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Geometric Bounds on the Ornstein-Uhlenbeck Velocity Process

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Summary. Let $X: \Omega \to \mathscr{C}(\mathbb{R}_+; \mathbb{R}^n)$ be the Ornstein-Uhlenbeck velocity process in equilibrium and denote by $\tau_A = \tau_A(X)$ the first hitting time of $A \subseteq \mathbb{R}^n$. If $A, B \in \mathscr{R}^n$ and $\mathbb{P}(X(0) \in A) = \mathbb{P}(X_n(0) \leqq a)$, $\mathbb{P}(X(0) \in B) = \mathbb{P}(X_n(0) \geqq b)$ we prove that

$$\mathbb{P}(\tau_A \leqq t) \geqq \mathbb{P}(\tau_{\{x_n \leqq a\}} \leqq t)$$

and

$$\mathbb{E}\left(\int_0^{t \wedge \tau_A} 1_B(X(s)\, ds\right) \leqq \mathbb{E}\left(\int_0^{t \wedge \tau_{\{x_n \leqq a\}}} 1_{\{x_n \geqq b\}}(X(s))\, ds\right).$$

Here X_n denotes the n-th component of X.

1. Introduction

In a recent paper [5] Ehrhard proves some very interesting inequalities for Gaussian Dirichlet integrals using symmetrization in Gauss space (Ehrhard [6]). Here we shall give further attention to this new subject by also inserting time.

Let $N = -\Delta + x \cdot \nabla$ (be the number operator) and consider the following Cauchy problem

$$\begin{gathered}(\partial_t + \tfrac{1}{2} N + c(t, x))\, u = g(t, x), \qquad t > 0,\ x \in \mathbb{R}^n \\ u = f \quad \text{on } t = 0\end{gathered} \tag{1.1}$$

where $c, f, g \geqq 0$ (we will specify exact conditions on sure functions from Sect. 2 on). The standard solution of (1.1) is written u_S. Among other things we are going to show that if $h: \mathbb{R}^n \to [0, +\infty[$ is given and $p \geqq 1$, then the average

$$\int_{\mathbb{R}^n} u_S^p(t, x)\, h(x)\, e^{-|x|^2/2}\, dx/(2\pi)^{n/2} \tag{1.2}$$

increases under appropriate Gauss symmetrizations of c, f, g, and h (Theorem 3.1).

The bound so obtained may be expressed in terms of the Ornstein-Uhlenbeck velocity process U in $\mathbb{R}^n$, normalized so that

$$dU(t) = -\tfrac{1}{2}U(t)\,dt + dW(t), \qquad t \geqq 0.$$

Here W is the Wiener process ($\mathbb{E}|W(t)|^2 = nt$). Indeed, the function

$$u^c_{f,g}(t,x) = \mathbb{E}_x\Big[f(U(t))\, e^{-\int_0^t c(t-\lambda,\, U(\lambda))\,d\lambda}\Big] + \mathbb{E}_x\left[\int_0^t g(t-s,\, U(s))\, e^{-\int_0^s c(t-\lambda,\, U(\lambda))\,d\lambda}\, ds\right] \tag{1.3}$$

agrees with $u_S(t,x)$. In particular, choosing $h=p=1$ in (1.2) we get estimates on certain hitting probabilities of the Ornstein-Uhlenbeck velocity process in equilibrium. Denoting the latter process by X,

$$\mathbb{P}(X \in \cdot) = \int \mathbb{P}_x(U \in \cdot)\, e^{-|x|^2/2}\, dx/(2\pi)^{n/2}$$

so that X is mean zero Gaussian and

$$\mathbb{E} X_i(s)\, X_j(t) = \delta_{ij}\, e^{-\frac{1}{2}|s-t|}.$$

Corollaries 3.1 and 3.2 are the main contributions of this paper.

A reader who wants more background material on X may consult the very charming books by Nelson [11] and Simon [13].

2. Some Notation

Throughout, $Q =]0, +\infty[\times \mathbb{R}^n$. For $M = \partial Q (= \mathbb{R}^n)$, Q, or $\bar{Q}$, we introduce

$$\mathscr{B}(M) = \{f: M \to \mathbb{R};\ f \text{ Borel measurable}\}$$

$$\mathscr{C}(M) = \{f: M \to \mathbb{R};\ f \text{ continuous}\}$$

$$\mathscr{L}(M) = \{f: M \to \mathbb{R};\ \sup_{\xi, \eta \in M} |f(\xi) - f(\eta)|/|\xi - \eta| < +\infty\}$$

$$\mathscr{K}_b(M) = \{f \in \mathscr{K}(M);\ \sup_{\substack{(t,x) \in M \\ t \leqq T}} |f(t,x)| < +\infty,\ 0 \leqq T < +\infty\}, \qquad \mathscr{K} = \mathscr{B}, \mathscr{C}, \mathscr{L}$$

and

$$\mathscr{K}_b^+(M) = \{f \in \mathscr{K}_b(M);\ f \geqq 0\}, \qquad \mathscr{K} = \mathscr{C}, \mathscr{L}.$$

Moreover, we will often make use of the following notation

$$\nabla = (\partial_{x_1}, \ldots, \partial_{x_n}) = (\nabla', \partial_{x_n})$$

and

$$\Delta = \partial^2_{x_1} + \ldots + \partial^2_{x_n} = \Delta' + \partial^2_{x_n}.$$

3. The Main Results

Suppose G is a k-dimensional linear subspace of $\mathbb{R}^n$, $k \geq 1$, and let $\theta \in G$ be a fixed unit vector. The generic point in $\mathbb{R}^n$ is written $x=(x', x'')$, $x' \in G^\perp$, $x'' \in G$. Furthermore, γ_G denotes the canonical Gaussian measure in G,

$$\gamma_G(dx'')=e^{-|x''|^2/2}\, dx''/(2\pi)^{k/2}.$$

Then, by [5], for each $f \in \mathscr{C}(\mathbb{R}^n)$ there exists a unique $f^\theta = S_{G,\theta}^{\mathbb{R}^n} f \in \mathscr{C}(\mathbb{R}^n)$ possessing the following properties for every $x \in \mathbb{R}^n$:

(i) $f^\theta(x)=f^\theta(x', \langle \theta, x'' \rangle \theta)$

(ii) $\lambda \curvearrowright f^\theta(x', \lambda\theta)$, $\lambda \in \mathbb{R}$, increases

and

(iii) $\gamma_G(f^\theta(x', \cdot) \geq \lambda)=\gamma_G(f(x', \cdot) \geq \lambda)$, $\lambda \in \mathbb{R}$.

Moreover,

$$S_{G,\theta}^{\mathbb{R}^n} \mathscr{L}(\mathbb{R}^n) \subseteq \mathscr{L}(\mathbb{R}^n) \tag{3.1}$$

and

$$\|\nabla f\|_{2,\gamma_{\mathbb{R}^n}} \geq \|\nabla f^\theta\|_{2,\gamma_{\mathbb{R}^n}}, \quad f \in \mathscr{L}(\mathbb{R}^n). \tag{3.2}$$

For short, let us write

$$f^\theta = S_{G,\theta}^G f, \quad \text{if } f \in \mathscr{C}(G)$$

and

$$f^\theta = \{(t,x) \curvearrowright [S_{G,\theta}^{\mathbb{R}^n} f(t,\cdot)](x)\}, \quad \text{if } f \in \mathscr{C}(Q) \cup \mathscr{C}(\bar{Q}).$$

Our main result may then be stated as follows

Theorem 3.1. *Suppose $c, g \in \mathscr{L}_b^+(Q)$, $f \in \mathscr{L}_b^+(\partial Q)$, and let $u_{f,g}^c$ be as in* (1.3).
If $h \in \mathscr{L}_b^+(G)$ and p: $[0, +\infty[\to \mathbb{R}$ is increasing and convex, then

$$\langle p(u_{f,g}^c(t,x',\cdot)), h \rangle_{\gamma_G} \leq \langle p(u_{f^\theta,g^\theta}^{c^{-\theta}}(t,x',\cdot)), h^\theta \rangle_{\gamma_G}, \quad t \geq 0,\ x' \in G^\perp.$$

From $u_{f,0}^0(t,\cdot)=\left[\exp\left(-\frac{t}{2}N\right)\right] f$ and Theorem 3.1 it follows that

$$t^{-1}\langle f - e^{-\frac{t}{2}N} f, f\rangle_{\gamma_{\mathbb{R}^n}} \geq t^{-1}\langle f^\theta - e^{-\frac{t}{2}N} f^\theta, f^\theta\rangle_{\gamma_{\mathbb{R}^n}}.$$

Thus by letting $t \to 0^+$ we have Ehrhard's basic inequality (3.2) (some details are excluded here).

We next discuss some other corollaries.

Let $\mathscr{R}^n = \{\text{Borel sets in } \mathbb{R}^n\}$ and set $\tau_A = \tau_A(X) = \inf\{t>0; X(t) \in A\}$.

Corollary 3.1. *Suppose $A \in \mathscr{R}^n$. If $\mathbb{P}(X(0) \in A) = \mathbb{P}(X_n(0) \leq a)$, then*

$$\mathbb{P}(\tau_A \leq t) \geq \mathbb{P}(\tau_{\{x_n \leq a\}} \leq t), \quad t \geq 0. \tag{3.3}$$

Proof. Suppose first that A is open and choose compacts $K_i \subseteq A$ with $\gamma_{\mathbb{R}^n}(A \setminus K_i) \downarrow 0$. Moreover, let $\mathscr{L}_b^+(\mathbb{R}^n) \ni c_i \uparrow +\infty 1_A$ and $c_i \geq i 1_{K_i}$. Now by dominated convergence

$$\mathbb{P}(\tau_A \geqq t) = \lim_{i \to +\infty} \langle u^{c_i}_{1,0}(t, \cdot), 1 \rangle_{\gamma_{\mathbb{R}^n}}, \quad t \geqq 0$$

and

$$\mathbb{P}(\tau_{\{x_n < a\}} \geqq t) = \lim_{i \to +\infty} \langle u^{c_i^{-e_n}}_{1,0}(t, \cdot), 1 \rangle_{\gamma_{\mathbb{R}^n}}, \quad t \geqq 0,$$

where $e_n = (0, \ldots, 0, 1) \in \mathbb{R}^n$. Hence, from Theorem 3.1

$$\mathbb{P}(\tau_A < t) \geqq \mathbb{P}(\tau_{\{x_n \leqq a\}} < t), \quad t \geqq 0.$$

Since $\mathbb{P}_x(\tau_B = t) = 0$, $t > 0$, $B \in \mathscr{R}^n$ (compare Port and Stone [12, Theorem 4.7]) we have proved (3.3) for A open.

The general case requires some caution.

Let $A \in \mathscr{R}^n$ be fixed and introduce $A^r = \{\mathbb{P}_\cdot(\tau_A = 0) = 1\}$. As in [12, Theorem 3.7] one verifies that $\gamma_{\mathbb{R}^n}(A \setminus A^r) = 0$. Therefore, by Blumenthal and Getoor [3, Chapter 1, Theorem 11.2], there exist open $A_i \supseteq A$ satisfying $\tau_{A_i} \uparrow \tau_A$ a.s. $\mathbb{P}_X$. The inequality (3.3) is now obvious. □

Corollary 3.2. *Let* $A, B \in \mathscr{R}^n$ *and suppose* $\mathbb{P}(X(0) \in A) = \mathbb{P}(X_n(0) \leqq a)$ *and* $\mathbb{P}(X(0) \in B) = \mathbb{P}(X_n(0) \geqq b)$. *Then*

$$\mathbb{E}\left(\int_0^{t \wedge T_A} 1_B(X(s))\, ds\right) \leqq \mathbb{E}\left(\int_0^{t \wedge T_{\{x_n \leqq a\}}} 1_{\{x_n \geqq b\}}(X(s))\, ds\right), \quad t \geqq 0.$$

Corollary 3.2 follows as Corollary 3.1 does and the proof is omitted.

The main ideas in our proof of Theorem 3.1 were initiated by Baernstein [1]. Actually, Baernstein treats Δ-subharmonic functions and Steiner (radial) symmetrization but, as will be seen, his elegant method fits very well in the present situation, too.

4. Preparations

This section collects various theorems which are needed for the proof of Theorem 3.1. Most of them are well-known or have appeared previously.

Theorem 4.1. *Suppose* $c, g \in \mathscr{L}_b^+(Q)$ *and* $f \in \mathscr{L}_b^+(\partial Q)$.
a) *The function* $u = u^c_{f,g}$ *is the unique classical solution of* (1.1) *with* $u \in \mathscr{C}_b(\bar{Q})$.
b) *If* c, g *are real analytic, then* $u^c_{f,g}(t, \cdot)$ *is real analytic for each* $t > 0$.

By a classical solution is here meant a solution which is $\mathscr{C}^{1,2}$ in the interior of its domain of definition.

Proof. a) Let $B_r = \{|x| < r\}$ and $Q_r =]0, +\infty[\times B_r$ $(0 < r < +\infty)$. The Cauchy problem

$$\begin{cases} (\partial_t + \frac{1}{2} N + c)\, v = g & \text{in } Q_r \\ v = \psi \quad \text{on } \partial Q_r, & v \in \mathscr{C}(\bar{Q}_r) \ (\psi \in \mathscr{C}(\partial Q_r)) \end{cases}$$

has a unique classical solution $v=v(\cdot, \psi, g)$. Introducing $\sigma_r=\tau_{B_r}(U)\wedge t$, we have

$$\begin{aligned} v(t,x) &= \mathbb{E}_x\left[\psi(U(t))\, e^{-\int_0^{\sigma_r} c(t-\lambda,\, U(\lambda))\,d\lambda};\ \sigma_r=t\right] \\ &+ \mathbb{E}_x\left[\psi(t-\sigma_r, U(\sigma_r))\, e^{-\int_0^{\sigma_r} c(t-\lambda,\, U(\lambda))\,d\lambda};\ \sigma_r<t\right] \\ &+ \mathbb{E}_x\left[\int_0^{\sigma_r} g(t-s, U(s))\, e^{-\int_0^{s} c(t-\lambda,\, U(\lambda))\,d\lambda}\, ds\right], \quad (t,x)\in\bar{Q}_r \end{aligned}$$

(see Friedman [8, Theorem 5.2, p. 147]).

It is obvious that $u^c_{f,g}$ is continuous. Set $v_R=v(\cdot, u^c_{f,g\,|\,\partial Q_R}, g)$. Then

$$v_R(t,x)=v(t,x,v_{R\,|\,\partial Q_r}, g), \quad (t,x)\in\overline{Q_r},\ r<R.$$

We next let R tend to $+\infty$ and use dominated convergence to obtain

$$u^c_{f,g}(t,x)=v(t,x,u^c_{f,g\,|\,\partial Q_r}, g), \quad (t,x)\in\overline{Q_r}.$$

Accordingly, $u^c_{f,g}$ is a classical solution of (1, 1). Uniqueness now results from Friedman [9, Theorem 10, p. 44].

b) For a proof, see Friedman [10].

In what follows, we write

$$M=-\Delta+x'\cdot\nabla'-x_n\,\partial_{x_n}=-\Delta+x_1\,\partial_{x_1}+\dots+x_{n-1}\,\partial_{x_{n-1}}-x_n\,\partial_{x_n}$$

and denote by V a solution of

$$dV(t)=-\tfrac{1}{2}(V_1(t),\dots,V_{n-1}(t),\,-V_n(t))\,dt+dW(t), \quad t\geqq 0.$$

Note that

$$\mathbb{E}_x V_i(t)=\begin{cases} e^{-t/2}x_i, & i<n \\ e^{t/2}x_n, & i=n \end{cases} \tag{4.1}$$

and

$$\mathbb{E}_x(V_i(t)-\mathbb{E}_x V_i(t))^2=\begin{cases} 1-e^{-t}, & i<n \\ e^{t}-1, & i=n. \end{cases} \tag{4.2}$$

Theorem 4.2. *Suppose $c\in\mathscr{L}^+_b(Q)$. If*

$$\begin{cases} (\partial_t+\tfrac{1}{2}M+c)\,u\geqq 0 & \text{in } Q \quad (\text{weak sense}) \\ u\geqq 0 \quad \text{on } \partial Q, & u\in\mathscr{C}_b(\bar{Q}) \end{cases}$$

then $u\geqq 0$.

Here the members of $\mathscr{C}^\infty_0(Q)$ serve as test functions. Recall from distribution theory that a positive distribution is a positive Radon measure. Although Theorem 4.2 should be regarded as folklore, we submit a detailed proof of it.

Proof. Set

$$a(x)=e^{(-|x'|^2+x_n^2)/4}, \quad b(x)=(|x|^2-2n+4)/8, \quad x\in\mathbb{R}^n.$$

Let $0 \leqq \kappa \in \mathscr{C}_0^\infty(\mathbb{R}^{1+n})$, $\int \kappa\, dt\, dx = 1$, and $\operatorname{supp} \kappa \subseteq]0, 1[\times \mathbb{R}^n$. Next suppose $0 < \varepsilon < 1$ is fixed and set $\kappa_\varepsilon = \varepsilon^{-(1+n)} \kappa(\cdot/\varepsilon)$, $\check{\kappa}_\varepsilon = \kappa_\varepsilon(-(\cdot))$, and $Q(\varepsilon) = (\varepsilon, 0, \dots, 0) + Q$.

Now consider the following linear transformation

$$\begin{cases} A_\varepsilon \colon \mathscr{D}'(Q) \to \mathscr{D}'(Q(\varepsilon)) \\ \langle A_\varepsilon f, \varphi \rangle = \langle a f, \check{\kappa}_\varepsilon * (\varphi/a) \rangle. \end{cases}$$

If $f \in \mathscr{C}(Q)$, then

$$(A_\varepsilon f)(t, x) = \frac{1}{a(x)} \int_Q a(x-y) f(t-s, x-y)\, \kappa_\varepsilon(s, y)\, ds\, dy.$$

Moreover, as $(\partial_t - \frac{1}{2}\Delta)(a\varphi) = a(\partial_t + \frac{1}{2}M - b)\varphi$, $\varphi \in \mathscr{C}_0^\infty(Q)$, we have

$$A_\varepsilon(\partial_t + \tfrac{1}{2}M - b) f = (\partial_t + \tfrac{1}{2}M - b) A_\varepsilon f, \qquad f \in \mathscr{D}'(Q).$$

Hence

$$(\partial_t + \tfrac{1}{2}M + c) A_\varepsilon u = A_\varepsilon(\partial_t + \tfrac{1}{2}M + c) u + h_\varepsilon,$$

where

$$h_\varepsilon = b A_\varepsilon u - A_\varepsilon(bu) + c A_\varepsilon u - A_\varepsilon(cu).$$

In what follows, let $\delta > \varepsilon$. Then, for all $(t, x) \in \overline{Q_r}$,

$$\begin{aligned} (A_\varepsilon u)(\delta + t, x) \geqq\; & \mathbb{E}_x\Big[(A_\varepsilon u)(\delta, V(t))\, e^{-\int_0^{\sigma_r} c(\delta+t-\lambda, V(\lambda))\, d\lambda}; \sigma_r = t\Big] \\ & + \mathbb{E}_x\Big[(A_\varepsilon u)(\delta + t - \sigma_r, V(\sigma_r))\, e^{-\int_0^{\sigma_r} c(\delta+t-\lambda, V(\lambda))\, d\lambda}; \sigma_r < t\Big] \\ & + \mathbb{E}_x\Big[\int_0^{\sigma_r} h_\varepsilon(\delta + t - s, V(s))\, e^{-\int_0^{s} c(\delta+t-\lambda, V(\lambda))\, d\lambda}\, ds\Big] \qquad (\sigma_r = \tau_{B_r}(V) \wedge t). \end{aligned}$$

As $|A_\varepsilon u| + |h_\varepsilon|$ is uniformly bounded in each $\overline{Q_r} \cap \{t \leqq T\}$ $(0 < r, T < +\infty)$ we get by first letting $\varepsilon \to 0^+$ and then $\delta \to 0^+$,

$$u(t, x) \geqq \mathbb{E}_x\Big[u(t - \sigma_r, V(\sigma_r))\, e^{-\int_0^{\sigma_r} c(t-\lambda, V(\lambda))\, d\lambda}; \sigma_r < t\Big].$$

Finally, if $r \to +\infty$, the last inequality reduces to $u(t, x) \geqq 0$. □

Below we write $\gamma_{\mathbb{R}} = \gamma$, for simplicity.

Theorem 4.3. *Let $c \in \mathscr{L}_b^+(Q)$ be such that $\partial_{x_n} c \geqq 0$. Suppose $u \in \mathscr{C}_b(\bar{Q})$ and set*

$$v(t, x) = \int_{-\infty}^{x_n} u(t, x', \lambda)\, d\gamma(\lambda), \qquad (t, x) \in \bar{Q}.$$

If

$$\begin{cases} (\partial_t + \frac{1}{2}M + c)\, v(t, x) \geqq \int_{-\infty}^{x_n} v(t, x', \lambda)\, dc(t, x', \lambda) & \text{in } Q \text{ (weak sense)} \\ v \geqq 0 & \text{on } \partial Q \end{cases}$$

then $v \geqq 0$.

Proof. We first extend c to $\bar{Q}$ by continuity and introduce the following operators on $\mathscr{B}_b(\bar{Q})$:

$$(Af)(t,x)=\int_{-\infty}^{x_n} f(t,x',\lambda)\,dc(t,x',\lambda)$$

and

$$(Bf)(t,x)=\mathbb{E}_x\left[\int_0^t f(t-s,V(s))\,e^{-\int_0^s c(t-\lambda,V(\lambda))\,d\lambda}\,ds\right].$$

Of course, $Av\in\mathscr{C}_b(\bar{Q})$ as being clear from

$$(Av)(t,x)=v(t,x)\,c(t,x)-\int_{-\infty}^{x_n} u(t,x',\lambda)\,c(t,x',\lambda)\,d\gamma(\lambda).$$

In addition,

$$(\partial_t+\tfrac{1}{2}M+c)\,BAv=Av \quad \text{in } Q \text{ (weak sense).}$$

Thus $v\geqq BAv$ from Theorem 4.2 and the premise for v of the theorem and by iteration

$$v\geqq(BA)^k v,\qquad k=1,2,\ldots.$$

But now assuming $c_{|[0,T_0]\times\mathbb{R}^n}\leqq C_{T_0}(0<T_0<+\infty)$,

$$\sup_{\substack{0\leqq t\leqq T\\ x\in\mathbb{R}^n}}|[BAf](t,x)|\leqq T\,C_{T_0}\sup_{\substack{0\leqq t\leqq T\\ x\in\mathbb{R}^n}}|f(t,x)|,\qquad T\leqq T_0,$$

and it follows that $v_{|[0,T]\times\mathbb{R}^n}\geqq 0$ if $T\,C_{T_0}<1$. By repetition, $v\geqq 0$.

We next discuss some geometrical points.

Theorem 4.4. *Suppose G is a k-dimensional linear subspace of $\mathbb{R}^n$, $k\geqq 1$, and let $\theta\in G$ be a fixed unit vector.*
a) *If $k=2$, there exists a sequence $\theta_i\in G$, $|\theta_i|=1$, $i\in\mathbb{N}$, such that*

$$[S^{\mathbb{R}^n}_{G,\theta}f](x)=\lim_{i\to+\infty}[(S^{\mathbb{R}^n}_{\operatorname{span}\theta_0,\theta_0}\circ\ldots\circ S^{\mathbb{R}^n}_{\operatorname{span}\theta_i,\theta_i})f](x),\qquad x\in\mathbb{R}^n,$$

for each $f\in\mathscr{L}(\mathbb{R}^n)$, where span θ *denotes the* 1*-dimensional subspace spanned by θ.*
b) *If $k\geqq 3$, there exist $k-1$ 2-dimensional linear subspaces $G_i\ni\theta$ of $\mathbb{R}^n$ such that*

$$S^{\mathbb{R}^n}_{G,\theta}=S^{\mathbb{R}^n}_{G_1,\theta}\circ\ldots\circ S^{\mathbb{R}^n}_{G_{k-1},\theta}$$

Theorem 4.4 is a simple consequence of [5] and [6] and we do not go into details.

Before stating the next theorem we discuss some simple estimates, which will be useful throughout the paper.

Suppose $f_i\in\mathscr{C}_b^+(\mathbb{R})$ and set $f_i^\theta=S^{\mathbb{R}}_{\mathbb{R},\theta}f_i$, $i=0,1$. Then for all $s,t\geqq 0$,

$$\gamma(f_0\geqq s,f_1\geqq t)\leqq\gamma(f_0^\theta\geqq s)\wedge\gamma(f_1^\theta\geqq t)=\gamma(f_0^\theta\geqq s,f_1^\theta\geqq t).\tag{4.3}$$

By integrating this inequality with respect to the measure $ds\,dt$ over the region $0\leqq s,t<+\infty$, we have

$$\langle f_0,f_1\rangle_\gamma\leqq\langle f_0^\theta,f_1^\theta\rangle_\gamma.$$

Since $(a-f_1)^\theta = a - f_1^{-\theta}$, $a\in\mathbb{R}$, it also follows that

$$\langle f_0, f_1\rangle_\gamma \geqq \langle f_0^\theta, f_1^{-\theta}\rangle.$$

Below $\|\cdot\|_{1,\gamma}$ ($\|\ \ \|_\infty$) means $L_1(\gamma)$-norm (supremum norm).

Theorem 4.5. *Let* $\theta=-1$. *Suppose* $f_i\in\mathscr{C}_b^+(\mathbb{R})$ *and set* $f_i^\theta = S^{\mathbb{R}}_{\mathbb{R},\theta} f_i$ *and* $\tilde{f}_i = \int\limits_{-\infty}^{\bullet} f_i^\theta\, d\gamma$, $i=0,1$. *Then*

a) $\|f_0^\theta - f_1^\theta\|_{1,\gamma} \leqq \|f_0 - f_1\|_{1,\gamma}$.

In particular,

$$\|\tilde{f}_0 - \tilde{f}_1\|_\infty \leqq \|f_0 - f_1\|_{1,\gamma}.$$

b) *If* $p\colon [0,+\infty[\to\mathbb{R}$ *is increasing and convex, then*

$$\tilde{f}_0 \leqq \tilde{f}_1 \Rightarrow \langle p(f_0), h\rangle_\gamma \leqq \langle p(f_1^\theta), h^\theta\rangle_\gamma, \qquad h\in\mathscr{C}_b^+(\mathbb{R}).$$

Proof. a) Writing

$$|f_0 - f_1| = f_0 + f_1 - 2\int\limits_0^\infty 1_{[0,f_0]}(s)\, 1_{[0,f_1]}(s)\, ds$$

the result follows from (4.3).

b) Let

$$p(f_0^\theta) - p(f_1^\theta) = a(f_0^\theta, f_1^\theta)\,(f_0^\theta - f_1^\theta)$$

where $a\geqq 0$ increases in each variable separately. But then noting that the function $a(f_0^\theta, f_1^\theta)\, h^\theta$ decreases,

$$\langle p(f_0^\theta) - p(f_1^\theta), h^\theta\rangle_\gamma \leqq 0.$$

Now using

$$\langle p(f_0), h\rangle_\gamma \leqq \langle p(f_0^\theta), h^\theta\rangle_\gamma$$

Part b) follows at once. □

Theorem 4.6. *Suppose* $\varepsilon\colon \mathbb{R}\to\mathbb{R}$, $\varepsilon(0)=0$, $x\in\mathbb{R}$, *and* $a_0<b_0<\ldots<a_m<b_m$. *Set* $\Phi = \gamma(]-\infty,\cdot])$. *The equation*

$$\sum_{\nu=0}^{m}\ \int\limits_{a_\nu-\varepsilon(r)}^{b_\nu+\varepsilon(r)} d\gamma(\lambda) = \Phi(x+r), \qquad r\in\mathbb{R}$$

implies $0\leqq \varepsilon'(0)\leqq 1$.

See Borell [4].

If $f\in\mathscr{C}_b^+(\mathbb{R})$ is real analytic and non-constant and $\theta=-1$, the reader should note the following relations between $f, f^\theta, \tilde{f}$, and Φ:

$$\tilde{f}(x) = \int\limits_{\{f\geqq f^\theta(x)\}} f\, d\gamma = \sup\{\int\limits_A f\, d\gamma;\ A\in\mathscr{R},\ \gamma(A)=\Phi(x)\}, \qquad x\in\mathbb{R}.$$

5. Proof of Theorem 1.1

By Theorem 4.4 it is enough to treat the special case $G=\operatorname{span}\{e_n\}$, where $e_0, \ldots, e_n$ is the standard basis in $\mathbb{R}\times\mathbb{R}^n$. Choose $\theta=-e_n$. To simplify the notation let

$$u=u^c_{f,g}$$

and

$$\tilde{u}(t,x)=\int_{-\infty}^{x_n} u^\theta(t,x',\lambda)\,d\gamma(\lambda), \qquad (t,x)\in\bar{Q}.$$

Moreover, we set $v=u^{c^{-\theta}}_{f^\theta,g^\theta}$ and introduce v^θ and $\tilde{v}$ as above with u replaced by v. In view of Theorem 4.5 b), Theorem 1.1 now follows from

$$\tilde{u}\leqq\tilde{v} \quad\text{and}\quad v^\theta=v. \tag{5.1}$$

The proof of (5.1) occupies the rest of this paper; the reasoning below stems from Baernstein [1]. However, we choose a form more close to Essén [7, Theorem 9.3] (compare also Bandle [2, Theorem 4.17]).

Lemma 5.1

$$(\partial_t+\tfrac{1}{2}M+c^{-\theta})\,\tilde{u}(t,x)\leqq\int_{-\infty}^{x_n}\tilde{u}(t,x',\lambda)\,dc^{-\theta}(t,x',\lambda)+\int_{-\infty}^{x_n} g^\theta(t,x',\lambda)\,d\gamma(\lambda) \quad \textit{in } Q.$$

Here and from now on all derivatives are in the weak sense and we do not explicitly mention this each time.

Proof. First note that

$$\int_{-\infty}^{x_n} u^\theta(t,x',\lambda)\,c^{-\theta}(t,x',\lambda)\,d\gamma(\lambda)=c^{-\theta}(t,x)\,\tilde{u}(t,x)-\int_{-\infty}^{x_n}\tilde{u}(t,x',\lambda)\,dc^{-\theta}(t,x',\lambda). \tag{5.2}$$

By approximation and use of Theorem 4.5 we may without loss of generality assume that (i) $c, g\in\mathscr{L}_b^+(Q)$ are real analytic (ii) $g(t,x)\leqq\text{const.}_+\exp(-x_n^2)$ and that (iii) $0\not\equiv f\in\mathscr{L}_b^+(\partial Q)$ has compact support. In view of these assumptions and (4.1) and (4.2), $\sup\{u(t,x',x_n);\ 0<t\leqq T,\ x'\in\mathbb{R}^{n-1}\}\to 0$ as $|x_n|\to+\infty$ for all $T<+\infty$. Moreover, by Theorem 4.1 b), $u(t,x',\cdot)$ is real analytic so that

$$\tilde{u}(t,x)=\int_{C(t,x)} u(t,x',\lambda)\,d\gamma(\lambda)$$

where $C(t,x)=\{u(t,x',\cdot)\geqq u^\theta(t,x)\}$ is compact. The reader should note that $u^\theta(t,x',\cdot)$ is strictly decreasing. From the continuity of u^θ we now also conclude that the set

$$\bigcup_{(t,x)\in K} C(t,x)$$

is compact for all compacts $K\subseteq Q$.

We next compute or estimate derivatives. For $i<n$ and $r>-t$

$$\tilde{u}((t,x)+re_i)\geqq\int_{C(t,x)} u((t,x',\lambda)+re_i)\,d\gamma(\lambda)$$

and, consequently,

$$\partial_{x_i} \tilde{u}(t, x) = \int_{C(t,x)} \partial_{x_i} u(t, x', \lambda)\, d\gamma(\lambda), \quad i = 0, \ldots, n-1 \qquad (x_0 = t). \tag{5.3}$$

Using

$$\partial_{x_i}^2 \psi(t, x) = \lim_{r \to 0} r^{-2} (\psi((t, x) + r e_i) + \psi((t, x) - r e_i) - 2\psi(t, x)), \qquad \psi \in \mathscr{C}_0^\infty(Q)$$

we also have

$$\partial_{x_i}^2 \tilde{u}(t, x) \geqq \int_{C(t,x)} \partial_{x_i}^2 u(t, x', \lambda)\, d\gamma(\lambda), \qquad i = 1, \ldots, n-1. \tag{5.4}$$

In order to handle derivatives containing x_n, we write

$$C(t, x) = \bigcup_{\nu=0}^{m} [a_\nu, b_\nu] \qquad (-\infty < a_0 < b_0 < \ldots < a_m < b_m < +\infty).$$

Remember that $\mathbb{R} \setminus C(t, x)$ only has finitely many connected components because $u(t, x', \cdot)$ is real analytic and $C(t, x)$ compact.

For each fixed $r \in \mathbb{R}$, let $\varepsilon(r)$ satisfy

$$\sum_{\nu=0}^{m} \int_{a_\nu - \varepsilon(r)}^{b_\nu + \varepsilon(r)} d\gamma(\lambda) = \Phi(x_n + r).$$

Then, setting $\varphi = \Phi'$,

$$\varepsilon'(r) \sum_{\nu=0}^{m} (\varphi(b_\nu + \varepsilon(r)) + \varphi(a_\nu - \varepsilon(r))) = \varphi(x_n + r)$$

and, from this,

$$\varepsilon'(0) \sum_{\nu=0}^{m} (\varphi(b_\nu) + \varphi(a_\nu)) = \varphi(x_n) \tag{5.5}$$

and

$$-\varepsilon'(0)^2 \sum_{\nu=0}^{m} (b_\nu \varphi(b_\nu) - a_\nu \varphi(a_\nu)) + \varepsilon''(0) \sum_{\nu=0}^{m} (\varphi(b_\nu) + \varphi(a_\nu)) = -x_n \varphi(x_n). \tag{5.6}$$

Now let $u_0 = u(t, x', a_0) = \ldots = u(t, x', b_m)$. We claim that

$$\partial_{x_n} \tilde{u}(t, x) = u_0\, \varphi(x_n). \tag{5.7}$$

Indeed,

$$\Big(\int_{C((t,x) + r e_n)} - \int_{C(t,x)} \Big) u(t, x', \lambda)\, d\gamma(\lambda)$$
$$\geqq \sum_{\nu=0}^{m} \left(\int_{a_\nu - \varepsilon(r)}^{b_\nu + \varepsilon(r)} - \int_{a_\nu}^{b_\nu} \right) u(t, x', \lambda)\, d\gamma(\lambda)$$

and (5.7) follows from (5.5).

We next show that

$$\partial_{x_n}^2 \tilde{u}(t,x) \geqq \varepsilon'(0)^2 \sum_{\nu=0}^{m} (u'_{x_n}(t,x',b_\nu)\,\varphi(b_\nu) - u'_{x_n}(t,x',a_\nu)\,\varphi(a_\nu)) - u_0\, x_n\, \varphi(x_n). \tag{5.8}$$

To this end, we use

$$\begin{aligned}
&\Big(\int_{C((t,x)+r e_n)} + \int_{C((t,x)-r e_n)} - 2 \int_{C(t,x)} \Big) u(t,x',\lambda)\, d\gamma(\lambda) \\
&\geqq \sum_{\nu=0}^{m} \left(\int_{a_\nu - \varepsilon(r)}^{b_\nu + \varepsilon(r)} + \int_{a_\nu - \varepsilon(-r)}^{b_\nu + \varepsilon(-r)} - 2 \int_{a_\nu}^{b_\nu} \right) u(t,x',\lambda)\, d\gamma(\lambda)
\end{aligned}$$

and have

$$\begin{aligned}
\partial_{x_n}^2 \tilde{u}(t,x) \geqq \varepsilon'(0)^2 &\sum_{\nu=0}^{m} (\partial_\lambda (u(t,x',\lambda)\,\varphi(\lambda))|_{\lambda=b_\nu} - \partial_\lambda (u(t,x',\lambda)\,\varphi(\lambda))|_{\lambda=a_\nu}) \\
&+ \varepsilon''(0) \sum_{\nu=0}^{m} (u(t,x',b_\nu)\,\varphi(b_\nu) + u(t,x',a_\nu)\,\varphi(a_\nu)).
\end{aligned}$$

Now (5.8) results from (5.6).

Since $u'_{x_n}(t,x',b_\nu) \leqq 0$ and $u'_{x_n}(t,x',a_\nu) \geqq 0$, Theorem 4.6 and (5.8) give

$$\partial_{x_n}^2 \tilde{u}(t,x) \geqq \sum_{\nu=0}^{m} (u'_{x_n}(t,x',b_\nu)\,\varphi(b_\nu) - u'_{x_n}(t,x',a_\nu)\,\varphi(a_\nu)) - u_0 x_n \varphi(x_n). \tag{5.9}$$

It is now simple to complete the proof of Lemma 5.1. First from (5.4)

$$\begin{aligned}
\tfrac{1}{2}\Delta' \tilde{u}(t,x) &\geqq \int_{C(t,x)} \tfrac{1}{2}\Delta' u(t,x',\lambda)\, d\gamma(\lambda) \\
&= \int_{C(t,x)} \big(-\tfrac{1}{2}\partial_\lambda^2 u(t,x',\lambda) + \tfrac{1}{2} x' \cdot \nabla' u(t,x',\lambda) + \tfrac{1}{2}\lambda\, \partial_\lambda u(t,x',\lambda) \\
&\qquad + \partial_t u(t,x,\lambda) + c(t,x',\lambda)\, u(t,x',\lambda) - g(t,x',\lambda)\big)\, d\gamma(\lambda)
\end{aligned}$$

and then using (5.3)

$$\begin{aligned}
\tfrac{1}{2}\Delta' \tilde{u}(t,x) \geqq &-\tfrac{1}{2} \int_{C(t,x)} \partial\, (u'_{x_n}(t,x',\lambda)\,\varphi(\lambda))\, d\lambda + \tfrac{1}{2} x' \cdot \nabla' \tilde{u}(t,x) + \partial_t \tilde{u}(t,x) \\
&+ \int_{C(t,x)} (c(t,x',\lambda)\, u(t,x',\lambda) - g(t,x',\lambda))\, d\gamma(\lambda).
\end{aligned}$$

Accordingly,

$$\begin{aligned}
\tfrac{1}{2}\Delta' \tilde{u}(t,x) \geqq &-\tfrac{1}{2} \sum_{\nu=0}^{m} (u'_{x_n}(t,x',b_\nu)\,\varphi(b_\nu) - u'_{x_n}(t,x',a_\nu)\,\varphi(a_\nu)) + \tfrac{1}{2} x' \cdot \nabla' \tilde{u}(t,x) + \partial_t \tilde{u}(t,x) \\
&+ \int_{-\infty}^{x_n} c^{-\theta}(t,x',\lambda)\, u^\theta(t,x',\lambda)\, d\gamma(\lambda) - \int_{-\infty}^{x_n} g^\theta(t,x',\lambda)\, d\gamma(\lambda).
\end{aligned}$$

Finally, using (5.7) and (5.9), Lemma 5.1 follows from (5.2). □

Lemma 5.2. *$v^\theta = v$ in $\bar{Q}$ and*

$$(\partial_t + \tfrac{1}{2}M + c^{-\theta})\,\tilde{v}(t,x) = \int_{-\infty}^{x_n} \tilde{v}(t,x',\lambda)\,dc^{-\theta}(t,x',\lambda) + \int_{-\infty}^{x_n} g^\theta(t,x',\lambda)\,d\gamma(\lambda) \quad \text{in } Q. \tag{5.10}$$

Actually, for the proof of (5.1) we only need (5.10) with "=" replaced by "$\geqq$".

Proof. From

$$U(t) = (e^{-t/2},\ldots,e^{-t/2},e^{-t/2})\cdot U(0) + \int_0^t (e^{-(t-s)/2},\ldots,e^{-(t-s)/2},e^{-(t-s)/2})\cdot dW(s)$$

it is plain that $\partial_{x_n} v \leqq 0$. Hence $v^\theta = v$.

Set

$$\tilde{v}_a(t,x) = \int_a^{x_n} v(t,x',\lambda)\,d\gamma(\lambda).$$

Certainly, $\tilde{v}_a \to \tilde{v}$ in the distribution sense as $a \to -\infty$. Furthermore, a straightforward calculation yields

$$\begin{aligned}(\partial_t + \tfrac{1}{2}M)\,\tilde{v}_a(t,x) = &-\int_a^{x_n} c^{-\theta}(t,x,\lambda)\,v(t,x',\lambda)\,d\gamma(\lambda) + \int_a^{x_n} g^\theta(t,x',\lambda)\,d\gamma(\lambda)\\ &-v'_{x_n}(t,x',a)\,\varphi(a).\end{aligned}$$

In view of (3.1) we may now apply Lemma 5.1 to complete the proof of Lemma 5.2. □

We finally prove that $\tilde{u} \leqq \tilde{v}$.

Let $w = \tilde{v} - \tilde{u}$. Then from Lemmas 5.1 and 5.2

$$\begin{cases}(\partial_t + \tfrac{1}{2}M + c^{-\theta})\,w(t,x) \geqq \int_{-\infty}^{x_n} w(t,x',\lambda)\,dc^{-\theta}(t,x',\lambda) & \text{in } Q\\ w = 0 \quad \text{on } \partial Q.\end{cases}$$

By applying Theorem 4.3 we now conclude that $w \geqq 0$. The statements in (5.1) are thereby completely proved. □

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