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Harmonic Renewal Measures

P. Greenwood[1], E. Omey[2], and J.L. Teugels

[1] University of British Columbia, Vancouver, Canada
[2] Catholic University of Louvain, Dept. of Mathematics, Celestijnenlaan 200B, B-3030 Leuven, Belgium

Summary. If C is a distribution function on $(0, \infty)$ then the harmonic renewal function associated with C is the function $G(x)=\sum_1^\infty n^{-1} C^{(n)}(x)$. We link the asymptotic behaviour of G to that of $1-C$. Applications to the ladder index and the ladder height of a random walk are included.

§1. Notations, Introduction

Assume $C(x)$ to be a distribution function on $(0, \infty)$ with $C(0+)=0$; let $c(s)$ be its Laplace-Stieltjes transform (LST) i.e. for $\operatorname{Re} s \geqq 0$

$$c(s)=\int_0^\infty e^{-sx} dC(x).$$

For convenience LST's will be denoted by corresponding lower case letters. For $s>0$ then

$$\begin{aligned} 1-c(s) &= \exp\{\log[1-c(s)]\} \\ &= \exp -\sum_{n=1}^\infty \frac{1}{n}[c(s)]^n \\ &= \exp -\int_0^\infty e^{-sx} dG(x) \end{aligned}$$

where $G(x)=\sum_{n=1}^\infty \frac{1}{n} C^{(n)}(x)$ is non-decreasing, right-continuous and satisfies $G(0+)=0$. Hence for $\operatorname{Re} s \geqq 0$

$$1-c(s)=e^{-g(s)} \tag{1}$$

since (1) holds for $s \downarrow 0$ by monotone convergence. $C(x)$ will be non-defective iff $G(\infty)=\infty$.

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Functions of the form $\sum_{n=0}^{\infty} a_n C^{(n)}(x)$ are often called *generalized renewal functions*. If $a_n = 1$ for all n one obtains the *renewal function*

$$H(x) = \sum_{n=0}^{\infty} C^{(n)}(x)$$

which will be extensively used in the sequel. If $a_0 = 0$ and $a_n = n^{-1}$ ($n \geqq 1$) we get the *harmonic renewal function*

$$G(x) = \sum_{n=1}^{\infty} \frac{1}{n} C^{(n)}(x) \tag{2}$$

to which this paper is devoted.

Similarly for a discrete distribution $\{c_n, n \in \mathbb{N}\}$ we define the (discrete) *harmonic renewal sequence*

$$g_m = \sum_{n=1}^{\infty} \frac{1}{n} c_m^{(n)}$$

where $\{c_m^{(n)}, m \in \mathbb{N}\}$ is the n-fold convolution of the original sequence.

We investigate the asymptotic connection between $G(x)$ and $1 - C(x)$ for $x \to \infty$. Basically we focus attention on so-called regular behavior.

Definition. A measurable function $R: \mathbb{R}^+ \to \mathbb{R}^+$ *varies regularly at infinity* (r.v.) if there exists a $\rho \in \mathbb{R}$ such that for all $x \in \mathbb{R}^+$ $R(tx)/R(t) \to x^\rho$ as $t \to \infty$. The number ρ is called the *exponent* of regular variation. If $\rho = 0$, R is called *slowly varying* (s.v.). The family of r.v. functions at infinity with exponent ρ is denoted by RV_ρ.

Generalized renewal functions have been studied in a number of papers [11, 15, 23, 24]; for renewal functions we refer i.a. to [8, 17, 25]. For r.v. functions we refer to [3, 8, 21]. For r.v. sequences see [1, 26].

The next paragraph contains our main theorems. Proofs are given in § 3, 4. In § 5 we apply our results to random walk theory. Some concluding remarks are given in § 6.

§ 2. Main Theorems

We assume that $C(\infty) = 1$ or $G(\infty) = \infty$. Euler's constant is denoted by γ. If the mean of C is finite, we write $\mu = \int_0^\infty x\, dC(x)$; similarly if C has a finite second moment it is denoted by $\mu_2 = \int_0^\infty x^2\, dC(x)$.

Theorem 1. *Let* $\alpha \in [0, 1)$ *and* $L \in RV_0$. *Then the two statements are equivalent as* $x \to \infty$

(i) $1 - C(x) \sim x^{-\alpha} L(x)$,
(ii) $G(x) - \alpha \log x + \log L(x) \to \alpha\gamma - \log \Gamma(1-\alpha)$.

Theorem 2. *Let $L \in RV_0$. Then the two statements are equivalent as $x \to \infty$*

(i) $\int_0^x \{1-C(y)\}\,dy \sim L(x)$,
(ii) $G(x) - \log x + \log L(x) \to \gamma$.

Moreover C has finite mean μ iff

$$D \equiv \lim_{x\to\infty} \{\log x - G(x)\} < \infty$$

and then $\mu = \exp\{\gamma + D\}$.

Theorem 3. *Let C be non-lattice. Let $\alpha \in (1,2)$ and $L \in RV_0$. Then the two statements are equivalent as $x \to \infty$*

(i) $1 - C(x) \sim x^{-\alpha} L(x)$,
(ii) $G(x) - \log x + \log \mu - \gamma \sim \{\mu(\alpha-1)\}^{-1} x^{1-\alpha} L(x)$.

Theorem 4. *Let C be non-lattice. Let $L \in RV_0$ such that $L(x) \to L(\infty) \leqq \infty$. Then the two statements are equivalent as $x \to \infty$*

(i) $\int_0^x y^2\, dC(y) \sim L(x)$,
(ii) $x - \int_0^x y\, dG(y) \sim \frac{1}{2\mu} L(x)$.

For the next theorem we assume $\{c_n, n \in \mathbb{N}\}$ to be such that the greatest common divisor of those n for which $c_n > 0$ equals 1.

Theorem 5. *Let C be discrete as above. Let $\alpha > 1$ and $L \in RV_0$. Then the following two statements are equivalent as $n \to \infty$*

(i) $\sum_{k=n+1}^{\infty} c_k \sim n^{-\alpha} L(n)$,
(ii) $\frac{1}{n} - g_n \sim \frac{1}{\mu} n^{-\alpha} L(n)$.

Finally the analogue of Theorem 4.

Theorem 6. *Let C be discrete. Let $L \in RV_0$ such that $L(x) \to L(\infty) \leqq \infty$. Then the following two statements are equivalent as $n \to \infty$*

(i) $\sum_{k=1}^{n} k^2 c_k \sim L(n)$,
(ii) $n - \sum_{m=1}^{n} m g_m \sim \left(\frac{1}{2\mu} - \frac{1}{2L(\infty)}\right) L(n)$.

§3. Proof for the Non-Lattice Case

To simplify future references we list here the auxiliary functions with corresponding transforms when useful.

$$G_1(x)=\int_0^x y\,dG(y) \qquad g_1(s)=-g'(s), \tag{3}$$

$$\tilde{G}(x)=x-G_1(x) \qquad \tilde{g}(s)=\frac{1}{s}-g_1(s), \tag{4}$$

$$M(x)=\int_0^x \{1-C(y)\}\,dy \qquad m(s)=s^{-1}\{1-c(s)\}, \tag{5}$$

$$C_1(x)=\int_0^x y\,dC(y) \qquad c_1(s)=-c'(s), \tag{6}$$

$$C_2(x)=\int_0^x y^2\,dC(y) \qquad c_2(s)=c''(s), \tag{7}$$

$$H(x)=\sum_0^\infty C^{(n)}(x) \qquad h(s)=\{1-c(s)\}^{-1}, \tag{8}$$

$$\tilde{C}(x)=\frac{1}{\mu}\int_0^x \{1-C(y)\}\,dy \qquad \tilde{c}(s)=\mu^{-1}m(s), \tag{9}$$

$$V(x)=G(x)-\log x+\log\mu-\gamma. \tag{10}$$

§ 3.1.a. Preliminaries to the Proof of Theorems 1 and 2

The first lemma is interesting in its own right.

Lemma 3.1.1. (i) $\forall x\geqq 0,\ G_1(x)\leqq x$;

(ii) *there exists a constant* K *such that for all* $x\geqq 0,\ G(x)\leqq K+\log\{\max(1,x)\}$.

Proof. (i) Since $-g(s)=\log[1-c(s)]$, $g_1(s)=-e^{g(s)}c'(s)$. Hence

$$g_1(s)=\frac{c_1(s)}{1-c(s)}=c_1(s)\cdot h(s),$$

so that $G_1(x)=C_1\times H(x)=\int_0^x C_1(x-y)\,dH(y)$. On the other hand $s^{-1}=m(s)\,h(s)$ so that also $x=M\times H(x)$.

Hence

$$x-G_1(x)=M\times H(x)-C_1\times H(x)=(M-C_1)\times H(x).$$

Observe that

$$M(x)-C_1(x)=x\{1-C(x)\}$$

to conclude that $x-G_1(x)\geqq 0$.

(ii) First let $x\in[0,1]$. Then $G(x)\leqq G(1)$. Let $Y_1, Y_2, \ldots$ be i.i.d. with distribution C. By a result of Rosén [20] there exists a constant K_0 not depending on n for which

$$G(1)=\sum_1^\infty n^{-1} P\left\{\sum_{d=1}^n Y_d \in [0,1]\right\} \leqq K_o \sum_1^\infty n^{-3/2} < \infty.$$

Now $x>1$. From the definition of G_1 then

$$G(x)=x^{-1} G_1(x)-\lim_{x\downarrow 0} x^{-1} G_1(x)+\int_0^x y^{-2} G_1(y)\,dy.$$

But $G_1(x) \leqq x G(x)$. Hence by (i) above

$$G(x) \leqq 1+\int_0^1 y^{-2} G_1(y)\,dy+\int_1^x y^{-2} G_1(y)\,dy$$

or

$$G(x) \leqq 1+G(1)+\log x.$$

Indeed by Fubini's theorem

$$\int_0^1 y^{-2} G_1(y)\,dy=\int_0^1 (1-x)\,dG(x) \leqq G(1).$$

The result follows. □

The key to the proof of Theorem 1 lies in the following result of de Haan [4, 6, 10].

Lemma 3.1.2. *Let* $B\colon(0,\infty)\to(0,\infty)$ *be non-decreasing, right-continuous with* $B(0+)=0$. *Assume* $b(s)$ *finite for all* $s>0$. *Then for any* $c \geqq 0$ *and* $L \in RV_0$ *the following statements are equivalent*

(i) $\forall x>0\colon \lim_{t\to\infty} \{B(tx)-B(t)\}/L(t)=c \log x$

(ii) $\forall x>0\colon \lim_{t\downarrow 0} \{b(tx)-b(t)\}/L\left(\frac{1}{t}\right)=-c \log x.$

Both imply $\lim_{t\to\infty} \{B(t)-b(1/t)\}/L(t)=\gamma$.

§3.1.b. Proof of Theorems 1 and 2

Part 1: (i)⇒(ii). By Karamata's theorem in case either $0 \leqq \alpha<1$ or $\alpha=1$ (i) is equivalent to

$$1-c(s) \sim K_\alpha s^\alpha L(1/s) \quad \text{as } s\downarrow 0,$$

$$K_\alpha=\begin{cases}\Gamma(1-\alpha) & \text{if } 0 \leqq \alpha<1\\ 1 & \text{if } \alpha=1.\end{cases}$$

But then as $s\downarrow 0$

$$g(s)+\alpha \log s+\log L(1/s) \to -\log K_\alpha. \tag{11}$$

This implies for any $t>0$ as $s\downarrow 0$

$$g(st)-g(s)\to -\alpha\log t. \tag{12}$$

From Lemma 3.1.2. this implies that as $x\to\infty$

$$G(x)-g(1/x)\to\alpha\gamma.$$

Hence (ii) follows from the latter relation and (11).

Part 2: (ii)$\Rightarrow$(i). Obviously (ii) implies that for $t>0$ as $x\to\infty$

$$G(xt)-G(x)\to\alpha\log t.$$

By Lemma 3.1.2, as $s\downarrow 0$

$$G(1/s)-g(s)\to\alpha\gamma.$$

This implies (11) by (ii). But then by (12)

$$\frac{1-c(st)}{1-c(s)}=\exp-\{g(ts)-g(s)\}\to t^{\alpha}$$

or for some $L_1\in RV_0$, $1-c(s)=s^{\alpha}L_1(1/s)$. By the first part of the proof as $s\downarrow 0$

$$g(s)+\alpha\log s+\log L_1(1/s)\to -\log K_{\alpha}.$$

Compare this expression with (11) to see that $L_1(x)\sim L(x)$ for $x\to\infty$ so that (i) follows.

Part 3: Remaining statement in Theorem 2.

C has finite mean iff $L(x)\to\mu$ in (i). This is then equivalent to

$$G(x)-\log x\to\gamma-\log\mu. \quad \square$$

§3.2.a. Preliminaries to the Proofs of Theorems 3 and 4

We first rephrase statements (i) in Theorems 3 and 4 in alternative forms.

Lemma 3.2.1. *Let $L_i\in RV_0 (i=1,2,3)$ then for $x\to\infty$*

(i) *if $1<\alpha\leqq 2$ then* $1-C(x)\sim x^{-\alpha}L_1(x)\Leftrightarrow 1-\tilde{C}(s)\sim\dfrac{1}{\mu(\alpha-1)}x^{1-\alpha}L_1(x)$;

(ii) *if $1<\alpha\leqq 2$ then*

$$C_2(x)\sim\alpha x^{2-\alpha}L_2(x)\Leftrightarrow\int_0^x[1-\tilde{C}(y)]\,dy\sim\frac{1}{\mu(\alpha-1)}x^{2-\alpha}L_2(x);$$

(iii) *if $1<\alpha<2$ and $L_1(x)=(2-\alpha)L_2(x)$ then all four statements are equivalent;*

(iv) *if $\alpha=2$ then $C_2(x)\sim L_3(x)$ implies* $\dfrac{x\{1-\tilde{C}(x)\}}{\int_0^x\{1-\tilde{C}(y)\}\,dy}\to 0.$

Proof. Most statements are classical [8, 21]. For (ii) and (iv) we refer to Lemma 2.1 in Mohan [14]. □

Next we restate the conclusions (ii) in terms of the function

$$V(x)=G(x)-\log x+\log\mu-\gamma. \tag{13}$$

Lemma 3.3.2. *Let $\alpha\in(1,2)$ and let $L_4\in RV_0$. Then the two statements are equivalent as $x\to\infty$*

(i) $\tilde{G}(x)=x-G_1(x)\sim(\alpha-1)x^{2-\alpha}L_4(x)$;
(ii) $V(x)\sim(2-\alpha)x^{1-\alpha}L_4(x)$.

Proof. The basic connection between $\tilde{G}$ and V can easily be derived from the definitions of $\tilde{G}$, V and G_1: for all $x\geqq 0$

$$\int_0^x V(y)\,dy=\tilde{G}(x)+xV(x). \tag{14}$$

But then the implication (ii)⇒(i) is obvious.

To prove (i)⇒(ii) we first show that (i) implies that C has a finite mean. By Theorem 2 this then yields $V(x)\to 0$ as $x\to\infty$. Now (i) and the Abelian theorem for LST's proves the existence of an $L_5\in RV_0$ for which $\tilde{g}(s)=s^{\alpha-2}L_5(1/s)$. By (1) and (4) then $s^{-1}+g'(s)=s^{\alpha-2}L_5(1/s)$. By (1) for $s\downarrow 0$

$$1-\frac{s[1-c(s)]'}{1-c(s)}=s^{\alpha-1}L_5(1/s).$$

Since $\alpha>1$, the left side tends to zero and Lamperti's theorem [13] applies; there exists a $L_6\in RV_0$ for which $1-c(s)=sL_6(1/s)$. Substitute back in the above equation to see that for $x\geqq 1$

$$\frac{L_6'(x)}{L_6(x)}=x^{-\alpha}L_5(x)$$

or $L_6(x)=L_6(1)\exp\int_1^x y^{-\alpha}L_5(y)\,dy\to L_6(1)\exp\int_1^\infty y^{-\alpha}L_5(y)\,dy<\infty$ since $\alpha>1$. Hence $L_6(x)\to\lambda<\infty$ or $\dfrac{1-c(s)}{s}\to\lambda<\infty$ as $s\downarrow 0$. But then C has finite mean $\lambda=\mu$. We go back to (14). Equation (14) has the converse $(0<x<y)$

$$V(x)-V(y)=\int_x^y t^{-2}\tilde{G}(t)\,dt-x^{-1}\tilde{G}(x)+y^{-1}\tilde{G}(y).$$

Upon taking $y\to\infty$ we get

$$V(x)=\int_x^\infty t^{-2}\tilde{G}(t)\,dt-x^{-1}\tilde{G}(x).$$

But then (i)⇒(ii) follows. □

Lemma 3.2.3. *The following relations hold*

$$\tilde{G} \times \tilde{C}(x) = \int_0^x y \, d\tilde{C}(y), \tag{15}$$

$$2\int_0^x \{1-\tilde{C}(y)\}\, dy - x\{1-\tilde{C}(x)\} = \tilde{G}(x) + \int_0^x \{1-\tilde{C}(x-y)\}\, dG_1(y). \tag{16}$$

Proof. (15) follows from $\tilde{g}(s)\tilde{c}(s) = -\tilde{c}'(s)$. For by (1) and (3)

$$\mu s \tilde{c}(s) = 1 - c(s) = \exp - g(s).$$

Differentiate this relation to obtain

$$\tilde{c}'(s) = -\tilde{c}(s)\{s^{-1} + g'(s)\} = -\tilde{c}(s)\tilde{g}(s).$$

The other relation (16) follows from (15) by an integration by parts and obvious rearrangements. □

Our final preliminary result is used to estimate the last integral in (16). The result only slightly generalizes Theorem 2.1. of Mohan [14].

Lemma 3.2.4. *Assume* $Q(x) \geqq 0$, *bounded, non-increasing and such that* $\int_0^\infty Q(x)\, dx = \infty$.

Assume $R(x) \geqq 0$, *non-decreasing, non-lattice and such that*

$$\lim_{x\to\infty} \{R(x+1) - R(x)\} = A \in (0, \infty).$$

If for some $\delta \in [0, 1)$, $\int_0^x Q(y)\, dy \in RV_\delta$ *then as* $x \to \infty$

$$Q \times R(x) \sim A \int_0^x Q(y)\, dy.$$

Typically R is similar to a renewal function.

§ 3.2.b. Proof of Theorem 3

In view of Lemmas 3.2.1 and 3.2.2 it suffices to prove the equivalence of $(1 < \alpha < 2)$

$$\tilde{G}(x) \sim \{\mu(2-\alpha)\}^{-1} x^{2-\alpha} L(x) \quad \text{as } x \to \infty \tag{17}$$

and

$$C_2(x) \sim \alpha(2-\alpha)^{-1} x^{2-\alpha} L(x) \quad \text{as } x \to \infty. \tag{18}$$

Part 1: (17)⇒(18). Clearly (17) implies that as $s \downarrow 0$

$$\tilde{g}(s) \sim \frac{1}{\mu} \Gamma(2-\alpha) s^{\alpha-2} L(1/s).$$

As in the proof of (15), $\tilde{g}(s)\sim -\tilde{c}'(s)$ as $s\downarrow 0$ since $\tilde{c}(0)=1$.
Or by integration as $s\downarrow 0$

$$1-\tilde{c}(s)\sim\{\mu(\alpha-1)\}^{-1}\Gamma(2-\alpha)\,s^{\alpha-1}L(1/s).$$

This is equivalent via Karamata's Tauberian theorem to

$$1-\tilde{C}(x)\sim\{\mu(\alpha-1)\}^{-1}x^{1-\alpha}L(x).$$

Now apply (i) of Lemma 3.1.2.

Part 2: (18)⇒(17). For convenience put $1-\tilde{C}(x)=Q(x)$ and $\bar{Q}(x)=\int_0^x Q(y)\,dy$. By (16) we obtain

$$2\bar{Q}(x)-xQ(x)=\tilde{G}(x)+Q\times G_1(x).$$

(18) together with Lemma 3.2.1 yields $xQ(x)/\bar{Q}(x)\to 2-\alpha$. To estimate $Q\times G_1(x)$ we use Lemma 3.2.4 with $R=G_1$. As was noted in the proof of Lemma 3.1.1 (i), $G_1=C_1\times H$ so that

$$G_1(x+1)-G_1(x)=\int_0^x\{H(x+1-y)-H(x-y)\}\,dC_1(y)+\int_x^{x+1}H(x+1-y)\,dC_1(y).$$

By Blackwell's theorem the first integral tends to $\frac{1}{\mu}C_1(\infty)=1$. The second is bounded by $H(1)\{C_1(x+1)-C_1(x)\}\to 0$ as $x\to\infty$. We conclude that (18) implies for $x\to\infty$

$$\tilde{G}(x)/\bar{Q}(x)\to 2-(2-\alpha)-1=\alpha-1. \quad\square$$

§ 3.2.c. Proof of Theorem 4

Observe that (i) is of the form

$$C_2(x)\sim L(x) \tag{19}$$

while the expression for G is

$$\tilde{G}(x)=x-G_1(x)\sim\frac{1}{2\mu}L(x). \tag{20}$$

Part 1: (20)⇒(19). Clearly (20) implies that $\tilde{g}(s)\sim(2\mu)^{-1}L(1/s)$. From here on the proof is the same as in Part 1 of § 3.2.b.

Part 2: (19)⇒(20) for $L(x)\to\infty$.
With Q and $\bar{Q}$ as defined in § 3.2.b. Part 2 we obtain

$$\tilde{G}(x)/\bar{Q}(x)=2-\{xQ(x)/\bar{Q}(x)\}-\{Q\times G_1(x)/\bar{Q}(x)\}.$$

By (iv) of Lemma 3.2.1 $xQ(x)/\bar{Q}(x)\to 0$ as $x\to\infty$. To the remaining integral Lemma 3.2.4 is still applicable since $\bar{Q}(x)\sim\frac{1}{2\mu}L(x)\to\infty$ by assumption. Hence (20) follows.

Part 3: (19)⇒(20) for $L(x)\to\mu_2<\infty$.
Since Lemma 3.2.4 is not applicable we proceed in a slightly different fashion. Since $G_1=C_1\times H$ we write

$$G_1(x)-x=C_1\times H(x)-\frac{1}{\mu}x\,C_1(x)-x\{1-\mu^{-1}C_1(x)\}$$
$$=\int_0^x\left\{H(x-y)-\frac{x-y}{\mu}\right\}dC_1(x)-\frac{1}{\mu}C_2(x)-x\{1-\mu^{-1}C_1(x)\}.$$

An application of the renewal asymptote theorem yields that $H(x)-\frac{x}{\mu}\to\mu_2/(2\mu^2)$. Further $C_2(x)\to\mu_2$ by assumption while $C_1(x)\to\mu$; so $1-\mu^{-1}C_1(x)=o(x^{-1})$ since $\mu_2<\infty$. Combining terms we obtain by Lebesgue's theorem

$$G_1(x)-x\to\frac{\mu_2}{2\mu^2}\cdot\mu-\frac{\mu_2}{\mu}=-\frac{1}{2\mu}\mu_2. \quad\square$$

§4. Proofs for the Lattice Case

We start again with a list of auxiliary quantities. Whenever convenient we denote by $a(z)$ the generating function of a discrete sequence $\{a_n\}_0^\infty$, i.e. for $|z|<1$

$$a(z)=\sum_{n=0}^\infty a_n z^n.$$

$$g_n=\sum_{m=1}^\infty\frac{1}{m}c_n^{(m)};\qquad g(z)=-\log\{1-c(z)\},\tag{21}$$

$$h_n=\sum_{m=0}^\infty c_n^{(m)};\qquad h(z)=\{1-c(z)\}^{-1},\tag{23}$$

$$q_n=\sum_{m=n+1}^\infty c_m;\qquad q(z)=\{1-c(z)\}/(1-z).\tag{23}$$

§ 4.1.a. Preliminaries to the Proof of Theorem 5

We start with the following result

Lemma 4.1. *Suppose μ the mean of $\{c_n\}_0^\infty$ is finite. For any $\alpha>1$ and any $L\in RV_0$ the following two statements are equivalent as $n\to\infty$*

(i) $q_n \sim n^{-\alpha} L(n)$,

(ii) $n\{h_n - h_{n+1}\} \Big/ \left\{h_n - \frac{1}{\mu}\right\} \to \alpha - 1$.

They both imply as $n \to \infty$

(iii) $h_n - \frac{1}{\mu} \sim \{\mu^2(\alpha-1)\}^{-1} n q_n$.

Proof. Since $\mu = \sum_{m=1}^{\infty} m c_m$ is finite, $q(z) \to \mu > 0$ as $z \uparrow 1$. Now for $|z| \leqq 1$, $z \neq 1$, $1 - c(z) \neq 0$ and $q(1) = \mu \neq 0$; by a theorem of Wiener [2, p. 258] for $|z| \leqq 1$

$$\frac{1}{q(z)} = \sum_{n=0}^{\infty} \lambda_n z^n \quad \text{with} \quad \sum_{n=0}^{\infty} |\lambda_n| < \infty. \tag{24}$$

Since $h(z)\, q(z) = (1-z)^{-1}$ we also have

$$h_n = \sum_{k=0}^{n} \lambda_k \tag{25}$$

and

$$h_n - \frac{1}{\mu} = -\sum_{k=n+1}^{\infty} \lambda_k. \tag{26}$$

Part 1: (i)⇒(ii); (i)⇒(iii)

From (i), (24) and [2, p. 258] it follows that for $n \to \infty$

$$\lambda_n \sim -\mu^{-2} q_n \sim -\mu^{-2} n^{-\alpha} L(n).$$

Using this, (ii) and (iii) follow from (25) and (26).

Part 2: (ii)⇒(i).

Since

$$\frac{1}{q(z)} = (1-z)\, h(z) = 1 + \sum_{n=1}^{\infty} (h_n - h_{n-1})\, z^n$$

another application of [2, p. 258] yields for $n \to \infty$

$$q_n \sim -\mu^2 (h_n - h_{n-1})$$

or, using (ii) we get as $n \to \infty$

$$q_n \sim \mu^2(\alpha - 1) \left\{h_n - \frac{1}{\mu}\right\}.$$

Since (ii) implies regular variation of $h_n - \frac{1}{\mu}$ [1], we get (i). □

§ 4.1.b. Proof of Theorem 5

Part 1: (i)⇒(ii)

Relations (21) and (22) yield $g'(z) = h(z)\, c'(z)$ for $|z| < 1$. Equating coefficients we obtain

$$n g_n = \sum_{m=1}^{n} m c_m h_{n-m}.$$

But then

$$1 - n g_n = \left\{1 - \frac{1}{\mu} \sum_{m=1}^{n} m c_m\right\} - \left(h_n - \frac{1}{\mu}\right) \sum_{m=1}^{n} m c_m + \sum_{m=1}^{n} m c_m \{h_{n-m} - h_n\}$$

$$\equiv I_n - J_n + K_n$$

(a) $I_n = \frac{1}{\mu} \sum_{m=n+1}^{\infty} m c_m = \frac{1}{\mu} \left(n q_n + \sum_{k=n+1}^{\infty} q_k\right).$

By assumption then $I_n \sim \frac{1}{\mu} \left(n q_n + \frac{1}{\alpha - 1} n q_n\right) \sim \frac{\alpha}{\mu(\alpha-1)} n q_n.$

(b) $J_n \sim \mu \left(h_n \sim \frac{1}{\mu}\right) \sim \frac{1}{\mu(\alpha-1)} n q_n$ by Lemma 4.1.

(c) It remains to show that $K_n = o(n q_n)$. To accomplish this let $0 < u < v < 1$ and write

$A = \{1, 2, \ldots, n - [n v]\}, \quad B = \{n - [n v] + 1, \ldots, n - [n u]\},$
$C = \{n - [n u] + 1, \ldots, n\}$

We apply Lemma 4.1. (ii) and (iii). Take n so large that for $m \in A \cup B$ and any given $\varepsilon \in (0, 1)$

$$h_{n-m} - h_n = \sum_{k=n-m}^{n-1} \{h_k - h_{k+1}\} \leqq (\alpha - 1 + \varepsilon) \sum_{k=n-m}^{n-1} k^{-1} \left\{h_k - \frac{1}{\mu}\right\} \leqq C_1 \sum_{k=n-m}^{n-1} q_k$$

where $C_1 > (1+\varepsilon)\{\mu^2(\alpha-1)\}^{-1}(\alpha - 1 + \varepsilon)$.

Now $K_n = (\sum_A + \sum_B + \sum_C) m c_m (h_{n-m} - h_n) \equiv K_1 + K_2 + K_3$. Then

$$K_1 \leqq C_1 \sum_{m \in A} m c_m \sum_{k=n-m}^{n-1} q_k \leqq C_1 \sum_{m \in A} m c_m \sum_{k=[nv]}^{n-1} q_k.$$

Since for large n, $K_1 \geqq 0$ and by assumption (i)

$$\sum_{k=[nv]}^{n-1} q_k / n q_n \to (\alpha - 1)^{-1} \{v^{1-\alpha} - 1\},$$

$$0 \leqq \limsup_{n \to \infty} K_1 / (n q_n) \leqq C_2 \{v^{1-\alpha} - 1\}$$

for a constant C_2.

Similarly by the monotonicity of $\{q_k\}$

$$0 \leqq K_2 \leqq C_1 \sum_{m \in B} m c_m \sum_{k=n-m}^{n-1} q_k \leqq C_1 \sum_{m \in B} m c_m q_{n-m} \cdot m$$

$$\leqq C_1 \{n - [n u]\} q_{[nu]} \sum_{m \in B} m c_m.$$

We divide by $n q_n$, apply (i) and note that as $n\to\infty$, $\sum_{m\in B} m c_m \to 0$ since $\mu<\infty$. Hence $K_2=o(n q_n)$.

For K_3, put $M=\sup_k h_k$; then $K_3 \leqq 2M \sum_{m\in C} m c_m$. By an argument similar to (a) and K_1 we find

$$0\leqq \limsup_{n\to\infty} K_3/(n q_n) \leqq C_3\{(1-u)^{1-\alpha}-1\}$$

for a constant C_3. Hence

$$0\leqq \limsup_{n\to\infty} K_n/(n q_n) \leqq C_2\{v^{1-\alpha}-1\}+C_3\{(1-u)^{1-\alpha}-1\}.$$

Let $u\downarrow 0$ and $v\uparrow 1$. This proves part 1.

Part 2: (ii)⇒(i). From (21) we notice that

$$q(z)=\exp \sum_{n=1}^{\infty}\left(\frac{1}{n}-g_n\right) z^n$$

to which [2, p. 258] can again be applied. We find by (ii) that $q_n \sim \left(\frac{1}{n}-g_n\right)\cdot q(1)$ which yields (i). □

The following auxiliary functions will be used together with those introduced in (21)–(23).

$$\begin{aligned}
c_{1,n}&=\mu^{-1} q_n & c_1(z)&=\{1-c(z)\}/\mu(1-z),\\
c_{2,n}&=\sum_{m=n+1}^{\infty} c_{1,m} & c_2(z)&=\{1-c_1(z)\}/(1-z).\\
g_{1,n}&=\sum_{m=1}^{n} m g_m & g_1(z)&=z g'(z)/(1-z), && (27)\\
w_n&=\sum_{m=0}^{n} m c_m & w(z)&=z c'(z)/(1-z), && (28)\\
p_n&=\sum_{m=0}^{n} w_{n-m}\left(h_m-\frac{1}{\mu}\right) & p(z)&=w(z)\left\{h(z)-\frac{1}{\mu(1-z)}\right\}.
\end{aligned}$$

§ 4.2.a. Preliminaries to the Proof of Theorem 6

The next lemma brings together some elementary facts about the above defined sequences.

Lemma 4.2. *Let $L\in RV_0$, $L(n)\to L(\infty)\leqq\infty$. Then $\sum_{k=1}^{n} k^2 c_k \sim L(n)$ implies any of the following statements:*

(i) $p_n \sim \mu \sum_{m=0}^{n}\left(h_m-\frac{1}{\mu}\right)$;

(ii) $\sum_{k=0}^{n-1} \{1-\mu^{-1} w_k\} \sim \mu^{-1} L(n)$;

(iii) $\sum_{k=0}^{n} c_{2,k} \sim \left\{\frac{1}{2\mu} - \frac{1}{2L(\infty)}\right\} L(n)$;

(iv) $\sum_{k=0}^{n} \left(h_k - \frac{1}{\mu}\right) \sim \left\{\frac{1}{2\mu^2} - \frac{1}{2\mu L(\infty)}\right\} L(n)$.

Proof. The proofs of (i), (ii) and (iii) are easy; they follow from known facts about regularly varying sequences. See for example [1, 26]. To obtain (iv) first note that

$$\sum_{k=0}^{n} \left(h_k - \frac{1}{\mu}\right) = \sum_{m=0}^{n} h_{n-m} c_{2,m}$$

as follows from the generating functions. But then standard techniques using (iii) yield the result. □

Lemma 3.1.1. (i) has an analogue for the lattice case.

Lemma 4.3. *For all $n \geq 1$, $g_{1,n} \leq n$.*

Proof. Using generating functions again it easily follows that

$$n - g_{1,n} = \sum_{k=1}^{n} h_{n-k} \left\{\mu \sum_{m=0}^{k-1} c_{1,m} - \sum_{m=0}^{k} m c_m\right\}.$$

A summation by parts shows that the term inside the brackets is non-negative. □

§ 4.2.b. Proof of Theorem 6

Part 1: (i)⇒(ii).

From (27) and (28) one derives that

$$g_{1,n} = \sum_{m=0}^{n} w_{n-m} h_m$$

and hence

$$n - g_{1,n} = \sum_{k=0}^{n-1} \left(1 - \frac{1}{\mu} w_k\right) - \frac{1}{\mu} w_n - p_n.$$

Now apply Lemma 4.2. (ii), (i) and (iv).

Part 2: (ii)⇒(i). Define

$$t_n \equiv \left(\frac{1}{2\mu} - \frac{1}{2L(\infty)}\right) L(n) \sim n - g_{1,n} \equiv g_{2,n}$$

then (ii) implies by Lemma 4.3 and [8, p. 447] as $z \uparrow 1$

$$g_2(z) \sim (1-z)^{-1} t\{(1-z)^{-1}\}.$$

However

$$(1-z)\,g_2(z)\,c_1(z)=z\,c_1'(z)$$

by easy algebra. Hence $c_1'(z)\sim t\{(1-z)^{-1}\}$ as $z\uparrow 1$. By [8, p. 447]

$$\sum_{m=0}^{n} m\,c_{1,m}\sim t_n,$$

and

$$n^2\,c_{1,n}=o(t_n)$$

as $n\to\infty$.

However using the definition of $\{c_{1,n}\}$ we can derive

$$\sum_{m=0}^{n+1} m^2\,c_m=2\mu\sum_{m=1}^{n} m\,c_{1,m}+w_{n+1}-\mu\,n(n+1)\,c_{1,n+1}.$$

Since $w_{n+1}\to\mu$, $\sum_{m=0}^{n+1} m^2\,c_m\sim 2\mu\,t_n+\mu\sim L(n)$, which was to be proved. □

§5. Applications to Random Walk Theory

Let $X_1, X_2, \ldots$ be independent with common distribution F. Let $S_0=0$ a.s. and for $n\geqq 1$, $S_n=X_1+\ldots+X_n$. Define the *first (upgoing) ladder index* of the random walk $\{S_n\}_0^\infty$ by

$$N=\inf\{n\colon S_n>0\}.$$

Define the *(first ascending) ladder height* by S_N. We recall Spitzer-Baxter identities

$$1-E[z^N]=\exp-\sum_{n=1}^{\infty}\frac{z^n}{n}P[S_n>0],$$

$$1-E[e^{-\lambda S_N}]=\exp-\int_{0+}^{\infty} e^{-\lambda x}\,d\left\{\sum_{n=1}^{\infty}\frac{1}{n}P[0<S_n\leqq x]\right\}.$$

§5.1. *Ladder Index*

Choose $C(x)=P\{N\leqq x\}$, then we can identify the discrete harmonic renewal measure of C with

$$g_n=n^{-1}P(S_n>0)$$

so that $1-n\,g_n=P(S_n\leqq 0)$. We obtain the following information partly available in i.a. [17, 25].

Theorem 5.1. (i) *Let* $0 \leqq \beta \leqq 1$. *Then* $\sum_{m=1}^{n} P(N>m) \sim n^{1-\beta} L(n)$ *where* L *is a s.v. sequence* iff *Spitzer's condition holds, i.e.* $\lim \frac{1}{n} \sum_{m=1}^{n} P(S_m>0)=\beta$;

(ii) *N has finite mean* iff $\sum_{1}^{\infty} m^{-1} P[S_m \leqq 0]<\infty$. *Then*

$$\mu=EN=\exp \sum_{1}^{\infty} \frac{1}{m} P[S_m \leqq 0].$$

(iii) $1<\beta$: $P[N>n] \sim n^{-\beta} L(n) \Leftrightarrow P(S_m \leqq 0) \sim \frac{1}{\mu} n^{1-\beta} L(n)$;

(iv) EN^2 *is finite* iff $\sum_{1}^{\infty} P(S_n \leqq 0)<\infty$ *and then*

$$EN^2=\mu\left\{1+2 \sum_{n=1}^{\infty} P(S_n \leqq 0)\right\}.$$

Example 1. If for some normalizing constants $B_n \to \infty$, $S_n/B_n \xrightarrow{\mathscr{D}} Y_\alpha$ where Y_α is stable, then $\beta=P[Y_\alpha>0]$. If we use the canonical representation then if $\alpha \neq 1$ [8]

$$\log E[e^{itY_\alpha}]=-\Delta|t|^\alpha \exp -i \frac{t}{|t|} \frac{\pi \delta}{2}$$

where $\Delta>0$, $|\delta| \leqq 1-|1-\alpha|$, $0<\alpha \leqq 2$, $\alpha \neq 1$. It is easy to show that

$$\beta=P[Y_\alpha>0]=\frac{1}{2}+\frac{\delta}{2\alpha}.$$

For $\alpha=1$ see Emery [7] or Doney [5].

§ 5.2. Ladder Height

By identifying $C(x)=P(S_N \leqq x)$ and $G(x)=\sum_{1}^{\infty} \frac{1}{n} P\{0<S_n \leqq x\}$ we get another illustration of our theorems. The role played by Spitzer's condition for N is played for S_N by the condition that for all $t>0$

$$\lim_{x \to \infty} \sum_{n=1}^{\infty} \frac{1}{n} P\{x<S_n \leqq x t\}=\beta \log t \tag{39}$$

which we call *Sinai's condition* in view of [22].

Theorem 5.2. (i) *Let* $0 \leqq \beta \leqq 1$. *Then* $\int_0^x P(S_N>u)\, du \sim x^{1-\beta} L(x)$ *where* L *is s.v.* iff *Sinai's conditon holds.*

(ii) S_N *has finite mean* iff $D = \lim_{x\to\infty}\left\{\log x - \sum_1^\infty \frac{1}{n} P[0 < S_n \leqq x]\right\}$ *is finite. Then*

$$E[S_N] = \exp\{\gamma + D\}.$$

It would be interesting to compare this result with the approach of Lai [12], especially for higher moments of S_N. An alternative form for $G(x)$ however seems necessary.

Example 2. In [18, 22] it is proved that if $X_i \stackrel{\mathscr{D}}{=} Y_\alpha$ for all i then Sinai's condition holds. We provide an easier proof using Lemma 3.1.2 and an approach of Heyde [11]. Put for a fixed $t > 0$

$$G_t(x) = G(t\,x) - G(x).$$

Then since $S_n \stackrel{\mathscr{D}}{=} n^{1/\alpha} Y_\alpha$

$$s\int_0^\infty e^{-sx} G_t(x)\,dx = \int_0^\infty f_s(y)\,P[y < Y_\alpha \leqq y\,t]\,dy$$

where

$$f_s(y) = s\sum_{n=1}^\infty n^{\frac{1-\alpha}{\alpha}} \exp[-syn^{1/\alpha}].$$

If we write

$$H(v) = \sum_{n\in[1,\,v^\alpha]\cap\mathbb{N}} n^{\frac{1-\alpha}{\alpha}}$$

then as $v\to\infty$, $H(v)\to\infty$ and $H(v) \sim \alpha(v-1)$ as $v\to\infty$.

Moreover

$$f_s(y) = s\int_{0+}^\infty e^{-syv}\,dH(v).$$

By an abelian theorem $f_s(y) \sim \frac{\alpha}{y}$ as $s\downarrow 0$. Moreover

$$|f_s(y)| = s\int_{0+}^\infty e^{-t} H\left(\frac{t}{sy}\right) dt \leqq K s \int_0^\infty e^{-t}\left(\frac{t}{sy}\right) dt \leqq K y^{-1}$$

for some constant K. We can apply Lebesgue's theorem to obtain that

$$\lim_{s\downarrow 0} s\int_0^\infty e^{-sx} G_t(x)\,dx = \alpha\int_0^\infty y^{-1} P[y < Y_\alpha \leqq y\,t]\,dy.$$

The latter integral equals $\alpha P[Y_\alpha > 0]\log t$.

Karamata's theorem yields for $x\to\infty$

$$\int_0^x G_t(y)\,dy \sim x\,\alpha P[Y_\alpha > 0]\log t.$$

From this relation one derives as in [11] that

$$G_t(x) \to \alpha P[Y_\alpha > 0] \log t$$

so that Sinai's condition holds for $\beta = \alpha P[Y_\alpha > 0]$.

Example 3. Assume that the random walk is generated by X for which $E|X| < \infty$ and $EX = \mu > 0$. Then [17, p. 207] $\sum_1^\infty n^{-1} P[S_n \leqq 0] < \infty$ so that $EN < \infty$. By Wald's identity then $E[S_N] = \mu \cdot EN$.

By Theorem 5.2 (ii)

$$\lim_{x \to \infty} \{\log x - G(x)\} = \log ES_N - \gamma.$$

Hence

$$\begin{aligned}\sum_1^\infty \frac{1}{n} P[S_n \leqq x] - \log x &= \sum_1^\infty \frac{1}{n} P[S_n \leqq 0] + G(x) - \log x \\ &= \log EN + G(x) - \log x \\ &\to \log EN + \gamma - \log ES_N \\ &= \gamma - \log \mu.\end{aligned}$$

This sharpens somewhat a result of Heyde [11] where it was shown that

$$\sum_1^\infty \frac{1}{n} P[S_n \leqq x] \sim \log x.$$

The latter result should be combined with the consequences of Lemma 3.1.1.

§ 6. Concluding Remarks

§ 6.1. Lemma 4.1. links the behaviour of a renewal sequence to the tail behaviour of the underlying generating discrete distribution. The complete result will be published separately [16].

§ 6.2. Instead of assuming slowly varying functions in Theorems 2 and 4 we could have restricted ourselves to functions belonging to the subclass Π of slowly varying functions introduced in [3] by de Haan and essentially used in Lemma 3.1.3. Results of this type will be discussed in a forthcoming paper.

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