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Ergodic Theorems for Subadditive Superstationary Families of Convex Compact Random Sets

Klaus Schürger

Department of Economics, University of Bonn, Adenauerallee 24–42, D-5300 Bonn
Federal Republic of Germany

1. Introduction

Recently, several authors have shown that it is possible to derive random set analogues of some of the classical convergence theorems in probability theory. The first paper in this direction is apparently due to Artstein and Vitale [5] who proved a strong law of large numbers for independent and identically distributed random compact subsets of a Euclidean space. This result has been reproved by Cressie [7] for certain special types of random sets (e.g. those having an atmost countable number of realizations) by utilizing a different approach which yields explicit expressions for the limit sets. A random set analogue of Kolmogorov's three series theorem is proved in Lyašenko [20]. An extension of the results in [5] is due to Hess [12] who studied e.g. stationary sequences of random compact subsets of certain infinite-dimensional spaces. In [4] it is shown that the strong law of large numbers of [5] remains true if the random sets are merely supposed to be closed (but not necessarily bounded). Finally, central limit theorems for random sets were derived in [8] and [20].

One might hope to derive pointwise ergodic theorems for certain families of random sets satisfying rather weak conditions by looking at generalizations of Birkhoff's pointwise ergodic theorem. A complete generalization of that theorem has been developed by Kingman [16] (see also [17], [18]) who considered stationary subadditive stochastic processes. An extension of Kingman's results to the Banach valued case has been achieved by Ghoussoub and Steele [10]. Another striking generalization of Birkhoff's pointwise ergodic theorem is due to Krengel [19] who derived a pointwise ergodic theorem for superstationary processes. Finally, Abid [1] arrived at a pointwise ergodic theorem generalizing the ergodic theorems of Kingman and Krengel. Based on Abid's results, we derive pointwise as well as mean ergodic theorems for certain families of random convex compact subsets of a Euclidean space, which are subadditive and superstationary (see Sect. 3 for definitions).

Section 4 contains the desired pointwise and mean ergodic theorems (see Theorems (4.1), (4.16), (4.32) and (4.35)). Theorems (4.16) and (4.35) give con-

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ditions which ensure that the limits in our ergodic theorems are constant. Theorem (4.1) extends corresponding results of [5] and [12] in that it shows that a pointwise ergodic theorem holds for families of random sets which are merely supposed to be subadditive and superstationary (see Sect. 3 for definitions). On the other hand, it turns out that, in general, Theorem (4.1) does not hold for random compact sets which are not convex (see Sect. 5). We should like to point out that there is also a connection between Theorem (4.1) and a subadditive ergodic theorem in the Banach valued case (see [10]). It is easy to see that, in general, the pointwise ergodic theorem of [10] is not valid in the $C(K)$-valued case ($C(K)$ denoting the Banach space of all continuous real-valued functions defined on a compact set K, the norm being the supremum norm) provided K is not "trivial". Theorem (4.1) implies, however, that the pointwise ergodic theorem of [10] does hold even for *superstationary* $C(S_1)$-valued processes (S_1 denoting the unit sphere in R^d) provided the values of the random variables involved are support functions of nonvoid convex compact sets (compare, however, Remark (4.10) as well as the remark following (2.4)). Section 2 collects some basic concepts (Hausdorff metric, support functions, random sets) and results which are used in subsequent sections.

2. The Space $(\mathscr{C}, \rho)$; Random Sets

Let $\mathscr{C}$ denote the family of all nonvoid compact subsets of R^d ($d \geqq 1$), and let $\operatorname{co}\mathscr{C}$ be the family of all convex sets in $\mathscr{C}$. On $\mathscr{C}$, the *Hausdorff metric* ρ is defined by

$$(2.1) \qquad \rho(C,D) = \inf\{\varepsilon \geqq 0 \colon C \subset D + \varepsilon B_1, D \subset C + \varepsilon B_1\}, \qquad C, D \in \mathscr{C},$$

where B_1 is the closed unit ball in R^d with respect to the Euclidean norm $\|\cdot\|$. Furthermore, we put $\alpha A = \{\alpha a \colon a \in A\}$, $\alpha \in R$, $A \subset R^d$, and

$$A_1 + \ldots + A_n = \{a_1 + \ldots + a_n \colon a_i \in A_i, \ 1 \leqq i \leqq n\}, \qquad A_i \subset R^d, \ 1 \leqq i \leqq n.$$

One can show that $(\mathscr{C}, \rho)$ is a Polish space (see [6] or [21]). It is well-known that $\operatorname{co}\mathscr{C}$ is a closed subset of $\mathscr{C}$. For any $C \in \mathscr{C}$ put

$$(2.2) \qquad \|C\| = \sup\{\|c\| \colon c \in C\}$$

(there will be no danger to confuse this with the Euclidean norm). It can be easily seen that, for all $C \in \mathscr{C}$, $\|C\| = \rho(C, \{0\})$. Furthermore, $|\|C\| - \|D\|| \leqq \rho(C, D)$, C, $D \in \mathscr{C}$, which shows that the mapping $C \mapsto \|C\|$ from $\mathscr{C}$ into R is continuous. Let the *support function* $s(\cdot, C)$ of a set $C \in \mathscr{C}$ be defined by

$$(2.3) \qquad s(p, C) = \sup\{pc \colon c \in C\}, \qquad p \in S_1$$

(S_1 denoting the unit sphere $\{x \in R^d \colon \|x\| = 1\}$). It is easily seen that $s(\cdot, C)$ is continuous on S_1, and that

$$(2.4) \qquad \sup_{p \in S_1} |s(p, C) - s(p, D)| = \rho(C, D), \qquad C, D \in \operatorname{co}\mathscr{C}$$

(see [6]). It follows from (2.4) that the map $C \mapsto s(\cdot, C)$ defines an isometric embedding of the space $(\text{co}\,\mathscr{C}, \rho)$ into the Banach space $C(S_1)$ of continuous functions $f\colon S_1 \to R$, the norm of f being the supremum norm. Note that (2.4) also implies that, for each $p \in S_1$, the map $\sigma_p\colon \mathscr{C} \to R$, defined by

$$(2.5) \qquad \sigma_p C = s(p, C) = s(p, \text{co}\, C)$$

($\text{co}\, C$ denoting the convex hull of C) is continuous.

In [20], it has been shown that

$$(2.6) \qquad |s(p_1, C) - s(p_2, C)| \leqq \sqrt{2}\, \|C\|\, \|p_1 - p_2\|, \qquad C \in \text{co}\,\mathscr{C},\ p_1, p_2 \in S_1.$$

It follows from (2.6), (2.4) and the proof of Theorem 11 of [20] that the map $C \mapsto s(\cdot, C)$ defines an embedding of the space $(\text{co}\,\mathscr{C}, \rho)$ into $L^1(S_1)$ (the measure being the uniform distribution on S_1).

In this paper, we are interested in random sets taking their values in $\mathscr{C}$. More precisely, let $(\Omega, \mathscr{A}, P)$ be a probability space. A *random set* Y is a mapping $Y\colon \Omega \to \mathscr{C}$ such that Y is measurable with respect to the σ-algebra $\mathscr{B}(\mathscr{C})$ generated by the open sets in $(\mathscr{C}, \rho)$. It can be shown (see [9]) that $Y\colon \Omega \to \mathscr{C}$ is a random set iff $\{\omega \in \Omega\colon Y(\omega) \cap C \neq \emptyset\} \in \mathscr{A}$, $C \in \mathscr{C}$. If $Y, Y_1, \ldots, Y_n$ are random sets, so are αY $(\alpha \in R)$ and $Y_1 + \ldots + Y_n$, while $\|Y\|$ and $\sigma_p Y$ $(p \in S_1)$ are (real) random variables.

3. Subadditive and Superstationary Families of Random Sets

In this section, we introduce the concepts of subadditivity and superstationarity for certain families $X = (X_{s,t})$ of $\mathscr{C}$-valued random sets defined on a common probability space $(\Omega, \mathscr{A}, P)$. Throughout, the index set of families like $(X_{s,t})$ equals $I = \{(s,t) \in N_0 \times N_0\colon s < t\}$, where $N_0 = \{0, 1, 2, \ldots\}$.

(3.1) *Definition.* A family $(X_{s,t})$ of $\mathscr{C}$-valued random sets is called *subadditive* if

$$(3.2) \qquad X_{s,t} \subset X_{s,u} + X_{u,t} \qquad \text{whenever } (s,u), (u,t) \in I.$$

In order to introduce the concept of superstationarity for families $(X_{s,t})$ of $\mathscr{C}$-valued random sets, we first remark that the set-theoretical inclusion "$\subset$" is a closed partial order relation on the Polish space $\mathscr{C}$. Hence, $(\mathscr{C}, \subset)$ is a *partially ordered Polish space* (p.o. Polish space) in the sense of Kamae, Krengel and O'Brien [14] (see also Kamae and Krengel [13]). Consider the p.o. Polish space $\mathscr{D} = (\mathscr{C}^N)^N$ $(N = \{1, 2, \ldots\})$ being the product of the p.o. Polish spaces $(\mathscr{C}, \subset)$, endowed with the product topology and the coordinate-wise partial ordering which will be denoted by "$\lesssim$". If $C \in \mathscr{D}$, then $C = (C_1, C_2, \ldots)$ where $C_i = (C_{i1}, C_{i2}, \ldots) \in \mathscr{C}^N$, $i \in N$. Define the projections $\Pi_{ij}\colon \mathscr{D} \to \mathscr{C}$ by $\Pi_{ij}(C) = C_{ij}$, $i, j \in N$, $C \in \mathscr{D}$. For $C, D \in \mathscr{D}$ we then have $C \lesssim D$ iff $\Pi_{ij}(C) \subset \Pi_{ij}(D)$, $i, j \in N$. We shall conceive a family $X = (X_{s,t})$ of $\mathscr{C}$-valued random sets as a random element of $\mathscr{D}$, where $\Pi_{ij}(X) = X_{j-1, i+j-1}$, $i, j \in N$. Let the *shift* $T\colon \mathscr{D} \to \mathscr{D}$ be given by $\Pi_{ij}(T(C)) = C_{i, j+1}$, $i, j \in N$, $C \in \mathscr{D}$.

Let $\mathscr{M}(\mathscr{D})$ denote the family of probability measures defined on the family $\mathscr{B}(\mathscr{D})$ of Borelian subsets of $\mathscr{D}$. A probability measure $P_1 \in \mathscr{M}(\mathscr{D})$ is called *stochastically smaller* than $P_2 \in \mathcal{M}(\mathcal{D})$ (notation: $P_1 \prec P_2$) if $\int_{\mathcal{D}} f\,dP_1 \leqq \int_{\mathcal{D}} f\,dP_2$ for all bounded measurable functions $f: \mathcal{D} \to R$ which are *increasing*, i.e., for which $C \lesssim D$ implies $f(C) \leqq f(D)$, $C, D \in \mathcal{D}$ (see [14] as well as [13]). Kamae and Krengel [13] have shown that the relation "$\prec$" on $\mathcal{M}(\mathcal{D})$ is a closed partial order relation with respect to the topology of weak convergence.

(3.3) *Definition.* Let $X = (X_{s,t})$ be a family of $\mathscr{C}$-valued random sets defined on a common probability space $(\Omega, \mathcal{A}, P)$. Let Q_i denote the probability distribution of $T^i X$, i.e., $Q_i(A) = P\{T^i X \in A\}$, $A \in \mathcal{B}(\mathcal{D})$, $i \in N_0$. The family X is called *superstationary* if $Q_1 \prec Q_0$. (Note that $Q_1 \prec Q_0$ entails $Q_{i+1} \prec Q_i$, $i \in N$.)

4. Ergodic Theorems

In this section, we consider subadditive superstationary families $X = (X_{s,t})$ of co$\mathscr{C}$-valued random sets. We first show that under additional assumptions on X, a pointwise ergodic theorem holds for X.

(4.1) **Theorem.** *Let $X = (X_{s,t})$ be a subadditive superstationary family of co$\mathscr{C}$-valued random sets defined on a common probability space $(\Omega, \mathcal{A}, P)$. Assume that there exists a constant $\tilde{K} > 0$ such that*

$$E(\|X_{s,s+1}\|) \leqq \tilde{K}, \qquad s \in N_0. \tag{4.2}$$

Then $\lim_{t\to\infty} \frac{1}{t} X_{0,t}$ exists a.e. in $(\mathrm{co}\,\mathscr{C}, \rho)$.

In [5], the proof of a strong law of large numbers for independent and identically distributed co$\mathscr{C}$-valued random sets was based on a strong law of large numbers for Banach valued random variables (due to Mourier [22], [23]) by embedding the space $(\mathrm{co}\,\mathscr{C}, \rho)$ into $C(S_1)$ (compare the remark following (2.4)). Theorem (4.1) cannot be proved in this way since, in general, a pointwise ergodic theorem is not available for superstationary subadditive families of $C(S_1)$-valued random variables (see [10]). A proof of Theorem (4.1) can, however, be based on Abid's pointwise ergodic theorem (see [1]) together with the following simple result.

(4.3) **Lemma.** *Let $(C_n) \subset \mathrm{co}\,\mathscr{C}$ and $(p_n) \subset S_1$ be sequences such that (p_n) is dense in S_1 and, for all $k \in N$, $\lim_{n\to\infty} \sigma_{p_k}(C_n)$ exists and is finite. Then $\lim_{n\to\infty} C_n$ exists in* $(\mathrm{co}\,\mathscr{C}, \rho)$.

Proof. First note that the assumptions imply that $\sup_{n\in N} \|C_n\| < \infty$. In fact, if $0 < \varepsilon < \frac{1}{d}$ and if, for $i = 1, \ldots, d$, vectors $q_i, r_i \in (p_n)$ are chosen in such a way that $\|q_i - e_i\| \leqq \varepsilon$ and $\|r_i + e_i\| \leqq \varepsilon$ (e_i denoting the i-th unit vector in R^d), we have for all $C \in \mathrm{co}\,\mathscr{C}$ and $m \in N$

$$\|C\|^m \leqq \frac{d^{m-1}}{1 - \varepsilon d} \sum_{i=1}^{d} \max(|\sigma_{q_i}(C)|^m, |\sigma_{r_i}(C)|^m) \tag{4.4}$$

($\|C\|$ given by (2.2)). Now apply Blaschke's selection theorem (see, e.g., [27]).

Proof of Theorem (4.1). First note that the assumptions of Theorem (4.1) imply

(4.5) $$P\{\sigma_p(X_{s,t})>u\}\geqq P\{\sigma_p(X_{s+1,t+1})>u\}, \quad p\in S_1,\ u\in R,\ (s,t)\in I.$$

Utilizing (4.5) and (4.4) it is not difficult to verify that Condition (4.2) can (equivalently) be replaced by the following conditions:

(4.6) $$E(\sigma_p(X_{0,t}))<\infty, \quad p\in S_1,\ t\in N$$

(σ_p given by (2.5)), and there exists a constant $K>0$ such that

(4.7) $$\inf_{s\geqq 0} E(\sigma_p(X_{s,s+t}))\geqq -Kt, \quad p\in S_1,\ t\in N.$$

Now consider, for each $p\in S_1$, the family $X^{(p)}=(X^{(p)}_{s,t})=(\sigma_p(X_{s,t}))$. It is not difficult to check that each $X^{(p)}$ is a subadditive superstationary process in the sense of Abid [1]. Hence it follows from Abid's [1] pointwise ergodic theorem that

(4.8) $$\lim_{t\to\infty}\frac{1}{t}X^{(p)}_{0,t} \quad \text{a.e. exists and is finite, } p\in S_1.$$

Utilizing Lemma (4.3), we deduce from (4.8) that $\left(\frac{1}{t}X_{0,t}\right)$ is a.e. convergent in $(\mathrm{co}\,\mathscr{C},\rho)$.

We note that it follows from the proof of Theorem (4.1) and Abid's [1] mean ergodic theorem that, under the assumptions of Theorem (4.1),

(4.9) $$\lim_{t\to\infty}\frac{1}{t}\sigma_p(X_{0,t}) \quad \text{converges in } L^1,\ p\in S_1.$$

(4.10) *Remark.* Let $X=(X_{s,t})$ satisfy the assumptions of Theorem (4.1) and assume that X is even *stationary* (in the obvious sense). In this case, the assertion of Theorem (4.1) can be also deduced from Theorem 3 of [10] in view of the remark following (2.6).

Call a sequence (C_n) of nonvoid sets $C_n\subset R^d$ *subadditive* if

(4.11) $$C_{m+n}\subset C_m+C_n, \quad m,n\in N.$$

Then we have the following set analogue of a classical result on subadditive sequences of real numbers (see [24], p. 17).

(4.12) **Corollary.** *Let $(C_n)\subset\mathrm{co}\,\mathscr{C}$ be subadditive. Then we have*

(4.13) $$\left(\frac{1}{n}C_n\right) \textit{ converges in } (\mathrm{co}\,\mathscr{C},\rho)$$

and

(4.14) $$\lim_{n\to\infty}\frac{1}{n}C_n=\bigcap_{n=1}^{\infty}\frac{1}{n}C_n.$$

Proof. Define the random sets $X_{s,t}$ on some probability space $(\Omega,\mathscr{A},P)$ by

(4.15) $$X_{s,t}(\omega)=C_{t-s}, \quad \omega\in\Omega,\ (s,t)\in I.$$

Clearly (3.2) and (4.2) hold. Hence (4.13) is a consequence of Theorem (4.1). (Here, we could also apply Remark (4.10).) Finally, (4.14) can be proved by utilizing Theorem 1.4.1 of [21].

The following result gives conditions which ensure that the almost sure limit occurring in Theorem (4.1) is constant a.e. It should be noted that conditions like (4.18) and (4.19) given below (suggested by [15] and [11]) have been recently applied to show that certain types of interacting particle systems have an asymptotic shape (see [25], [26]).

(4.16) **Theorem.** *Let $X=(X_{s,t})$ be a family or random sets satisfying the assumptions of Theorem (4.1). Furthermore assume that Conditions (4.17), (4.18) and (4.19) given below are satisfied for all $p\in S_1$ ("Var" denoting variance).*

(4.17) $$E(\sigma_p^2(X_{0,t}))<\infty, \qquad t\in N;$$

(4.18) $$\lim_{t\to\infty}\frac{1}{t^2}E(\sigma_p^2(X_{0,t}))=\beta(p) \text{ exists and is finite};$$

(4.19) $$\operatorname{Var}(\sigma_p(X_{0,2t}))+E^2(\sigma_p(X_{0,2t})) \leqq 2(1+\delta_p)\operatorname{Var}(\sigma_p(X_{0,t}))+4E^2(\sigma_p(X_{0,t})), \qquad t\in N$$

($0\leqq\delta_p<1$ being a constant depending on $p\in S_1$). Then there exists a set $C\in\operatorname{co}\mathscr{C}$ such that $\lim_{t\to\infty}\frac{1}{t}X_{0,t}=C$ a.e. in $(\operatorname{co}\mathscr{C},\rho)$ and $\beta(p)=\sigma_p^2(C)$, $p\in S_1$. Putting $\alpha(p)=\sigma_p(C)$, $p\in S_1$, we have

(4.20) $$\lim_{t\to\infty}\frac{1}{t}\sigma_p(X_{0,t})=\alpha(p) \quad \text{in} \quad L^2, \qquad p\in S_1.$$

Proof. Let $X^{(p)}=(\sigma_p(X_{s,t}))$ be defined for all $p\in S_1$ as in the proof of Theorem (4.1). It follows from (4.9) that, for all $p\in S_1$,

(4.21) $$\lim_{t\to\infty}\frac{1}{t}E(\sigma_p(X_{0,t}))=\tilde{\alpha}(p)$$

exists and is finite. Following the proof in [11], p. 675, it is not difficult to see that (4.21), (4.17), (4.18) and (4.19) together imply

(4.22) $$\lim_{n\to\infty}\frac{1}{2^n m}\sigma_p(X_{0,2^n m})=\tilde{\alpha}(p) \quad \text{a.e.}, \qquad p\in S_1,\ m\in N,$$

and, finally,

(4.23) $$\lim_{t\to\infty}\frac{1}{t}\sigma_p(X_{0,t})=\tilde{\alpha}(p) \quad \text{in} \quad L^2, \qquad p\in S_1.$$

Since, by Theorem (4.1), $\lim_{t\to\infty}\frac{1}{t}\sigma_p(X_{0,t})$ exists a.e. for all $p\in S_1$, it follows from (4.22) and Lemma (4.3) that there exists a set $C\in\operatorname{co}\mathscr{C}$ such that $\lim_{t\to\infty}\frac{1}{t}X_{0,t}=C$

a.e. in $(\mathrm{co}\,\mathscr{C}, \rho)$. By (4.22), this implies $\tilde{\alpha}(p)=\sigma_p(C)=\alpha(p)$, $p\in S_1$, which, together with (4.23), yields (4.20).

We will conclude this section by proving two mean ergodic theorems for certain families of subadditive superstationary families of random sets (see Theorems (4.32) and (4.35) below). Hess [12] derived a mean ergodic theorem for stationary sequences of $\mathrm{co}\,\mathscr{C}$-valued random sets by embedding $(\mathrm{co}\,\mathscr{C}, \rho)$ into $C(S_1)$ (compare the remark following (2.4)) and utilizing an idea of Ahmad [2]. Since it does not seem possible to argue along these lines in the case of subadditive families of random sets, we proceed differently and base the proofs of the desired mean ergodic theorems on the following result being an analogue of Lemma (4.3) with respect to convergence in the mean.

(4.24) **Lemma.** *Let (Y_n) be a sequence of $\mathrm{co}\,\mathscr{C}$-valued random sets on some probability space $(\Omega, \mathscr{A}, P)$, and let $(p_n)\subset S_1$ be a sequence which is dense in S_1. Assume that, for some $m\in N$,*

(4.25) $$\lim_{n\to\infty} \sigma_{p_k}(Y_n) \text{ exists in } L^m, \qquad k\in N.$$

Then there exists a $\mathrm{co}\,\mathscr{C}$-valued random set Y having the following properties:

(4.26) $$\int_\Omega \|Y\|^m\, dP<\infty;$$

(Y_n) converges to Y in the m-th mean, i.e.,

(4.27) $$\lim_{n\to\infty}\int_\Omega \rho^m(Y_n, Y)\, dP=0.$$

The proof of Lemma (4.24) will be based on the following auxiliary result depending on certain geometrical properties of convex sets.

(4.28) **Lemma.** *Let $C, D\in\mathrm{co}\,\mathscr{C}$ and let $0<\varepsilon<1$ be fixed. Let $M(\varepsilon)\subset S_1$ denote a finite set with the property that, for any $p\in S_1$, there exists some $p_\varepsilon\in M(\varepsilon)$ such that $\|p-p_\varepsilon\|\leqq\sqrt{2\varepsilon}$. Then we have*

(4.29) $$\rho(C,D)\leqq\frac{2}{1-\varepsilon}\left(2\sqrt{2\varepsilon}\max(\|C\|,\|D\|)+\max_{p\in M(\varepsilon)}|\sigma_p(C)-\sigma_p(D)|\right).$$

It would be possible to deduce inequalities similar to (4.29) by utilizing Theorem 6 of [20]. A short direct proof of (4.29) can, however, be obtained as follows. Fix $0<\varepsilon<1$. Pick $p_0\in S_1$, $x_0\in C$ and $y_0\in D$ such that $s(p_0, C)=p_0x_0$, $s(p_0, D)=p_0y_0$ and $\rho(C,D)=|p_0(x_0-y_0)|=\|x_0-y_0\|$ (this is possible by (2.4)). Then choose $p_\varepsilon\in M(\varepsilon)$ such that $\|p_0-p_\varepsilon\|\leqq\sqrt{2\varepsilon}$. Let $x_\varepsilon\in C$, $y_\varepsilon\in D$ be such that $\sigma_{p_\varepsilon}(C)=p_\varepsilon x_\varepsilon$, $\sigma_{p_\varepsilon}(D)=p_\varepsilon y_\varepsilon$. Now, (4.29) can be easily obtained by considering the case in which $|p_\varepsilon(x_\varepsilon-y_\varepsilon)|$ is bigger (smaller) than $\frac{1}{2}(1-\varepsilon)\,\rho(C,D)$.

Proof of Lemma (4.24). Applying Cantor's diagonal method, we easily get from (4.25) and Lemma (4.3) that there exists a subsequence (Y_{n_i}) and a $\mathrm{co}\,\mathscr{C}$-

valued random set Y such that $\lim_{i\to\infty} \rho(Y_{n_i}, Y)=0$ a.e. implying

(4.30) $$\lim_{n\to\infty} \int_\Omega |\sigma_{p_k}(Y_n)-\sigma_{p_k}(Y)|^m \, dP=0, \qquad k\in N$$

($m\in N$ occurring in (4.25)). From (4.30) we get $\sigma_{p_k}(Y)\in L^m$, $k\in N$, which, in view of (4.4), proves (4.26). On the other hand, it follows from (4.25) and (4.4) applied to (Y_n) that

(4.31) $$\sup_{n\in N} \int_\Omega \|Y_n\|^m \, dP<\infty.$$

Now let $M(\varepsilon)\subset(p_n)$, for each $0<\varepsilon<1$, be a finite set having the approximation property mentioned in Lemma (4.28). In order to derive (4.27), first note that, by (4.29), we have, for all $\omega\in\Omega$, $0<\varepsilon<1$ and $n\in N$,

$$\rho^m(Y_n(\omega), Y(\omega))\leqq \left(\frac{8\sqrt{2\varepsilon}}{1-\varepsilon}\right)^m \max(\|Y_n(\omega)\|^m, \|Y(\omega)\|^m)$$
$$+\left(\frac{4}{1-\varepsilon}\right)^m \max_{p\in M(\varepsilon)} |\sigma_p(Y_n(\omega))-\sigma_p(Y(\omega))|^m.$$

Taking into account (4.30), we therefore get, for $0<\varepsilon<1$,

$$\limsup_{n\to\infty} \int_\Omega \rho^m(Y_n, Y)\, dP\leqq \left(\frac{8\sqrt{2\varepsilon}}{1-\varepsilon}\right)^m \sup_{n\in N} \int_\Omega \max(\|Y_n\|^m, \|Y\|^m)\, dP.$$

Hence, (4.27) follows from (4.26) and (4.31) since $\varepsilon\in(0,1)$ is arbitrary.

We can now prove the following result which is the first of the desired mean ergodic theorems.

(4.32) **Theorem.** *Let $X=(X_{s,t})$ be a family of co$\mathscr{C}$-valued random sets satisfying the assumptions of Theorem (4.1). Then there exists a co$\mathscr{C}$-valued random set Y having the following properties:*

(4.33) $$\int_\Omega \|Y\|\, dP<\infty;$$

$\left(\frac{1}{t}X_{0,t}\right)$ *converges to Y in the mean, i.e.,*

(4.34) $$\lim_{t\to\infty} \int_\Omega \rho\left(\frac{1}{t}X_{0,t}, Y\right) dP=0.$$

Proof. Immediate from (4.9) and Lemma (4.24).

Another consequence of Lemma (4.24) is as follows.

(4.35) **Theorem.** *Let $X=(X_{s,t})$ be a family of co$\mathscr{C}$-valued random sets satisfying the assumptions of Theorem (4.16). Then there exists a set $C\in$co$\mathscr{C}$ such that*

(4.36) $$\lim_{t\to\infty} \int_\Omega \rho^2\left(\frac{1}{t}X_{0,t}, C\right) dP=0.$$

Proof. Immediate from (4.20) and Lemma (4.24).

5. Two Counterexamples

In [5], a strong law of large numbers is first proved for independent and identically distributed co$\mathscr{C}$-valued random sets. A clever application of a well-known theorem due to Shapley and Folkman (see [3], p. 396) then shows that a strong law of large numbers still holds for $\mathscr{C}$-valued random sets (provided, of course, they satisfy a certain first moment condition). The following example implies that, in general, such a line of argument does not work in the context of subadditive superstationary families of random sets.

(5.1) *Example.* For $n \in N$ let

$$n = \sum_{i=0}^{\infty} a_i(n) 2^i, \quad a_i(n) \in \{0, 1\}, \ i \in N_0, \tag{5.2}$$

be the dyadic representation of n. Put

$$C_n = \left\{ \sum_{i=0}^{\infty} b_i 2^i : b_i \in \{0, a_i(n)\}, \ i \in N_0 \right\}, \quad n \in N. \tag{5.3}$$

Clearly

$$\{0, n\} \subset C_n \subset \{0, 1, 2, \dots, n\}, \quad n \in N. \tag{5.4}$$

The sets $C_1, \dots, C_{16}$ look as follows. $C_1 = \{0, 1\}$, $C_2 = \{0, 2\}$, $C_3 = \{0, 1, 2, 3\}$, $C_4 = \{0, 4\}$, $C_5 = \{0, 1, 4, 5\}$, $C_6 = \{0, 2, 4, 6\}$, $C_7 = \{0, 1, 2, \dots, 7\}$, $C_8 = \{0, 8\}$, $C_9 = \{0, 1, 8, 9\}$, $C_{10} = \{0, 2, 8, 10\}$, $C_{11} = \{0, 1, 2, 3, 8, 9, 10, 11\}$, $C_{12} = \{0, 4, 8, 12\}$, $C_{13} = \{0, 1, 4, 5, 8, 9, 12, 13\}$, $C_{14} = \{0, 2, 4, \dots, 14\}$, $C_{15} = \{0, 1, 2, \dots, 15\}$, $C_{16} = \{0, 16\}$.

(5.5) **Lemma.** *The sequence (C_n) given by* (5.3) *is subadditive.*

It is easily seen that the sequence $\left(\frac{1}{n} C_n\right)$ (C_n given by (5.3)) is not convergent in $(\mathscr{C}, \rho)$. In fact, we have e.g. $\frac{1}{2^n} C_{2^n} = \{0, 1\}$, $n \in N$, and $\lim_{n \to \infty} \frac{1}{2^n - 1} C_{2^n - 1} = [0, 1]$. One checks that the family $(X_{s,t})$ given by (4.15) is a subadditive and even stationary family of $\mathscr{C}$-valued random sets satisfying Condition (4.2) of Theorem (4.1). However, $\left(\frac{1}{t} X_{0,t}(\omega)\right)$ does not converge in $(\mathscr{C}, \rho)$ for any $\omega \in \Omega$.

Proof of Lemma (5.5). The inclusion (4.11) will be proved by induction on $m + n$. Clearly (4.11) holds if $m + n = 2$. Assume that (4.11) holds for all $m, n \in N$ such that $2 \leqq m + n \leqq r - 1$ for some $r \geqq 3$. Let $m + n = r$ for some $r \geqq 3$. Let $0 \neq c \in C_{m+n}$. Then, for some $k \in N$,

$$c = 2^{i_1} + 2^{i_2} + \dots + 2^{i_k}, \quad 0 \leqq i_1 < i_2 < \dots < i_k. \tag{5.6}$$

Case 1.

$$a_{i_k}(m) = a_{i_k}(n) = 0. \tag{5.7}$$

Putting

$$m_1 = \sum_{i=0}^{i_k} a_i(m) 2^i, \qquad n_1 = \sum_{i=0}^{i_k} a_i(n) 2^i \tag{5.8}$$

and using the induction hypothesis, one shows

$$c \in C_{m_1} + C_{n_1} \tag{5.9}$$

which implies $c \in C_m + C_n$ since $C_{m_1} \subset C_m$ and $C_{n_1} \subset C_n$.

Case 2. $a_{i_k}(m) + a_{i_k}(n) \geqq 1$.

Again one can deduce that $c \in C_m + C_n$ by observing that (5.7) implies (5.9) (m_1, n_1 given by (5.8)) whenever c is given by (5.6).

The results of [4] (see Introduction) suggest the following question. Does a pointwise ergodic theorem still hold if the random sets $X_{s,t}$ occurring in Theorem (4.1) are supposed to be nonvoid, convex and closed (but not necessarily bounded)? In order to make this question more precise let $\mathscr{F}$ denote the family of all closed subsets of R^d ($d \geqq 1$), and put $\mathscr{F}_0 = \mathscr{F} - \{\emptyset\}$. If $\mathscr{F}$ is endowed with the *topology of closed convergence*, $\mathscr{F}$ becomes a compact metrizable space (see [21]). A sequence $(C_n) \subset \mathscr{C}$ converges in $(\mathscr{C}, \rho)$ iff (C_n) converges in $\mathscr{F}$, and there exists some $C \in \mathscr{C}$ such that $C_n \subset C$, $n \in N$ (see Theorem 1.4.1 of [21]). Let $\mathscr{B}(\mathscr{F})$ denote the family of Borelian subsets of $\mathscr{F}$. The following example implies that there exist subadditive stationary families $X = (X_{s,t})$ of $\mathscr{B}(\mathscr{F})$-measurable $\mathscr{F}_0$-valued convex random sets such that $\left(\frac{1}{t} X_{0,t}\right)$ is nowhere convergent in the topology of closed convergence.

(5.10) *Example.* Let the sets $C_n \in \mathscr{F}_0$, $n \in N$, by given by

$$C_n = \begin{cases} \{n\} & \text{if } n \in N \text{ is even} \\ [n, \infty) & \text{if } n \in N \text{ is odd.} \end{cases}$$

The sequence (C_n) is clearly subadditive but $\left(\frac{1}{n} C_n\right)$ does not converge in $\mathscr{F}$. The desired random sets $X_{s,t}$ are now defined by (4.15).

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