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Random Ergodic Theorem with Weighted Averages

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1. Introduction

In this paper we shall present some convergence theorems analogous to the random ergodic theorems, the novelty being that we use a system of weighted averages.

Suppose there is a measurable space $(\Phi, \mathscr{F})$ consisting of a set Φ of measurable point transformations of a σ-finite measure space $(X, \mathscr{B}, m)$ into itself and the σ-algebra $\mathscr{F}$ of all subsets of Φ.

The random ergodic theorem formulated by Pitt, Ulam, von Neumann and Kakutani was a statement about the average behaviors of measure preserving transformations chosen, at random with the same distribution, from the set Φ. But the random ergodic theorems concerning non-singular transformations have not been formulated up to now.

A natural problem is the following. If we choose a sequence of non-singular transformations from the set Φ at random, not necessarily with the same distribution, but independently, under what conditions do random ergodic theorems hold almost everywhere or in the L_p-mean, with probability one?

Révész [17] raised this question in the case where Φ consists of measure preserving transformations and presented a sufficient condition for the validity of the convergence theorems for a certain class of L_2-functions.

For a given sequence $\{\mu_n, n \geqq 1\}$ of probability measures defined on $\mathscr{F}$, we consider the product measure space $(\Phi^*, \mathscr{F}^*, \mu^*)$:

$$\Phi^* = \Phi_1 \times \Phi_2 \times \cdots, \qquad \mathscr{F}^* = \mathscr{F}_1 \times \mathscr{F}_2 \times \cdots, \qquad \mu^* = \mu_1 \times \mu_2 \times \cdots,$$
$$\Phi_1 = \Phi_2 = \cdots = \Phi, \qquad \mathscr{F}_1 = \mathscr{F}_2 = \cdots = \mathscr{F}.$$

Let H denote the space of all measurable functions $f(x)$ defined on X for which

$$\int_X |f(x)|^2 \, dm < \infty, \qquad \int_X f(x) \, dm = 0,$$

and H^* the space of bounded measurable functions $f(x)$ defined on X for which

$$\int_X |f(x)|^2 \, dm < \infty, \qquad \int_X |f(x)| \, dm < \infty, \qquad \int_X f(x) \, dm = 0.$$

Suppose further that the functions $f(\varphi_k \ldots \varphi_1 x)$, $k \geqq 1$, are measurable and integrable on $X \times \Phi^*$ for every $f(x) \in H$.

Révész's result is then stated as follows.

Theorem R. *If for every* $f(x)\in H$,

$$\left\|\sum_{k=0}^{\infty}\int_{\Phi^*} f(\varphi_{j+k}\cdots\varphi_j x)\,d\mu^*\right\|_{L_2(m)} \leqq K\cdot j^{1-\varepsilon}\cdot\|f\|_{L_2(m)}, \qquad j\geqq 1,$$

where K is an arbitrary positive constant and $0<\varepsilon\leqq 1$, *then*

$$\mu^*\left\{\lim_{n\to\infty}\left\|\frac{1}{n}\sum_{k=1}^{n} f(\varphi_k\cdots\varphi_1 x)\right\|_{L_2(m)}=0\right\}=1 \quad \text{for } f(x)\in H,$$

$$\mu^*\left\{\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f(\varphi_k\cdots\varphi_1 x)=0 \text{ for a.a. } x\right\}=1 \quad \text{for } f(x)\in H^*.$$

It should be noticed that in Theorem R, the limit function is zero and does not depend essentially upon the points (φ^*, x).

But it is not required in the Révész's problem that the limit functions must not depend on the random parameters.

In §2 we shall first prove some general results which include the ergodic theorems of Chacon [5] and of Hopf [14]. One of them yields a generalization of Beck and Schwartz's theorem [2].

Furthermore, we give a proof of a random ergodic theorem for operators which are not necessarily positive and which act in spaces of functions which take their values in an arbitrary reflexive Banach space. An extension of Cairoli's theorem [4] is obtained as an application of this proof.

The method of proof we have used seems best adapted to the systematical study of random ergodic theorems. The results obtained below answer the questions raised by Révész.

Our problem mentioned above is considered in §3 and several forms of the convergence theorems are presented for non-singular transformations.

The obtained results generalize and extend the ergodic theorems of Hurewicz [15], Halmos [12] and Dowker [8].

2. Random Ergodic Theorems Concerning Measure Preserving Transformations

Let $(X, \mathscr{B}, m)$ be a σ-finite measure space and let $\mathfrak{X}$ be a Banach space with the norm $\|\ \|$. By $L_p(m, \mathfrak{X})$, $1\leqq p<\infty$, we denote, as usual, the Banach space of all strongly $\mathscr{B}$-measurable $\mathfrak{X}$-valued functions $f(x)$ defined on X for which

$$\|f\|_{L_p(m,\mathfrak{X})}=\Big(\int_X \|f(x)\|^p\,dm\Big)^{1/p}<\infty.$$

Similarly, we denote by $L_\infty(m, \mathfrak{X})$ the Banach space of all strongly $\mathscr{B}$-measurable $\mathfrak{X}$-valued functions $f(x)$ defined on X for which

$$\|f\|_{L_\infty(m,\mathfrak{X})}=\operatorname{ess\,sup}\|f(x)\|<\infty.$$

If $\mathfrak{X}$ is the linear space of real or complex numbers, we shall use the notations $L_p(m)$ and $L_\infty(m)$ instead of $L_p(m, \mathfrak{X})$ and $L_\infty(m, \mathfrak{X})$ respectively.

We consider the measurable space $(N, \mathscr{A})$ obtained by taking N to be the positive integers and $\mathscr{A}$ the σ-algebra of all subsets of N. Let $\{w_k, k\geqq 1\}$ be a sequence of non-negative numbers whose sum is one and let $\{u_k, k\geqq 0\}$ be the sequence defined by

$$u_0=1, \quad u_k=w_1 u_{k-1}+\cdots+w_k u_0, \quad k\geqq 1.$$

Then for every k, $0\leqq u_k\leqq 1$ and

$$\lim_{n\to\infty} u_n=\begin{cases}\left(\sum_{k=1}^{\infty} k\,w_k\right)^{-1} & \text{if } \sum_{k=1}^{\infty} k\,w_k<\infty, \quad \gcd(k\colon w_k>0)=1,\\ 0 & \text{if } \sum_{k=1}^{\infty} k\,w_k=\infty,\end{cases}$$

(cf. Feller [10], Baxter [1]).

The following result plays a fundamental role in this section.

Theorem 1. *Let $\mathfrak{X}$ be reflexive and let U be a linear contraction operator on $L_1(m,\mathfrak{X})$ as well as on $L_\infty(m,\mathfrak{X})$. Then for every $f\in L_p(m,\mathfrak{X})$ with $1\leqq p<\infty$, there exists a function $f^*\in L_p(m,\mathfrak{X})$ such that*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} u_k(U^k f)(x)=f^*(x) \tag{2.1}$$

strongly in $\mathfrak{X}$ almost everywhere on X and

$$\lim_{n\to\infty}\left\|f^*-\frac{1}{n}\sum_{k=0}^{n-1} u_k(U^k f)\right\|_{L_p(m,\mathfrak{X})}=0, \quad 1<p<\infty. \tag{2.2}$$

Moreover, if m is a finite measure then

$$\lim_{n\to\infty}\left\|f^*-\frac{1}{n}\sum_{k=0}^{n-1} u_k(U^k f)\right\|_{L_1(m,\mathfrak{X})}=0. \tag{2.3}$$

If one takes $w_1=1$, $w_k=0$, $k\geqq 2$, then Theorem 1 is reduced to that of Chacon [5].

Proof of Theorem 1. We consider the product measure space $(N\times X, \mathscr{A}\times\mathscr{B}, \lambda\times m)$, where $(N,\mathscr{A},\lambda)$ is a measure space equipped with the measure given by

$$\lambda(\{1\})=1, \ \lambda(\{k\})=1-w_1-\cdots-w_{k-1}, \quad k\geqq 2.$$

Let $\{\beta_k, k\geqq 1\}$ be the sequence defined by $\beta_1=w_1$ and, for $k\geqq 2$, by

$$\beta_k=\begin{cases} w_k/(1-w_1-\cdots-w_{k-1}) & \text{if } w_1+\cdots+w_{k-1}<1\\ 0 & \text{if } w_1+\cdots+w_{k-1}=1.\end{cases}$$

It is evident that $0\leqq\beta_k\leqq 1$, $k\geqq 1$. Let V be the linear operator on $L_1(\lambda)$ such that

$$V\delta_1=\sum_{k=1}^{\infty}\beta_k\cdot\delta_k, \quad V\delta_k=(1-\beta_{k-1})\cdot\delta_{k-1}, \quad k\geqq 2,$$

where $\delta_k(i)$ stands for the Kronecker delta (cf. Chacon [7]). Then it is easy to verify that V is a positive linear contraction on $L_1(\lambda)$ as well as on $L_\infty(\lambda)$. Taking W to be the direct product of V and U, it follows that W is a linear contraction operator on $L_1(\lambda\times m,\mathfrak{X})$ as well as on $L_\infty(\lambda\times m,\mathfrak{X})$. If for any $f\in L_p(m,\mathfrak{X})$ we write $g(i,x)=\delta_1(i)\cdot f(x)$, then by iteration,

$$(W^k g)(1,x)=u_k\cdot(U^k f)(x), \quad k\geqq 0$$

since $(V^k \delta_1)(1) = u_k$ for all $k \geqq 0$. Therefore we may apply Chacon's ergodic theorem [5] to W with the function $g(i, x)$ to obtain (2.1) and (2.2) of the theorem. In particular, the L_1-convergence (2.3) results from (2.2) and the fact that if the measure m is finite $L_2(m, \mathfrak{X})$ is dense in $L_1(m, \mathfrak{X})$. This completes the proof of Theorem 1.

As a corollary, we have the following generalization of Beck and Schwartz's random ergodic theorem [2].

Corollary 2. *Let $\mathfrak{X}$ be reflexive and let there be defined on X a strongly $\mathscr{B}$-measurable function U_x with values in the Banach space $B(\mathfrak{X})$ of bounded linear operators on $\mathfrak{X}$. Suppose that $\|U_x\| \leqq 1$ for all $x \in X$. Let φ be a measure preserving transformation of X into itself. Then for each $f \in L_p(m, \mathfrak{X})$ there exists a function $f^* \in L_p(m, \mathfrak{X})$ such that the strong limit*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} u_k U_x U_{\varphi x} \cdots U_{\varphi^{k-1} x} f(\varphi^k x) = f^*(x)$$

almost everywhere on X and f^* is the limit in the mean of order p with $1 < p < \infty$. Furthermore, if m is finite then f^* is also the limit in the mean of order 1.

We assume in this section that Φ is a set of measure preserving transformations φ of X into itself. Let us consider the product measure space $(X_r^*, \mathscr{B}_r^*, m_r^*)$, $1 \leqq r \leqq \infty$:

$$X_r^* = X_1 \times \cdots \times X_r, \quad \mathscr{B}_r^* = \mathscr{B}_1 \times \cdots \times \mathscr{B}_r, \quad m_r^* = m_1 \times \cdots \times m_r,$$
$$X_1 = \cdots = X_r = X, \quad \mathscr{B}_1 = \cdots = \mathscr{B}_r = \mathscr{B}, \quad m_1 = \cdots = m_r = m$$

and denote by σ the one-sided shift transformation on $(\Phi^*, \mathscr{F}^*, \mu^*)$:

$$\sigma \varphi^* = \varphi^{*\prime}, \quad \varphi_n' = \varphi_{n+1}, \quad n \geqq 1,$$

where

$$\varphi^* = (\varphi_1, \varphi_2, \ldots), \quad \varphi^{*\prime} = (\varphi_1', \varphi_2', \ldots).$$

Here, and in what follows, φ_n will denote the n-th coordinate of φ^*.

Throughout this paper we suppose that for any $E \in \mathscr{B}$,

$$\{(x, \varphi^*) \colon \varphi_{n_k} \cdots \varphi_{n_1} x \in E\} \in \mathscr{B} \times \mathscr{F}^*, \tag{2.4}$$

where $(n_1, \ldots, n_k)$ is an arbitrary sequence of positive integers with $n_1 < \cdots < n_k$. Let $\{\psi_{\varphi^*} \colon \varphi^* \in \Phi^*\}$ be a family of measure preserving transformations of X_r^* into itself given by

$$\psi_{\varphi^*} = \psi_{(\varphi_1, \ldots, \varphi_r)} = \varphi_1 \times \cdots \times \varphi_r, \quad \varphi^* \in \Phi^*.$$

From the assumption (2.4) it follows that $\{\psi_{\varphi^*} \colon \varphi^* \in \Phi^*\}$ is a $\mathscr{B}_r^* \times \mathscr{F}^*$-measurable family. Thus, setting

$$S(x_r^*, \varphi^*) = (\psi_{\varphi^*} x_r^*, \sigma \varphi^*) \tag{2.5}$$

we obtain the so-called skew product transformation S on $(X_r^* \times \Phi^*, \mathscr{B}_r^* \times \mathscr{F}^*, m_r^* \times \mu^*)$ (cf. Kin [16]).

From now on, unless otherwise stated, we shall write

$$f_{(0)}(x_r^*) = f(x_r^*), \quad f_{(k)}(x_r^*) = f(\psi_{(\varphi_k \cdots \varphi_1, \ldots, \varphi_{r+k-1} \cdots \varphi_r)} x_r^*), \quad k \geqq 1.$$

The following corollary is an answer to the question raised by Révész and generalizes and extends the usual random ergodic theorems concerned with measure preserving transformations.

Corollary 3. *Let $\mathfrak{X}$ be reflexive and let $f(x_r^*) \in L_p(m_r^*, \mathfrak{X})$ with $1 \leqq p < \infty$. Then there is a set D^* of μ^*-measure zero such that for any $\varphi^* \in \Phi^* - D^*$ there exists a function $f_{\varphi^*}^*(x_r^*) \in L_p(m_r^*, \mathfrak{X})$ such that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} u_k f_{(k)}(x_r^*) = f_{\varphi^*}^*(x_r^*)$$

strongly in $\mathfrak{X}$ for almost all x_r^* and

$$\lim_{n\to\infty} \left\| f_{\varphi^*}^* - \frac{1}{n} \sum_{k=0}^{n-1} u_k f_{(k)} \right\|_{L_p(m_r^*, \mathfrak{X})} = 0, \qquad 1 < p < \infty,$$

and if further m is finite then

$$\lim_{n\to\infty} \left\| f_{\varphi^*}^* - \frac{1}{n} \sum_{k=0}^{n-1} u_k f_{(k)} \right\|_{L_1(m_r^*, \mathfrak{X})} = 0.$$

The proof of Corollary 3 is a simple application of Theorem 1 to the transformation S given by (2.5).

Theorem 4. *Let $\mathfrak{X}$ be reflexive and let U be in $B(\mathfrak{X})$. Suppose that $|||U||| \leqq 1$. If $g(k)$ is a scalar (real or complex) valued function defined on N satisfying*

$$\sum_{k=1}^{\infty} |g(k)|^p < \infty \qquad \text{for a } p \text{ with } 1 \leqq p < \infty, \tag{2.6}$$

then for such a p and every $f(x_r^*) \in L_p(m_r^*, \mathfrak{X})$ there is a set D^* of μ^*-measure zero such that for any $\varphi^* \in \Phi^* - D^*$ there exists a function $G_{(j,\varphi^*)}(x_r^*) \in L_p(m_r^*, \mathfrak{X})$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} u_k\, g(k+j)\, U^k f_{(k)}(x_r^*) = G_{(j,\varphi^*)}(x_r^*) \tag{2.7}$$

strongly in $\mathfrak{X}$ almost everywhere on X_r^*, for $j \geqq 1$. Furthermore, for p with $1 < p < \infty$ and $j \geqq 1$,

$$\lim_{n\to\infty} \left\| G_{(j,\varphi^*)} - \frac{1}{n} \sum_{k=0}^{n-1} u_k\, g(k+j)\, U^k f_{(k)} \right\|_{L_p(m_r^*, \mathfrak{X})} = 0 \qquad \mu^*\text{-a.e.} \tag{2.8}$$

Proof. Let $(N, \mathscr{A}, \nu)$ be a σ-finite measure space with the measure ν defined by $\nu(A) = \#(A)$ for $A \in \mathscr{A}$ and let ξ be a permutation of N given by $\xi(n) = n+1$ which preserves the measure ν. Taking (Ω, Σ, P) to be the direct product of $(N, \mathscr{A}, \nu)$, $(X_r^*, \mathscr{B}_r^*, m_r^*)$ and $(\Phi^*, \mathscr{F}^*, \mu^*)$ and T the direct product of ξ and S given by the formula

$$T(n, x_r^*, \varphi^*) = (\xi(n), S(x_r^*, \varphi^*)) = (n+1, \psi_{(\varphi_1, \ldots, \varphi_r)} x_r^*, \sigma\varphi^*), \tag{2.9}$$

it follows that T is a measure preserving transformation on (Ω, Σ, P). By iterations,

$$T^k(n, x_r^*, \varphi^*) = (n+k, \psi_{(\varphi_k \ldots \varphi_1, \ldots, \varphi_{r+k-1} \ldots \varphi_r)} x_r^*, \sigma^k \varphi^*), \qquad k \geqq 1 \qquad (T^0 = \text{identity}).$$

Let $f(x_r^*)\in L_p(m_r^*, \mathfrak{X})$. According to the hypothesis (2.4), we see that $f_{(k)}(x_r^*)$, $k\geqq 1$, are strongly $\mathscr{B}_r^* \times \mathscr{F}^*$-measurable and by Fubini's theorem (cf. Bochner [3]),

$$\iint_{X_r^* \times \Phi^*} |||f_{(k)}(x_r^*)|||^p \, dm_r^* \times \mu^* = \int_{X_r^*} |||f(x_r^*)|||^p \, dm_r^* < \infty .$$

Also, writing $F(n, x_r^*, \varphi^*) = g(n) f(x_r^*)$ we have, by (2.6) and Fubini's theorem,

$$\begin{aligned}\iiint_{N \times X_r^* \times \Phi^*} & |||F(n, x_r^*, \varphi^*)|||^p \, dv \times m_r^* \times \mu^* \\ &= \int_N |g(n)|^p \, dv \int_{\Phi^*} d\mu^* \int_{X_r^*} |||f(x_r^*)|||^p \, dm_r^* \\ &= \left(\sum_{n=1}^{\infty} |g(n)|^p\right) \int_{X_r^*} |||f(x_r^*)|||^p \, dm_r^* < \infty .\end{aligned}$$

Define a mapping τ from $L_1(P, \mathfrak{X})$ to itself as follows:

$$(\tau H)(n, x_r^*, \varphi^*) = U H(T(n, x_r^*, \varphi^*)), \qquad H\in L_1(P, \mathfrak{X}).$$

Then τ is a linear operator on $L_1(P, \mathfrak{X})$ with

$$\|\tau\|_{L_1(P,\mathfrak{X})} \leqq 1, \qquad \|\tau\|_{L_\infty(P,\mathfrak{X})} \leqq 1 .$$

Therefore, in view of Theorem 1, there exist a null set D in Ω and a function $F^*\in L_p(P, \mathfrak{X})$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} u_k(\tau^k F)(j, x_r^*, \varphi^*) = F^*(j, x_r^*, \varphi^*) \tag{2.10}$$

strongly in $\mathfrak{X}$ for all $(j, x_r^*, \varphi^*)\in \Omega - D$, and if $1<p<\infty$,

$$\lim_{n\to\infty} \left\| F^* - \frac{1}{n} \sum_{k=0}^{n-1} u_k(\tau^k F) \right\|_{L_p(P,\mathfrak{X})} = 0 . \tag{2.11}$$

Noticing that (2.10) holds almost everywhere on $X_r^* \times \Phi^*$, for all $j\geqq 1$, in consideration of the definition of v, the pointwise convergence (2.7) follows immediately from (2.10). To prove (2.8) it is convenient to use the operator V on $L_1(\lambda)$ given in the proof of Theorem 1. Let W be the direct product of V and τ. Then W is a linear contraction operator on $L_1(\lambda\times P, \mathfrak{X})$ as well as on $L_\infty(\lambda\times P, \mathfrak{X})$. By virtue of Theorem 1, the functions

$$\frac{1}{n} \sum_{k=0}^{n-1} (W^k g)(1, j, x_r^*, \varphi^*)$$

approach a limit in the norm of $L_p(\lambda\times P, \mathfrak{X})$ as $n\to\infty$ and are, for $k\geqq 1$, all dominated by a function in $L_p(\lambda\times P, \mathfrak{X})$ (Chacon [5]), where

$$g(i, j, x_r^*, \varphi^*) = \delta_1(i)\cdot F(j, x_r^*, \varphi^*) .$$

Hence, using this fact and (2.11), we get (2.8) and finish the proof of Theorem 4.

Applying Dowker's ergodic theorem [8] to the transformation T defined by (2.9), we have

Theorem 5. *Let $g(k)$ be an arbitrary real or complex valued function defined on N with $\sum_{k=1}^{\infty} |g(k)| < \infty$. Let $h(x_r^*)$ be a non-negative measurable function defined on X_r^**

such that except for a μ^**-null set,*

$$\sum_{k=0}^{\infty} h_{(k)}(x_r^*) = \infty$$

almost everywhere on X_r^*. *Then for every* $f(x_r^*) \in L_1(m_r^*)$ *there exists a set* D^* *of* μ^**-measure zero such that for any* $\varphi^* \in \Phi^* - D^*$ *the limits*

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} g(k+j)\cdot f_{(k)}(x_r^*)}{\sum_{k=0}^{n-1} h_{(k)}(x_r^*)}, \qquad j \geqq 1$$

exist and are finite for almost all $x_r^* \in X_r^*$.

Application of Baxter's ergodic theorem [1] to the transformation S defined by (2.8) gives the following generalization of Hopf's ratio ergodic theorem [14].

Theorem 6. *Let* $h(x_r^*)$ *be a non-negative measurable function defined on* X_r^* *such that excepting a* μ^**-null set in* Φ^*, $\sum_{k=0}^{\infty} u_k\, h_{(k)}(x_r^*) = \infty$ *for almost all* x_r^*. *Then for each* $f(x_r^*) \in L_1(m_r^*)$ *there is a set* D^* *with* μ^**-measure zero such that for any* $\varphi^* \in \Phi^* - D^*$ *the limit*

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} u_k\, f_{(k)}(x_r^*)}{\sum_{k=0}^{n-1} u_k\, h_{(k)}(x_r^*)}$$

exists and is finite almost everywhere on X_r^*.

In what follows, we assume that $(X, \mathscr{B}, m)$ is a probability measure space. Let $(Y, \mathscr{C}, \mu)$ be a σ-finite measure space on which a measure preserving transformation η is given.

Theorem 7. *Let* $\mathfrak{X}$ *be reflexive and let there be defined on* X_r^* *a strongly* $\mathscr{B}_r^*$*-measurable* $L_1(\mu, \mathfrak{X})$*-operator valued function* $U(x_r^*)$. *Denote*

$$\begin{aligned} &U(0, x_r^*) = U(x_r^*), \\ &U(k, x_r^*) = U(x_r^*) \ldots U(\psi_{(\varphi_k \ldots \varphi_1, \ldots, \varphi_{r+k-1} \ldots \varphi_r)}\, x_r^*), \qquad k \geqq 1. \end{aligned} \tag{2.12}$$

Suppose that $\|U(x_r^*)\|_{L_1(\mu,\mathfrak{X})} \leqq 1$, $\|U(x_r^*)\|_{L_\infty(\mu,\mathfrak{X})} \leqq 1$ *for all* x_r^* *and that the operators*

$$U(\psi_{(\varphi_k \ldots \varphi_1, \ldots, \varphi_{r+k-1} \ldots \varphi_r)}\, x_r^*), \qquad k \geqq 1$$

are strongly $\mathscr{B}_r^* \times \mathscr{F}^*$*-measurable. Then for every* $f(x_r^*, y) \in L_p(m_r^* \times \mu, \mathfrak{X})$ *with* $1 \leqq p < \infty$, *there is a* μ^**-null set* D^* *in* Φ^* *such that for any* $\varphi^* \in \Phi^* - D^*$ *there exist a function* $G_{\varphi^*}(x_r^*, y) \in L_p(m_r^* \times \mu, \mathfrak{X})$ *and a set* D_r^* *with* m_r^**-measure zero such that for all* $x_r^* \in X_r^* - D_r^*$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} u_k\, U(k, x_r^*)\, f_{(k)}(x_r^*, y) = G_{\varphi^*}(x_r^*, y)$$

strongly in $\mathfrak{X}$ almost everywhere on Y, and if $1<p<\infty$,

$$\lim_{n\to\infty}\left\|G_{\varphi^*}(x_r^*,\cdot)-\frac{1}{n}\sum_{k=0}^{n-1}u_k\,U(k,x_r^*)\,f_{(k)}(x_r^*,\cdot)\right\|_{L_p(\mu,\mathfrak{X})}=0,$$

and if furthermore μ *is finite*

$$\lim_{n\to\infty}\left\|G_{\varphi^*}(x_r^*,\cdot)-\frac{1}{n}\sum_{k=0}^{n-1}u_k\,U(k,x_r^*)\,f_{(k)}(x_r^*,\cdot)\right\|_{L_1(\mu,\mathfrak{X})}=0.$$

Here

$$\begin{aligned}&f_{(0)}(x_r^*,y)=f(x_r^*,y),\\&f_{(k)}(x_r^*,y)=f(\psi_{(\varphi_k\dots\varphi_1,\dots,\varphi_{r+k-1}\dots\varphi_r)}x_r^*,\eta^k y),\qquad k\geqq 1.\end{aligned}\tag{2.13}$$

Proof. If we let

$$\begin{aligned}&W_0(x_r^*,\varphi^*)=\text{identity},\qquad V(x_r^*,\varphi^*)=U(x_r^*),\\&W_n(x_r^*,\varphi^*)=V(x_r^*,\varphi^*)\dots V(S^{n-1}(x_r^*,\varphi^*)),\qquad n\geqq 1,\end{aligned}$$

we observe that

(i) for every $n\geqq 1$, $W_n(x_r^*,\varphi^*)$ is a strongly $\mathscr{B}_r^*\times\mathscr{F}^*$-measurable $L_1(\mu,\mathfrak{X})$-operator valued function defined on $X_r^*\times\Phi^*$,

(ii) $$\|W_n(x_r^*,\varphi^*)\|_{L_1(\mu,\mathfrak{X})}\leqq 1\quad\text{and}\quad\|W_n(x_r^*,\varphi^*)\|_{L_\infty(\mu,\mathfrak{X})}\leqq 1$$

for all (x_r^*,φ^*) and $n\geqq 1$,

(iii) $$W_{n+m}(x_r^*,\varphi^*)=W_n(x_r^*,\varphi^*)\,W_m(S^n(x_r^*,\varphi^*)),\qquad m,n\geqq 1.$$

With S and η, define

$$\Delta(x_r^*,\varphi^*,y)=(S(x_r^*,\varphi^*),\eta(y)).$$

At this point, we require the following

Lemma 8. *There exists a semigroup* $\{W_n^*,n\geqq 0\}$ *of contraction linear operators on* $L_1(m_r^*\times\mu^*\times\mu,\mathfrak{X})$ *as well as on* $L_\infty(m_r^*\times\mu^*\times\mu,\mathfrak{X})$, *such that for any*

$$F(x_r^*,\varphi^*,y)\in L_p(m_r^*\times\mu^*\times\mu,\mathfrak{X})$$

there is a null set E^* *in* $X_r^*\times\Phi^*$ *such that for any* $(x_r^*,\varphi^*)\in(X_r^*\times\Phi^*)-E^*$,

$$(W_n^*F)(x_r^*,\varphi^*,y)=W_n(x_r^*,\varphi^*)\,F(\Delta^n(x_r^*,\varphi^*,y))$$

holds for all points y of Y with the exception of a null set.

Proof. First we note that for every k and $F(x_r^*,\varphi^*,y)$ in $L_p(m_r^*\times\mu^*\times\mu,\mathfrak{X})$, the function $W_k(x_r^*,\varphi^*)\,F(\Delta^k(x_r^*,\varphi^*,\cdot))$ is strongly $\mathscr{B}_r^*\times\mathscr{F}^*$-measurable as an $L_p(\mu,\mathfrak{X})$-operator valued function defined on $X_r^*\times\Phi^*$. On account of the strong $\mathscr{B}_r^*\times\mathscr{F}^*$-measurability of the function, there exist countably $L_p(\mu,\mathfrak{X})$-valued $\mathscr{B}_r^*\times\mathscr{F}^*$-measurable functions $g_i^{(k)}(x_r^*,\varphi^*,\cdot)$, $i\geqq 1$, defined on $X_r^*\times\Phi^*$ and a negligible set E in $X_r^*\times\Phi^*$ such that

$$\lim_{i\to\infty}\|W_k(x_r^*,\varphi^*)\,F(\Delta^k(x_r^*,\varphi^*,\cdot))-g_i^{(k)}(x_r^*,\varphi^*,\cdot)\|_{L_p(\mu,\mathfrak{X})}=0\tag{2.14}$$

holds uniformly on $(X_r^* \times \Phi^*) - E$, wherefore

$$\lim_{i,j\to\infty} \|g_i^{(k)}(x_r^*, \varphi^*, \cdot) - g_j^{(k)}(x_r^*, \varphi^*, \cdot)\|_{L_p(\mu,\mathfrak{X})} = 0$$

uniformly on $(X_r^* \times \Phi^*) - E$ (Hille and Phillips [13]).

Indeed $W_k(x_r^*, \varphi^*) F(\Delta^k(x_r^*, \varphi^*, y))$ may not be $\mathscr{B}_r^* \times \mathscr{F}^* \times \mathscr{C}$-measurable, but $g_i^{(k)}(x_r^*, \varphi^*, y)$, $i \geqq 1$, are $\mathscr{B}_r^* \times \mathscr{F}^* \times \mathscr{C}$-measurable and belong to $L_p(m_r^* \times \mu^* \times \mu, \mathfrak{X})$ since the functions are countably $L_p(\mu, \mathfrak{X})$-valued and $\mathscr{B}_r^* \times \mathscr{F}^*$-measurable. Thus

$$\lim_{i,j\to\infty} \|g_i^{(k)} - g_j^{(k)}\|_{L_p(m_r^* \times \mu^* \times \mu, \mathfrak{X})} = 0,$$

so that there exists a function $G^{(k)}(x_r^*, \varphi^*, y) \in L_p(m_r^* \times \mu^* \times \mu, \mathfrak{X})$ satisfying

$$\lim_{i\to\infty} \|g_i^{(k)} - G^{(k)}\|_{Lp(m_r^* \times \mu^* \times \mu, \mathfrak{X})} = 0.$$

Hence, by choosing a subsequence of $\{g_i^{(k)}, i \geqq 1\}$ if necessary, we may assume that there is a set D_k^* of $m_r^* \times \mu^*$-measure zero such that

$$\lim_{i\to\infty} \|g_i^{(k)}(x_r^*, \varphi^*, \cdot) - G^{(k)}(x_r^*, \varphi^*, \cdot)\|_{L_p(\mu,\mathfrak{X})} = 0 \tag{2.15}$$

for all $(x_r^*, \varphi^*) \in (X_r^* \times \Phi^*) - D_k^*$. Inserting (2.15) into (2.14), we see that for any $(x_r^*, \varphi^*) \in (X_r^* \times \Phi^*) - \big(E \cup \bigcup_{k \geqq 0} D_k^*\big)$,

$$W_k(x_r^*, \varphi^*) F(\Delta^k(x_r^*, \varphi^*, y)) = G^{(k)}(x_r^*, \varphi^*, y), \qquad k \geqq 0$$

almost everywhere on Y. It is now easy to verify that such a function $G^{(k)}$ is uniquely determined except on a set of $m_r^* \times \mu^* \times \mu$-measure zero.

For every n, we define the mapping W_n^* from $L_1(m_r^* \times \mu^* \times \mu, \mathfrak{X})$ to itself as follows:

$$(W_n^* F)(\cdot, \cdot, \cdot) = G^{(n)}(\cdot, \cdot, \cdot), \qquad F \in L_1(m_r^* \times \mu^* \times \mu, \mathfrak{X}).$$

Then W_n^* turns out to be a linear contraction operator on $L_1(m_r^* \times \mu^* \times \mu, \mathfrak{X})$ as well as on $L_\infty(m_r^* \times \mu^* \times \mu, \mathfrak{X})$. Let $H(x_r^*, \varphi^*, y) \in L_1(m_r^* \times \mu^* \times \mu, \mathfrak{X})$. Then for almost all (x_r^*, φ^*),

$$\begin{aligned}
(W_m^* H)(x_r^*, \varphi^*, \cdot) &= W_m(x_r^*, \varphi^*) H(\Delta^m(x_r^*, \varphi^*, \cdot)),\\
(W_n^* W_m^* H)(x_r^*, \varphi^*, \cdot) &= W_n(x_r^*, \varphi^*)(W_m^* H)(\Delta^n(x_r^*, \varphi^*, \cdot))\\
&= W_n(x_r^*, \varphi^*)\, W_m(S^n(x_r^*, \varphi^*))\, H(\Delta^{n+m}(x_r^*, \varphi^*, \cdot)),\\
(W_{n+m}^* H)(x_r^*, \varphi^*, \cdot) &= W_{n+m}(x_r^*, \varphi^*) H(\Delta^{n+m}(x_r^*, \varphi^*, \cdot))
\end{aligned}$$

and so

$$(W_{n+m}^* H)(x_r^*, \varphi^*, \cdot) = (W_n^* W_m^* H)(x_r^*, \varphi^*, \cdot)$$

in $L_1(\mu, \mathfrak{X})$. Consequently, in $L_1(m_r^* \times \mu^* \times \mu, \mathfrak{X})$

$$(W_{n+m}^* H)(\cdot, \cdot, \cdot) = (W_n^* W_m^* H)(\cdot, \cdot, \cdot),$$

which implies that $\{W_n^*, n \geqq 0\}$ is a semigroup. This proves the lemma.

We are now in a position to prove Theorem 7. If in Lemma 8, we write $F(x_r^*, \varphi^*, y) = f(x_r^*, y)$ then $F(x_r^*, \varphi^*, y) \in L_p(m_r^* \times \mu^* \times \mu, \mathfrak{X})$. Taken with Theorem 1, we can conclude that there exists a function $F^*(x_r^*, \varphi^*, y) \in L_p(m_r^* \times \mu^* \times \mu, \mathfrak{X})$ such that the strong limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} u_k (W_k^* F)(x_r^*, \varphi^*, y) = F^*(x_r^*, \varphi^*, y)$$

except on a null set in $X_r^* \times \Phi^* \times Y$. In addition, the function F^* is the limit in the mean of order p with $1 < p < \infty$ and if μ is finite then F^* is also the limit in the mean of order 1. Noticing that for almost all (x_r^*, φ^*),

$$(W_k^* F)(x_r^*, \varphi^*, y) = U(k, x_r^*) f_{(k)}(x_r^*, y)$$

almost everywhere on Y, the conclusion of the theorem ensues from the facts observed above. The proof of Theorem 7 is herewith completed.

As an application of Lemma 8, we have the following generalization of Cairoli's ratio random ergodic theorem [4].

Theorem 9. *Let there be defined on X_r^* a strongly $\mathscr{B}_r^*$-measurable $L_1(\mu)$-operator valued function $U(x_r^*)$. Let $U(x_r^*)$ be positive and $\| U(x_r^*)\|_{L_1(\mu)} \leqq 1$ for all x_r^*. Assume that $U(\psi_{(\varphi_k \dots \varphi_1, \dots, \varphi_{r+k-1} \dots \varphi_r)} x_r^*)$, $k \geqq 1$, are strongly $\mathscr{B}_r^* \times \mathscr{F}^*$-measurable. If $f(x_r^*, y) \in L_1(m_r^* \times \mu)$ and if $h(x_r^*, y) \in L_1(m_r^* \times \mu)$ is non-negative then there exists a set D^* of $m_r^* \times \mu^*$-measure zero such that for any $(x_r^*, \varphi^*) \in (X_r^* \times \Phi^*) - D^*$, the limit*

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} u_k\, U(k, x_r^*) f_{(k)}(x_r^*, y)}{\sum_{k=0}^{n-1} u_k\, U(k, x_r^*) h_{(k)}(x_r^*, y)}$$

exists and is finite almost everywhere on the set

$$\left\{ y \colon \sum_{k=0}^{\infty} u_k\, U(k, x_r^*) h_{(k)}(x_r^*, y) > 0 \right\}.$$

Proof. As in the proof of Theorem 7, we construct the semigroup $\{W_n^*, n \geqq 0\}$ of linear contraction operators on $L_1(m_r^* \times \mu^* \times \mu)$. It is quite clear that for each n, W_n^* is positive. In order to substantiate the assertion of the theorem, we may therefore apply Baxter's ratio ergodic theorem [1] (cf. Chacon [7]) to $\{W_n^*, n \geqq 0\}$. Hence Theorem 7 follows.

Remark 10. Let Φ be a set of measure preserving transformations on a probability measure space $(X, \mathscr{B}, m)$ such that

$$\varphi_1 \times \cdots \times \varphi_r, \quad \varphi_j \in \Phi, \quad 1 \leqq j \leqq r$$

are ergodic. For example, one may take Φ to be a set of weakly mixing transformations on X. Then the family $\{\psi_{\varphi^*} \colon \varphi^* \in \Phi^*\}$ is ergodic. (If all φ in Φ are weakly mixing, the family is weakly mixing.) Thus, by Gladysz's theorems [11], S is ergodic. (If all φ in Φ are weakly mixing then S is weakly mixing). In this case, Révész's theorem mentioned in the introduction remains true without the assumption stated in the theorem. More generally, if we take $w_1 = 1$, $w_k = 0$, $k \geqq 2$, then for almost all (x_r^*, φ^*), the limit function in Corollary 3 is constant and equal to $\int_{X_r^*} f(x_r^*)\, dm_r^*$ and the limit function in Theorem 6 is constant and equal to $(\int_{X_r^*} f(x_r^*)\, dm_r^*)/(\int_{X_r^*} h(x_r^*)\, dm_r^*)$ of which the denominator is positive.

3. Random Ergodic Theorems Concerning Non-singular Transformations

In the sequel, $(X, \mathscr{B}, m)$ will denote a σ-finite measure space. A bimeasurable transformation φ of X onto itself is called positively non-singular if φ transforms sets of measure zero into sets of measure zero. It will be assumed in the following that φ is bimeasurable and positively non-singular.

Let $m_n(A) = m(\varphi^n A)$ for $A \in \mathscr{B}$ and $n \geqq 0$. Then m_n is easily seen to be a measure defined on $\mathscr{B}$. Since, for all $n \geqq 0$, φ^n is bimeasurable and positively non-singular, there exist measurable functions $\alpha_n(x)$, $n \geqq 0$, positive almost everywhere such that

$$m_n(A) = \int_A \alpha_n(x)\, dm, \qquad A \in \mathscr{B}, \quad n \geqq 0$$

in accordance with Radon-Nikodym's theorem. It is not difficult to see that $\alpha_n(x)$, $n \geqq 1$, satisfy the following equations:

$$\alpha_n(x) = \alpha(\varphi^{n-1} x)\, \alpha_{n-1}(x) \quad m\text{-a.e.}, \qquad n \geqq 1,$$

where $\alpha(x) = \alpha_1(x)$, and

$$\int_{\varphi^n A} f(x)\, dm = \int_A f(\varphi^n x)\, \alpha_n(x)\, dm, \quad A \in \mathscr{B}, \qquad n \geqq 0$$

for every $f(x) \in L_1(m)$. There is no loss of generality in supposing that for all x, $\alpha(x) > 0$, $\alpha_0(x) = 1$ and $\alpha_n(x) = \alpha(\varphi^{n-1} x)\, \alpha_{n-1}(x)$, $n \geqq 1$; then $\alpha_n(x) > 0$ everywhere on X for $n \geqq 1$.

For every measurable function $f(x)$, define

$$U\colon\ (Uf)(x) = f(\varphi x) \cdot \alpha(x)$$

which is positive and linear. An easy calculation shows that for $f \in L_1(m)$, $\|Uf\|_{L_1(m)} = \|f\|_{L_1(m)}$ which implies that $L_1(m)$ is invariant under U. Thus U is a positive linear contraction operator on $L_1(m)$. Moreover,

$$(U^n f)(x) = f(\varphi^n x)\, \alpha_n(x), \qquad n \geqq 0.$$

The following result extends the ergodic theorem of Dowker [8] who generalizes that of Hurewicz [15] and Halmos [12].

Theorem 11. *Let $h(x)$ be a non-negative measurable function defined on X. Then for any $f(x) \in L_1(m)$,*

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} u_k\, f(\varphi^k x)\, \alpha_k(x)}{\sum_{k=0}^{n-1} u_k\, h(\varphi^k x)\, \alpha_k(x)}$$

exists and is finite almost everywhere on $\left\{x\colon \sum_{k=0}^{\infty} u_k\, h(\varphi^k x)\, \alpha_k(x) > 0\right\}$.

Proof. Let $\{p_k(x), k \geqq 0\}$ be the sequence of non-negative measurable functions defined by

$$p_0(x) = h(x), \quad p_k(x) = (U p_{k-1})(x), \quad k \geqq 1.$$

Then $|(U\xi)(x)| \leqq p_{k+1}(x)$ almost everywhere whenever $|\xi(x)| \leqq p_k(x)$ almost everywhere and $\xi(x) \in L_1(m)$. As in the proof of Theorem 1, we utilize the direct product W of U with V given in that of Theorem 1. In this case, we know that

$$(W^k g)(1, x) = u_k (U^k f)(x), \quad k \geqq 0,$$

where $g(i, x) = \delta_1(i) \cdot f(x)$. If we write

$$p_k^*(i, x) = (V^k \delta_1)(i) \cdot p_k(x),$$

then

$$\begin{aligned} |(W\xi^*)(i, x)| \leqq (W p_k^*)(i, x) &= \big(V(V^k \delta_1)\big)(i) \cdot (U p_k)(x) \\ &= (V^{k+1} \delta_1)(i)\, p_{k+1}(x) = p_{k+1}^*(i, x) \end{aligned}$$

almost everywhere on $N \times X$ whenever $|\xi^*(i, x)| \leqq p_k^*(i, x)$ almost everywhere on $N \times X$ and $\xi^*(i, x) \in L_1(\lambda \times m)$. Hence application of Chacon's ergodic theorem [6] to W leads us to the conclusion of the theorem.

Theorem 11 obtained above allows, for example, the consideration of ratios of the forms

$$\frac{\sum_{k=0}^{n-1} u_k \exp(ik\beta) f(\varphi^k x)\, \alpha_k(x)}{\sum_{k=0}^{n-1} u_k \cdot \alpha_k(x)}, \qquad \frac{\sum_{k=0}^{n-1} \exp(ik\beta) f(\varphi^k x)\, \alpha_k(x)}{\sum_{k=0}^{n-1} \alpha_k(x)}$$

Now let us take Φ to be a set of one-to-one bimeasurable positively non-singular transformations of X onto itself. Let $\{\mu_n, -\infty < n < \infty\}$ be a sequence of probability measures defined on $(\Phi, \mathscr{F})$ and let $(\Phi^*, \mathscr{F}^*, \mu^*)$ be the two-sided product measure space:

$$\Phi^* = \cdots \times \Phi_{-1} \times \Phi_0 \times \Phi_1 \times \cdots, \qquad \mathscr{F}^* = \cdots \mathscr{F}_{-1} \times \mathscr{F}_0 \times \mathscr{F}_1 \times \cdots,$$

$$\mu^* = \cdots \times \mu_{-1} \times \mu_0 \times \mu_1 \times \cdots, \qquad \Phi_n = \Phi, \ \mathscr{F}_n = \mathscr{F}, \qquad -\infty < n < \infty,$$

on which the two-sided shift transformation σ is given.

It follows then by the assumption (2.4) that the functions $f(\varphi_k \ldots \varphi_1 x)$, $k \geqq 1$, are $\mathscr{B} \times \mathscr{F}^*$-measurable if $f(x)$ is $\mathscr{B}$-measurable. Since for any $\varphi^* \in \Phi^*$ and every $k \geqq 1$, the successive transformation $\varphi_k \ldots \varphi_1$ is positively non-singular, it follows from Radon-Nikodym's theorem that there exists a family $\{\beta_{(k, \varphi^*)}(x), k \geqq 1\}$ of non-negative measurable functions such that for any $E \in \mathscr{B}$,

$$m(\varphi_k \ldots \varphi_1 E) = \int_E \beta_{(k, \varphi^*)}(x)\, dm, \quad k \geqq 1. \tag{3.1}$$

Without loss of generality, we may assume that every function $\beta_{(k, \varphi^*)}(x)$ is positive everywhere on X. Considering approximating sums to the integrals in question, it is easily seen that

$$\int_X f(x)\, dm = \int_X f(\varphi_k \cdots \varphi_1 x)\, \beta_{(k, \varphi^*)}(x)\, dm$$

for $k \geqq 1$ and $f \in L_1(m)$. From (3.1) we have

$$\beta_{(i+j,\varphi^*)}(x) = \beta_{(i,\sigma^j\varphi^*)}(\varphi_j \dots \varphi_1 x)\, \beta_{(j,\varphi^*)}(x) \tag{3.2}$$

almost everywhere on X, for all $i, j \geqq 1$, and it can be supposed with no loss of generality that the Eq. (3.2) holds everywhere on X.

Lemma 12. *There exists a sequence $\{\beta_k^*(x, \varphi^*), k \geqq 1\}$ of $\mathscr{B} \times \mathscr{F}^*$-measurable versions of $\beta_{(k,\varphi^*)}(x)$, $k \geqq 1$ such that excepting a set of μ^*-measure zero,*

$$\beta_k^*(x, \varphi^*) = \beta_{(k,\varphi^*)}(x), \qquad k \geqq 1$$

almost everywhere on X.

Proof. Define

$$S_k(x, \varphi^*) = (\varphi_k \dots \varphi_1 x, \varphi^*), \qquad k \geqq 1.$$

Evidently S_k is a one-to-one bimeasurable positively non-singular transformation of $X \times \Phi^*$ onto itself. Because of Radon-Nikodym's theorem there exists a sequence $\{\beta_k^*(x, \varphi^*), k \geqq 1\}$ of positive $\mathscr{B} \times \mathscr{F}^*$-measurable functions defined on $X \times \Phi^*$ such that for any $\Lambda \in \mathscr{B} \times \mathscr{F}^*$,

$$m \times \mu^*(S_k \Lambda) = \iint_\Lambda \beta_k^*(x, \varphi^*)\, dm \times \mu^*, \qquad k \geqq 1.$$

So, for any $A \in \mathscr{B}$ and for any $B \in \mathscr{F}^*$,

$$m \times \mu^*\big(S_k(A \times B)\big) = \int_B d\mu^* \int_A \beta_k^*(x, \varphi^*)\, dm,$$

and also

$$m \times \mu^*\big(S_k(A \times B)\big) = \int_B m(\varphi_k \dots \varphi_1 A)\, d\mu^*.$$

Accordingly

$$m(\varphi_k \dots \varphi_1 A) = \int_A \beta_k^*(x, \varphi^*)\, dm, \qquad k \geqq 1 \tag{3.3}$$

for almost all $\varphi^* \in \Phi^*$. Hence the comparison of (3.3) with (3.1) proves the lemma.

Theorem 13. *Let $g(n)$ be an arbitrary function defined on the set Z of all integers by the requirement that $\sum_{n=-\infty}^{\infty} |g(n)| < \infty$. Let $h(x)$ be an arbitrary non-negative function measurable in X. Then for all $f(x) \in L_1(m)$ there exists a set D^* with μ^*-measure zero such that for any $\varphi^* \in \Phi^* - D^*$, the limits*

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} u_k g(k+j) f(\varphi_k \dots \varphi_1 x)\, \beta_{(k,\varphi^*)}(x)}{\sum_{k=1}^{n} u_k h(\varphi_k \dots \varphi_1 x)\, \beta_{(k,\varphi^*)}(x)}, \qquad j \in Z$$

exist and are finite almost everywhere on the set

$$\left\{x : \sum_{k=1}^{\infty} u_k h(\varphi_k \dots \varphi_1 x)\, \beta_{(k,\varphi^*)}(x) > 0\right\}.$$

Proof. Let $\mathscr{A}$ denote the σ-algebra of all subsets of Z, ν the measure on $\mathscr{A}$ given by $\nu(A)=\#(A)$ for $A\in\mathscr{A}$ and T the transformation of $(Z\times X\times\Phi^*, \mathscr{A}\times\mathscr{B}\times\mathscr{F}^*, \nu\times m\times\mu^*)$ onto itself, defined by the formula

$$T(n,x,\varphi^*)=(n+1,\varphi_1 x,\sigma\varphi^*)$$

which will be $\mathscr{A}\times\mathscr{B}\times\mathscr{F}^*$-measurable and positively non-singular with respect to $\nu\times m\times\mu^*$. Then Radon-Nikodym's theorem guarantees that there exists a sequence $\{\alpha_k(n,x,\varphi^*), k\geqq 1\}$ of positive measurable functions defined on $Z\times X\times\Phi^*$ such that for any $\Gamma\in\mathscr{A}\times\mathscr{B}\times\mathscr{F}^*$,

$$\nu\times m\times\mu^*(T^k\Gamma)=\iiint_\Gamma \alpha_k(n,x,\varphi^*)\,d\nu\times m\times\mu^*,\qquad k\geqq 1,$$

so that, using this, we get

$$\alpha_{i+j}(n,x,\varphi^*)=\alpha_i\big(T^j(n,x,\varphi^*)\big)\,\alpha_j(n,x,\varphi^*) \tag{3.4}$$

almost everywhere on $Z\times X\times\Phi^*$, for all $i,j\geqq 1$. There is no loss of generality in supposing that (3.4) holds everywhere on $Z\times X\times\Phi^*$. With the functions $g(n)$ and $h(x)$ described in the theorem, we write

$$F(n,x,\varphi^*)=g(n)f(x),\qquad H(n,x,\varphi^*)=h(x)$$

for $f(x)\in L_1(m)$. Obviously $F(n,x,\varphi^*)$ is integrable with respect to $\nu\times m\times\mu^*$ and $H(n,x,\varphi^*)$ is non-negative. Thus, by Theorem 11, the average

$$\frac{\sum_{k=1}^{n} u_k F\big(T^k(j,x,\varphi^*)\big)\,\alpha_k(j,x,\varphi^*)}{\sum_{k=1}^{n} u_k H\big(T^k(j,x,\varphi^*)\big)\,\alpha_k(j,x,\varphi^*)} \tag{3.5}$$

approaches a finite limit (as $n\to\infty$) almost everywhere on the set

$$\left\{(j,x,\varphi^*)\colon \sum_{k=1}^{\infty} u_k H\big(T^k(j,x,\varphi^*)\big)\,\alpha_k(j,x,\varphi^*)>0\right\}.$$

However, from the definition of the measure ν, we can infer that the limit of (3.5) exists for all $j\in Z$. Let $\Gamma\in\mathscr{A}\times\mathscr{B}\times\mathscr{F}^*$. Then by Lemma 12,

$$\begin{aligned}
\nu\times m\times\mu^*(T^k\Gamma) &= \iint_{Z\times\Phi^*} m\big([T^k\Gamma](n,\varphi^*)\big)\,d\nu\times\mu^* \;{}^{1}\\
&= \iint_{Z\times\Phi^*} m\big(\varphi_k\ldots\varphi_1[\Gamma](n-k,\sigma^{-k}\varphi^*)\big)\,d\nu\times\mu^*\\
&= \iint_{Z\times\Phi^*} d\nu\times\mu^* \int_{[\Gamma](n-k,\sigma^{-k}\varphi^*)} \beta_k^*(x,\sigma^{-k}\varphi^*)\,dm\\
&= \iiint_{Z\times X\times\Phi^*} \chi_\Gamma(n-k,x,\sigma^{-k}\varphi^*)\,\beta_k^*(x,\sigma^{-k}\varphi^*)\,d\nu\times m\times\mu^* \;{}^{2}\\
&= \iiint_{Z\times X\times\Phi^*} \chi_\Gamma(n,x,\varphi^*)\,\beta_k^*(x,\varphi^*)\,d\nu\times m\times\mu^*\\
&= \iiint_\Gamma \beta_k^*(x,\varphi^*)\,d\nu\times m\times\mu^*.
\end{aligned}$$

[1] $[\Gamma](n,\varphi^*)$ stands for the (n,φ^*)-section of the set Γ.

[2] $\chi_\Gamma(n,x,\varphi^*)$ denotes the indicator function of the set Γ.

Consequently
$$\alpha_k(n, x, \varphi^*) = \beta_k^*(x, \varphi^*), \qquad n \in Z,\ k \geqq 1 \tag{3.6}$$
almost everywhere on $X \times \Phi^*$. Therefore, inserting (3.6) into (3.5) and considering Lemma 12, we come up to the conclusion of Theorem 13.

Note that in Theorem 13, $g(n)$ cannot be replaced by a constant. But if we use the transformation S on $X \times \Phi^*$ given by
$$S(x, \varphi^*) = (\varphi_1 x, \sigma \varphi^*), \tag{3.7}$$
then, by the same way as that of the proof of Theorem 13, we can prove

Theorem 14. *Let* $h(x)$ *be as in Theorem* 13 *and* $f(x) \in L_1(m)$. *Then, excepting a* μ^**-null set in* Φ^*,
$$\lim_{n \to \infty} \frac{\sum_{k=1}^{n} u_k f(\varphi_k \dots \varphi_1 x)\, \beta_{(k, \varphi^*)}(x)}{\sum_{k=1}^{n} u_k h(\varphi_k \dots \varphi_1 x)\, \beta_{(k, \varphi^*)}(x)}$$
exists and is finite almost everywhere on $\left\{x\colon \sum_{k=1}^{\infty} u_k h(\varphi_k \dots \varphi_1 x)\, \beta_{(k, \varphi^*)}(x) > 0\right\}$.

This theorem is an extension of the ergodic theorem of Hurewicz and Halmos (cf. Theorem 11).

In what follows, we mention a few special cases of Theorem 13 and Theorem 14. Let $(Y, \mathscr{C}, \mu)$ be a probability measure space on which an automorphism η is given. Let $\{\psi_y\colon y \in Y\}$ be a $\mathscr{C} \times \mathscr{B}$-measurable family of one-to-one bimeasurable positively non-singular transformations of X onto itself. It follows then by Radon-Nikodym's theorem that there exist positive measurable functions $\beta_{(k, y)}(x)$, $k \geqq 1$ such that for any $A \in \mathscr{B}$,
$$m(\psi_{\eta^{k-1} y} \dots \psi_{\eta y} \psi_y A) = \int_A \beta_{(k, y)}(x)\, dm, \qquad k \geqq 1.$$
Let us set
$$\Phi^* = \{\varphi^* = (\dots, \psi_{\eta^{-1} y}, \psi_y, \psi_{\eta y}, \dots)\colon y \in Y\}$$
and define a mapping H of Y onto Φ^* by
$$H(y) = \varphi^*, \qquad \varphi^* = (\dots, \varphi_{-1}, \varphi_0, \varphi_1, \dots),$$
$$\varphi_n = \psi_{\eta^n y}, \qquad -\infty < n < \infty.$$
Taking
$$\mathscr{F}^* = \{B\colon H^{-1} B \in \mathscr{C}\}, \qquad \mu^*(B) = \mu(H^{-1} B), \qquad B \in \mathscr{F}^*,$$
we obtain a probability measure space $(\Phi^*, \mathscr{F}^*, \mu^*)$. Then the two-sided shift transformation σ on Φ^* is $\mathscr{F}^*$-measurable and preserves the measure μ^*. Considering this situation, Theorem 13 entails

Corollary 15. *Let* $g(n)$ *and* $h(x)$ *be as in Theorem* 13. *Then for every* $f(x) \in L_1(m)$ *there is a null set* D *in* Y *such that for any* $y \in Y - D$,
$$\lim_{n \to \infty} \frac{\sum_{k=1}^{n} u_k g(k+j) f(\psi_{\eta^{k-1} y} \dots \psi_{\eta y} \psi_y x)\, \beta_{(k, y)}(x)}{\sum_{k=1}^{n} u_k h(\psi_{\eta^{k-1} y} \dots \psi_{\eta y} \psi_y x)\, \beta_{(k, y)}(x)}, \qquad j \in Z$$

exist and are finite almost everywhere on the set $E(y)$, *where*

$$E(y)=\left\{x:\ \sum_{k=1}^{\infty} u_k h(\psi_{\eta^{k-1}y}\ldots\psi_{\eta y}\psi_y x)\,\beta_{(k,y)}(x)>0\right\}.$$

Theorem 14 *yields*

Corollary 16. *Let* $h(x)$ *be as in Theorem* 14 *and* $f(x)\in L_1(m)$. *Then there is a set* D *with* μ*-measure zero such that for any* $y\in Y-D$,

$$\lim_{n\to\infty}\frac{\sum_{k=1}^{n} u_k f(\psi_{\eta^{k-1}y}\ldots\psi_{\eta y}\psi_y x)\,\beta_{(k,y)}(x)}{\sum_{k=1}^{n} u_k h(\psi_{\eta^{k-1}y}\ldots\psi_{\eta y}\psi_y x)\,\beta_{(k,y)}(x)}$$

exists and is finite almost everywhere on the set $E(y)$.

Remark 17. If η is an endomorphism of Y and $\{\psi_y\colon y\in Y\}$ is a $\mathscr{C}\times\mathscr{B}$-measurable family of endomorphisms of X, we take $\Phi^*=\{\varphi^*=(\psi_y,\psi_{\eta y},\ldots)\colon y\in Y\}$ and σ to be the one-sided shift transformation on Φ^*. Such a dynamical system $(\Phi^*,\mathscr{F}^*,\mu^*,\sigma)$ displays great efficacy in reproducing the usual forms of random ergodic theorems concerning measure preserving transformations through the results obtained in § 2.

In the rest of this section, we assume that Φ is a set of measurable non-singular transformations φ of a finite measure space $(X,\mathscr{B},m)$ into itself, where φ is not necessarily one-to-one and its inverse mapping φ^{-1} is not necessarily measurable.

As in § 2, we consider the one-sided product measure space $(\Phi^*,\mathscr{F}^*,\mu^*)$ and the one-sided shift transformation σ on the space

Theorem 18. *Suppose there exists a positive constant* K *such that*

$$m(\varphi_1^{-1}\ldots\varphi_k^{-1}A)\leqq K\cdot m(A),\qquad k\geqq 1$$

for all $\varphi^*\in\Phi^*$ *and for any* $A\in\mathscr{B}$, *where* K *is independent of points* $\varphi^*\in\Phi^*$ *and measurable sets* $A\in\mathscr{B}$. *Then for every* $f(x)\in L_1(m)$ *there is a* μ^**-null set* D^* *such that for any* $\varphi^*\in\Phi^*-D^*$, *there exists a function* $G_{\varphi^*}(x)\in L_1(m)$ *such that*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f(\varphi_k\ldots\varphi_1 x)=G_{\varphi^*}(x)$$

almost everywhere on X.

Proof. Let S be the skew product transformation defined on $X\times\Phi^*$ by (3.7). It is quite clear that S is $\mathscr{B}\times\mathscr{F}^*$-measurable and non-singular with respect to $m\times\mu^*$. By assumption,

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n} m\times\mu^*(S^{-k}\Lambda)&=\frac{1}{n}\sum_{k=1}^{n}\int_{\Phi^*} m([S^{-k}\Lambda](\varphi^*))\,d\mu^*\\ &=\frac{1}{n}\sum_{k=1}^{n}\int_{\Phi^*} m(\varphi_1^{-1}\ldots\varphi_k^{-1}[\Lambda](\sigma^k\varphi^*))\,d\mu^*\\ &\leqq\frac{K}{n}\sum_{k=1}^{n}\int_{\Phi^*} m([\Lambda](\sigma^k\varphi^*))\,d\mu^*\\ &=\frac{K}{n}\sum_{k=1}^{n}\int_{\Phi^*} m([\Lambda](\varphi^*))\,d\mu^*\\ &=K\cdot m\times\mu^*(\Lambda),\qquad n\geqq 1\end{aligned}$$

for every $\Lambda \in \mathscr{B} \times \mathscr{F}^*$. Define $F(x, \varphi^*) = f(x)$. Then $F(x, \varphi^*) \in L_1(m \times \mu^*)$, so that by Dunford and Miller's theorem [9], there exists a function $G(x, \varphi^*) \in L_1(m \times \mu^*)$ such that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} F(S^k(x, \varphi^*)) = G(x, \varphi^*)$$

for almost all $(x, \varphi^*) \in X \times \Phi^*$. The desired conclusion follows from this by Fubini's theorem.

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