontinuous q.l.c. AF

Watanabe [8] has shown that every purely discontinuous q.l.c. AF of a special standard process X has the form $\sum_{s\leqq t}F(X_{s-},X_s)$, his proof depending on the existence of a Lévy system for the process, which relied in turn on fairly deep martingale techniques. In this section, we prove this result by simpler methods in the case where X is assumed to be in duality with another standard process $\hat{X}$, though it is not assumed that X is special standard. For the remainder of the paper, the duality assumptions stated at the beginning of Section 4 are in force.

Proposition 5.1. *Let $A \in \mathscr{A}$ be purely discontinuous and q.l.c. and suppose that a.s., the jumps of A are $\leqq \beta < \infty$. Then A is σ-integrable and $\nu_A \ll \nu$. If F is a Radon-Nikodym derivative of ν_A with respect to ν which vanishes on D, then $F \leqq \beta$ a.e. (ν) on $E \times E$ and A_t is equivalent to $C_t = \sum_{s \leqq t} F(X_{s-}, X_s)$.*

Proof. That A is σ-integrable follows from a previously discussed theorem of Revuz. In order to prove that $\nu_A \ll \nu$ and $F \leqq \beta$ a.e. (ν), it suffices to prove that if $\Gamma \in \mathscr{E} \times \mathscr{E}$ has compact closure disjoint from D and $B = {}^{\Gamma}A$, then $\nu_A(G) \leqq \beta\, \nu_B(G)$ for every $G \in b(\mathscr{E} \times \mathscr{E})_+$ which vanishes off Γ, and for this inequality it suffices to prove that $\mathscr{U}_A^\alpha G \leqq \beta\, \mathscr{U}_B^\alpha G$ for every $\alpha > 0$. But for such G,

$$\begin{aligned}\mathscr{U}_A^\alpha G(x) &= E^x \int_0^\infty e^{-\alpha t}\, G(X_{t-}, X_t)\, dA_t \\ &= E^x \sum_{n=1}^\infty e^{-\alpha J_\Gamma^n} G(X_{J_\Gamma^n -}, X_{J_\Gamma^n})\, \Delta A_{J_\Gamma^n} \\ &\leqq \beta E^x \sum_{n=1}^\infty e^{-\alpha J_\Gamma^n}\, G(X_{J_\Gamma^n -}, X_{J_\Gamma^n}) \\ &= \beta\, \mathscr{U}_B^\alpha\, G(x).\end{aligned}$$

Let $E \times E \setminus D = \bigcup_{n=1}^\infty \Gamma_n$ (disjoint) where each $\Gamma_n \in \mathscr{E} \times \mathscr{E}$ has compact closure disjoint from D. Let

$$B^n = {}^{\Gamma_n}A, \qquad C_t^n = \int_{(0,t]} F(X_{s-}, X_s)\, dB_s^n.$$

Then

$$C_t = \sum_{s \leqq t} F(X_{s-}, X_s) = \sum_{n=1}^\infty \sum_{s \leqq t} (1_{\Gamma_n} F)(X_{s-}, X_s) = \sum_{n=1}^\infty C_t^n$$

and it follows that $d\nu_C = F\, d\nu = d\nu_A$, and this proves that A and C are equivalent, by Proposition 4.4. □

Theorem 5.2. *Let $A \in \mathscr{A}$ be purely discontinuous and q.l.c. Then there exists a finite function $F \in (\mathscr{E} \times \mathscr{E})_+$ vanishing on D such that A_t is equivalent to $C_t = \sum_{s \leqq t} F(X_{s-}, X_s)$. Moreover, $d\nu_A = F\, d\nu$.*

Proof. Let $A_t^n = \sum_{s \leqq t} \Delta A_s\, 1_{\{\Delta A_s \in [n-1, n)\}}$. Then $A^n \in \mathscr{A}$ has jumps bounded by n and $A = \sum_{n=1}^\infty A^n$. By Proposition 5.1, $\nu_{A^n} \ll \nu$ and we can assume that the Radon-Nikodym derivative, F^n, of ν_A^n relative to ν everywhere $\leqq n$. Let $F = \sum_{n=1}^\infty F^n\, 1_{D^c}$. Since A_t^n is equivalent to $C_t^n = \sum_{s \leqq t} F^n(X_{s-}, X_s)$, $A_t = \sum_{n=1}^\infty A_t^n$ is equivalent to

$$\sum_n C_t^n = \sum_{s \leqq t} F(X_{s-}, X_s) = C_t.$$

To see that $F < \infty$ a.e. (ν), let $\Gamma \subset \{F = \infty\}$ be precompact in $E \times E \setminus D$. It will suffice to show that $\nu(\Gamma) = 0$, or equivalently $J_\Gamma \geqq \zeta$ a.s. But if this is not the case, then for some $x \in E$, $P^x\{C_t = \infty$ for some $t < \zeta\} > 0$, and this is known to be false, so $\nu(\Gamma) = 0$. We may therefore modify F on a set of ν-measure zero so that F is finite everywhere without changing C a.s. Finally for any $H \in b(\mathscr{E} \times \mathscr{E})_+$,

$$\nu_A(H) = \sum_{n=1}^{\infty} \nu_{A^n}(H) = \sum_{n=1}^{\infty} \nu(F^n H) = \nu(FH)$$

and therefore $d\nu_A = F\,d\nu$. ▯

Corollary. *Every finite AF of X is σ-integrable.*

Proof. If $A \in \mathscr{A}$, we can write

$$B_t = \int_{(0,t]} 1_D(X_{s-}, X_s)\, dA_s \quad \text{and} \quad C_t = \int_{(0,t]} 1_{D^c}(X_{s-}, X_s)\, dA_s$$

and $A = B + C$ is the decomposition of A into a natural part B and a purely discontinuous q.l.c. part C, each of which is in $\mathscr{A}$ and so each has only finite jumps a.s. By Theorem V.3 of Revuz [6], B is σ-integrable, and by the above theorem, $d\nu_C = F\,d\nu$ for a finite function F, so ν_C is σ-finite. ▯

The question of determining those functions $F \in (\mathscr{E} \times \mathscr{E})_+$ for which the corresponding AF $A_t = \sum_{s \leq t} F(X_{s-}, X_s)$ is in $\mathscr{A}$ is easily settled if one tries an analogue of the technique of Revuz [7], Theorems III.3 and III.4. See Getoor [3] for the exact statement. There is one simple situation however in which this approach is not necessary. If $F < \infty$ a.e. (ν) and F vanishes off a compact subset Γ of $E \times E \setminus D$, then $\sum_{s \leq t} F(X_{s-}, X_s)$ is clearly in $\mathscr{A}$ because it jumps by a finite amount a.s. at the times $J_\Gamma^1, J_\Gamma^2, \ldots$ which tend to ζ a.s.

We are now in a position to describe all finite additive functionals of a process X under duality hypotheses. Let X and ξ satisfy the ongoing duality assumptions. With X we associate the canonical measure ν on $\mathscr{E} \times \mathscr{E}$ and a measure λ defined on the σ-ring $\mathscr{H}$ of semipolar sets by

$$\lambda(B) = \sum_{i=1}^{\infty} \nu_{A^i}^2(B_i) \tag{5.1}$$

where $B = \bigcup_{i=1}^{\infty} B_i$ (disjoint) is a decomposition of B into totally thin sets (sets for which $\sup_{x \in E} E^x e^{-T_{B_i}} < 1$) and $A_t^i = \sum_{n=1}^{\infty} 1_{\{T_{B_i}^n \leq t\}}$. It is shown in [7] (Lemma II.1) that $\lambda(B)$ does not depend on the particular decomposition into totally thin sets, so that λ is indeed a well-defined measure on $\mathscr{H}$. It is clear that λ is σ-finite.

We remark that the canonical measures $\hat{\lambda}$ and $\hat{\nu}$ for the dual process satisfy.

$$\begin{aligned} &\text{(i)} & \hat{\lambda} &= \lambda \\ &\text{(ii)} & \hat{\nu}(K \times L) &= \nu(L \times K) \end{aligned} \tag{5.2}$$

because of Proposition II.2 of [7] and Proposition 4.6.

If $A \in \mathscr{A}$, we can express A as a sum $A^1 + A^2 + A^3$ where $A^1, A^2, A^3 \in \mathscr{A}$, A^1 is continuous, A^2 is natural and purely discontinuous and A^3 is purely discontinuous and q.l.c. The measure ν_A on $E \times E$ may be decomposed into a sum $\nu_1 + \nu_2 + \nu_3$ where $\nu_1 + \nu_2$ is the restriction of ν_A to the diagonal $D \subset E \times E$ and ν_3 is the restriction of ν_A to $E \times E \setminus D$, and where ν_2 is carried by a semipolar set K and ν_1 charges no semipolar set. We then have $\nu_{A^i} = \nu_i$, $i = 1, 2, 3$, and $\nu_2 \ll \lambda$, $\nu_3 \ll \nu$. Thus to each

$A\in\mathscr{A}$, there corresponds a triple (μ, f, F) where μ is a measure on E which doesn't charge semipolars, f is a Borel function on E which vanishes off some semipolar set K and F is a Borel function on $E\times E$ which vanishes on D such that $v_1^1=\mu$, $dv_2=f\,d\lambda$ and $dv_3=F\,dv$, and in the decomposition of A, one has

$$A_t^2=\sum_{i=1}^{\infty}\sum_{n=1}^{\infty} f(X(T_{K_i}^n))\,I_{(0,t]}(T_{K_i}^n)$$

where $K=\bigcup_{i=1}^{\infty} K_i$ (disjoint) is a partition of K into totally thin sets, and $A_t^3=\sum_{s\leqq t} F(X_{s-}, X_s)$ (up to equivalence, of course). The triple (μ, f, F) completely characterizes A and (μ_1, f_1, F_1) represents the same AF as (μ_2, f_2, F_2) if and only if $\mu_1=\mu_2$, $f_1=f_2$ a.e. (λ) and $F_1=F_2$ a.e. (v). The class of possible representing triples (μ, f, F) comprises those triples for which $\mu\in\mathscr{M}$, $f\in\mathscr{N}$ and $F\in\mathscr{D}$, where: $\mathscr{M}$ is the class of measures μ on E which don't charge semipolars and which are such that there exists an increasing sequence $\{E_n\}$ of Borel sets whose union is E and such that $\mu(E_n)<\infty$ for all n, $\int_{E_n} u^1(x,y)\,\mu(dy)$ is bounded in x for all n, and $\lim_{n\to\infty} T_{E_n^c}\geqq\zeta$ a.s.; $\mathscr{N}$ is the class of Borel functions f on E which vanish off some semipolar K and which are such that there exists an increasing sequence $\{E_n\}$ of Borel sets whose union is E and such that for all n,

$$\int_{E_n}(1-e^{-f(y)})\,\lambda(dy)<\infty, \qquad \int_{E_n} u^1(x,y)\,(1-e^{-f(y)})\,\lambda(dy)$$

is bounded in x, and $\lim_{n\to\infty} T_{E_n^c}\geqq\zeta$ a.s.; $\mathscr{D}$ is the class of Borel functions F on $E\times E$ which vanish on D and for which there exists an increasing sequence $\{E_n\}$ of Borel sets whose union is E and such that for all $n\geqq 1$,

$$\iint u^1(x,y)\,(1-e^{-F(y,z)})\,1_{E_n}(z)\,v(dy,dz)$$

is bounded and integrable in x, and $\lim_{n\to\infty} T_{E_n^c}\geqq\zeta$ a.s.

This summarizes work of Revuz ([6], [7]) in the first two cases and Getoor [3] in the last case.

6. Lévy Systems for Dual Processes

By a Lévy system (n, H) for a standard process X, we mean that H is a CAF of X and n is a kernel on $E\times E_\Delta$ (i.e. for all $B\in\mathscr{E}_\Delta$, $x\to n(x,B)$ is in $\mathscr{E}$, and for all $x\in E$, $B\to n(x,B)$ is a σ-finite measure) such that $n(x,\{x\})=0$ for every $x\in E$ and such that for every $F\in(\mathscr{E}\times\mathscr{E})_+$ which vanishes on D, every $x\in E$ and every $t\geqq 0$

$$E^x\sum_{s\leqq t} F(X_{s-}, X_s)=E^x\int_0^t\int_E n(X_s, dy)\,F(X_s, y)\,dH_s. \tag{6.1}$$

Watanabe [8] proved that a Lévy system exists for every special standard process, using the theory of square-integrable martingales. We shall prove here that a Lévy system exists for every process X under the duality assumptions in force here. The method depends only on factoring the canonical measure v for X in an appropriate way.

Proposition 6.1. *If K and L are nearly Borel subsets of E having disjoint compact closures, and if K is semipolar, then $\nu(K \times L)=0$.*

Proof. We may assume that K is thin. Since $E^x\{e^{-T_K}; T_K<\infty\} \in \mathcal{S}^1$ is strictly less than 1, it suffices to prove that $\nu(K\times L)=0$ if K has the property that $\sup_{x\in E} E^x\{e^{-T_K}; T_K<\infty\} \leqq \beta < 1$, for every thin set is a countable union of such nearly Borel sets. A standard argument shows that $\lim_{n\to\infty} T_K^n \geqq \zeta$ a.s., T_K^n being the n-th iterate of the hitting time T_K. Let $D_K=\inf\{t\geqq 0: X_t\in K\}$. It is shown in ([1], p. 59) that

$$D_K=\inf\{t\geqq 0:\ X_t\in K \text{ or } t>0 \text{ and } X_{t-} \text{ exists and is in } K\}.$$

It follows that a.s.

$$\{t>0:\ X_{t-}\in K\}\subset\{T_K, T_K^2, \ldots\}$$

and since $X_{T_K^n}\in K$ for every n, $\{t>0:\ X_{t-}\in K, X_t\in L\}$ is empty a.s., hence $A_t=\sum_{s\leqq t} 1_K(X_{s-})\, 1_\Gamma(X_s)$ is equivalent to the zero AF. This implies $\nu_A=0$, so $\nu(K\times L)=0$.

The simple proof of the following lemma was shown the author by R. K. Getoor, and is much simpler and needs fewer restrictive hypotheses than the version presented in Meyer [5].

Lemma 6.2. *Let λ be a finite measure on E which doesn't charge semipolars. There is an equivalent finite measure μ having a bounded* 1*-potential.*

Proof. Since λ is finite, $U^1\lambda$ is integrable and hence finite except on a polar set. Let $E_0=\{U^1\lambda=\infty\}$ and $E_n=\{U^1\lambda\in[n-1,n)\}$ for $n\geqq 1$, so that $E=\bigcup_{n=0}^{\infty} E_n$ (disjoint), and if we set $d\lambda_k=1_{E_k}\,d\lambda$, then $\lambda_0=0$ and if $k\geqq 1$, λ_k is a measure carried by E_k which doesn't charge semipolars. By the switching identity ([4], VI, 1.16)

$$P_{E_k}^1\, U^1\,\lambda_k = U^1\, \hat{P}_{E_k}^1\,\lambda_k,$$

and

$$\hat{P}_{E_k}^1\,\lambda_k(B)=\int_{E_k} \hat{P}_{E_k}^1(B,x)\,\lambda(dx)=\int_{{}^r E_k} \hat{P}_{E_k}^1(B,x)\,\lambda(dx)$$
$$=\int_{{}^r E_k} \delta_x(B)\,\lambda(dx)=\lambda_k(B),\qquad B\in\mathcal{E}.$$

Hence $P_{E_k}^1\, U^1\,\lambda_k=U^1\,\lambda_k$. But $U^1\,\lambda_k(x)\leqq k$ if x is in the fine closure of E_k, so $P_{E_k}^1\, U^1\,\lambda_k\leqq k$ everywhere, hence $U^1\,\lambda_k\leqq k$. Now let $\mu=\sum_{k=0}^{\infty} 2^{-k}\,\lambda_k$. Then μ is clearly equivalent to λ and it has a bounded 1-potential. □

Theorem 6.3. *The canonical measure ν for a standard process X which is in duality with another standard process $\hat{X}$ relative to the measure ξ can be represented in the form $\nu(dy,dz)=\mu(dy)\,n(y,dz)$ where μ is a finite measure on $(E,\mathcal{E})$ such that $\mu=\nu_H^1$ for some* CAF *$H\in\mathcal{A}$ and n is a kernel on $E\times\mathcal{E}_\Delta$ such that $n(x,\{x\})=0$ for all $x\in E$. Then (n,H) is a Lévy system for X.*

Proof. If $\nu=0$, there is nothing to prove, so we may assume $\nu>0$. Since ν is σ-finite, we may express $E\times E\setminus D$ as a finite or countably infinite union of disjoint

Borel sets Γ_k such that $0<\nu(\Gamma_k)<\infty$ for every k. Define λ_k on $(E,\mathscr{E})$ by

$$\lambda_k(B)=\nu((B\times B)\cap\Gamma_k)/\nu(\Gamma_k), \qquad B\in\mathscr{E}.$$

Each λ_k is a probability measure on $(E,\mathscr{E})$, and if $K\in\mathscr{E}$ is semipolar, $K\times E\setminus D$ may be written as a countable union of products $K_n\times L_n$ where K_n and L_n are Borel sets in E with disjoint compact closures and K_n is semipolar so that $\nu(K_n\times L_n)=0$ for all n because of Proposition 6.1. This shows that $\lambda_k(K)=0$, so λ_k doesn't charge semipolars. Moreover, each of the measures $B\to\iint_{B\times E} 1_{\Gamma_k}(y,z)\,\nu(dy,dz)$ is absolutely continuous so by the usual argument there exists a kernel h_k on $E\times\mathscr{E}_\Delta$ such that

$$1_{\Gamma_k}(y,z)\,\nu(dy,dz)=\lambda_k(dy)\,h_k(y,dz)$$

where $h_k(y,dz)=1_{\Gamma_k}(y,z)\,h_k(y,dz)$ so that in particular $h_k(y,\{y\})=0$ for all $y\in E$. Now define $\lambda=\sum_{k=1}^{\infty}2^{-k}\lambda_k$, a probability measure on $(E,\mathscr{E})$ which doesn't charge semipolars and such that for every k, there exists $f_k\in b\mathscr{E}_+$ with $d\lambda_k=f_k\,d\lambda$. We then have

$$1_{\Gamma_k}(y,z)\,\nu(dy,dz)=\lambda(dy)\,r_k(y,dz)$$

where $r_k(y,dz)=f_k(y)\,h_k(y,dz)$ is a kernel such that $1_{\Gamma_k}(y,z)\,r_k(y,dz)=r_k(y,dz)$. There exists therefore a kernel r on $E\times\mathscr{E}_\Delta$ such that $r(x,\{x\})\equiv 0$ and $I_{\Gamma_k}(y,z)\cdot r(y,dz)=r_k(y,dz)$. Then $\nu(dy,dz)=\lambda(dy)\,r(y,dz)$.

By Lemma 6.2, there is a finite measure μ on $(E,\mathscr{E})$ equivalent to λ and having a bounded 1-potential so there is a kernel n with $n(x,\{x\})\equiv 0$ such that $\nu(dy,dz)=\mu(dy)\,n(y,dz)$.

By the second theorem of V.6 in [6], there exists a CAF H such that $u_H^1=U^1\mu$. Notice that $H\in\mathscr{A}$ since $U^1\mu$ is finite, and $\nu_H^1=\nu_H^2=\mu$.

To prove that (n,H) is a Lévy system for X it suffices, by the monotone class theorem, to verify (6.1) for $F(x,y)=f(x)\,g(y)$ where $f,g\in b\mathscr{E}_+$ are carried by disjoint compact subsets of E.

Let $A_t=\sum_{s\leqq t}f(X_{s-})\,g(X_s)$, so that $A\in\mathscr{A}$. Because the jumps of A are bounded a.s., by Theorem I.3 of [7] E is the union of an increasing sequence $\{E_n\}$ of nearly Borel sets such that $U_A^1(x,E_n)$ is bounded in x for every $n\geqq 1$. Thus, modifying g if necessary, we may prove (6.1) in case u_A^1 is bounded. We have $\nu_A(dy,dz)=f(y)\,g(z)\,\nu(dy,dz)$ so if $h\in b\mathscr{E}_+$,

$$\begin{aligned}\nu_A^1(h)&=\iint h(y)\,f(y)\,g(z)\,\nu(dy,dz)\\&=\iint h(y)\,f(y)\,g(z)\,n(y,dz)\,\mu(dy)\\&=\nu_H^1(h\cdot f\cdot n\,g)\end{aligned}$$

where $n\,g(y)=\int n(y,dz)\,g(z)$. Therefore, $d\nu_A^1=f\cdot n\,g\cdot d\nu_H^1$. By Theorem 4.2,

$$\begin{aligned}u_A^1(x)=W_A^1\,1(x)&=\int_E u^1(x,y)\,f(y)\,n\,g(y)\,\nu_H^1(dy)\\&=U_H^1(f\cdot n\,g)\\&=E^x\int_0^\infty e^{-t}f(X_t)\,n\,g(X_t)\,dH_t\end{aligned}$$

and because $u_A^1(x)$ is finite for every $x \in E$,

$$E^x \int_{(0,t]} f(X_s)\, n g(X_s)\, dH_s < \infty$$

for every $t \geqq 0$, so

$$B_t = \int_{(0,t]} f(X_s)\, n g(X_s)\, dH_s$$

is in $\mathscr{A}$ and hence $d\nu_B^1 = d\nu_A^1$. By Proposition 4.5, A and B are associated, and this gives (6.1). ▯

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