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Stopped Distributions for Markov Processes in Duality

Neil Falkner*

Dept. of Mathematics, Ohio State University, 231 W 18th Ave, Columbus, Ohio 43210, USA

Summary. Let X and $\hat{X}$ be standard Markov processes in duality on a state space E and assume that semipolar sets are polar. Let μ be a measure on E whose X measure-potential μU is σ-finite. We characterize the measures ν on E which arise as the P^{μ}-distribution of X_T for some non-randomized stopping time T. We then apply this result to characterize the measures ν on E which satisfy $\nu U \leqq \mu U$.

Introduction

Given a stochastic process (X_t) and a measure ν, one may ask when there exists a stopping time T such that the distribution of X_T is ν. This problem has been studied by various people including Skorokhod [15; 16, Chap. 7], Dubins [5] and Root [11] for (X_t) = Brownian motion in one dimension, Rost [12–14] for (X_t) a standard Markov process, and Baxter and Chacon [3] and the author [6, 7] for (X_t) = Brownian motion in several dimensions. Under suitable conditions on ν, Skorokhod and Rost showed how to construct randomized stopping times T with X_T having distribution ν. (A randomized stopping time is a stopping time not of the natural filtration of (X_t) but rather of the natural filtration of $((X_t, Y))$ where Y is a continuously distributed real random variable independent of (X_t).) The other authors listed above concerned themselves with when a *non*-randomized stopping time T could be constructed which would give the desired distribution to X_T. The methods used to construct such non-randomized stopping times depended on special properties of Brownian motion. In this paper we consider this problem for the case where (X_t) is a standard Markov process in duality with another standard Markov process and satisfying Hunt's Hypothesis H; namely that semipolar sets are polar. In so doing, we generalize Theorem 3.1 and Corollary 3.2 of [7] and sharpen (in our more restricted context) the main result of [12]. It follows from our results

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that if μ is a measure whose potential μU is σ-finite and ν is a measure which does not charge polar sets and if there exists a randomized stopping time τ such that $\mu P_\tau = \nu$ (i.e., such that ν is the distribution of X_τ under P^μ) then there actually exists a non-randomized stopping time T such that $\mu P_T = \nu$. Contrast this with the following example:

Let (X_t) be uniform motion to the right on the real line. Then only the empty set is polar, so any measure ν on the state space does not charge polar sets. (But here any countable set is semipolar so Hypothesis H is violated.) Let μ be the unit point mass at 0. Then a randomized stopping time τ satisfying $\mu P_\tau = \nu$ exists iff $\nu((-\infty, 0)) = 0$ and $\nu(\mathbb{R}) \leqq 1$. But a non-randomized stopping time T satisfying $\mu P_T = \nu$ exists iff ν is the unit point mass at x for some $x \in [0, \infty)$ or $\nu = 0$. Thus in this case the set $\{\mu P_T : T$ a non-randomized stopping time$\}$ is much smaller than the set $\{\mu P_\tau : \tau$ a randomized stopping time$\}$. This in spite of the fact that (X_t) is standard and has a standard dual process (namely, uniform motion to the left).

Let us mention two questions which remain open and which we feel are interesting. First, can the results of this paper be proved without assuming that (X_t) has a dual process? Second, can one find a *formula* for a suitable T expressed in terms of ν? A nice formula of this short has been found by Azéma and Yor [1, 2] for (X_t) = Brownian motion in one dimension starting from 0.

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Notation. E will be a second countable locally compact Hausdorff space, $\mathscr{E} = \text{Borel}\, E$, $\mathscr{E}^* =$ universal completion of $\mathscr{E}$. ξ will be a σ-finite measure on $\mathscr{E}^*$. $X = (\Omega, \mathscr{M}, \mathscr{M}_t, X_t, \theta_t, P^x)$ and $\hat{X} = (\hat{\Omega}, \hat{\mathscr{M}}, \hat{\mathscr{M}}_t, \hat{X}_t, \hat{\theta}_t, \hat{P}^x)$ will be standard Markov processes with state space $(E, \mathscr{E})$ (augmented by the cemetery point ∂), in duality relative to ξ, as defined in [4, VI, 1]. ζ will be the lifetime [4, p. 21] of X and $\hat{\zeta}$ that of $\hat{X}$. We shall have need only of such duality theory as is developed in Sect. 1 of Chap. VI of Blumenthal and Getoor [4]. Hence we do *not* assume X or $\hat{X}$ to be strong Feller. $(\mathscr{F}_t)$ will denote the completed natural filtration of X; that is, for each $t \geqq 0$, $\mathscr{F}_t$ is the usual [4, p. 27] completion of $\sigma(X_s : 0 \leqq s \leqq t)$. $(\hat{\mathscr{F}}_t)$ is defined similarly in terms of $\hat{X}$. If T is an $(\mathscr{F}_t)$-stopping time then P_T will be the kernel on $(E, \mathscr{E}^*)$ defined by $P_T(x, A) = P^x(X_T \in A)$. Since $X_T = \partial \notin E$ on $\{T \geqq \zeta\}$, this may also be written as $P_T(x, A) = P^x(X_T \in A, T < \zeta)$. Thus if μ is a measure on $\mathscr{E}^*$ then $\mu P_T(A) = P^\mu(X_T \in A, T < \zeta)$. Thus μP_T is what the distribution of X_T under P^μ would be if X_T were regarded as undefined on $\{T \geqq \zeta\}$. Although we use the word "distribution" we do not suppose that $\mu(E) = 1$ (i.e., $P^\mu(\Omega) = 1$) or even that μ is finite. If $\hat{T}$ is an $(\hat{\mathscr{F}}_t)$-stopping time then $\hat{P}_{\hat{T}}$ will denote the *co*-kernel on $(E, \mathscr{E}^*)$ defined by $\hat{P}_{\hat{T}}(A, x) = \hat{P}^x(\hat{X}_{\hat{T}} \in A)$. Co-kernals act on functions on the right and on measures on the left. In general, for each kernel we associate with X, we associate with $\hat{X}$ the analogous co-kernel. This is customary and simplifies certain formulas. $(U^\alpha)_{\alpha \geqq 0}$ will denote the resolvent of X and $U = U^0$ the potential kernel of X. u: $E \times E \to [0, \infty]$ will denote the potential density. If μ is a measure on $\mathscr{E}^*$ then μU and $\hat{U}\mu$ are the measures on $\mathscr{E}^*$ satisfying $\mu U(A) = E^\mu \left[\int_0^\infty 1_A(X_t)\, dt \right]$, $\hat{U}\mu(A) = \hat{E}^\mu \left[\int_0^\infty 1_A(\hat{X}_t)\, dt \right]$ while $\mu \hat{U}$ is the *function* on E defined by $\mu \hat{U}(y) = \int \mu(dx)\, u(x, y)$. $\mu \hat{U}$ is the unique coexcessive version of the Radon-Nikodym derivative of μU with respect to ξ. For $A \subseteq E$, $T_A = \inf\{t>0\colon X_t \in A\}$ and $\hat{T}_A$ is defined similarly in terms of $\hat{X}$; if A is nearly Borel then P_A is short for P_{T_A} and $\hat{P}_A$ for $\hat{P}_{\hat{T}_A}$. A^r denotes the set of points regular for A and ${}^rA$ the set of points coregular for A. If μ is a measure and f a function then $\langle \mu, f\rangle$ denotes $\int f\,d\mu$ whenever the integral makes sense. Here are the statements of our two main results.

Theorem 1. *Assume semipolar sets are polar. Let μ be a measure on $\mathscr{E}^*$ such that μU is σ-finite. If ν is a measure on $\mathscr{E}^*$ such that $\mu U \geqq \nu U$ and there exists $C \in \mathscr{E}^*$ such that for every polar set $Z \in \mathscr{E}^*$, $\nu(Z) = \mu(Z \cap C)$ then there exists an $(\mathscr{F}_t)$-stopping time T such that $\mu P_T = \nu$ (and conversely). In particular, if $\mu U \geqq \nu U$ and ν does not charge polar sets, then such a T exists.*

Corollary 1. *Assume semipolar sets are polar. Let μ be a measure on $\mathscr{E}^*$ such that μU is σ-finite. Let ν be another measure on $\mathscr{E}^*$. The following are equivalent:*

(a) $\mu U \geqq \nu U$.

(b) *There exist measures α, β on $\mathscr{E}^*$ and an $(\mathscr{F}_t)$-stopping time T such that $\alpha + \beta = \mu$, β lives on a polar set, and $\alpha P_T + \beta = \nu$.*

We remark that (a)$\Rightarrow$(b) of Corollary 1 sharpens the result of Rost [12] since it may be interpreted as saying that $\nu = \mu P_\tau$ where τ is a randomized stopping time which is randomized only at time 0 (and only on a polar set). See Corollary 3.2 of [7] for further discussion of this point. Rost did not suppose duality or that semipolar sets were polar. Even in the presence of duality, our sharpening of Rost's result may fail if semipolar sets are not polar, as the example of uniform motion to the right on the line shows.

Now consider two measures μ and ν on $\mathscr{E}^*$ such that μU and νU are σ-finite and let us observe some facts which would be well-known if μ, ν were finite and X were transient [9], [10, IX, T64], [4, III, 5.11 and 6.22]. First, it is clear that there exists $h \in \mathscr{E}^*_+$ such that $h>0$ on E and $\langle \mu U + \nu U, h\rangle < \infty$. Then $Uh>0$ on E and $\langle \mu+\nu, Uh\rangle < \infty$. Thus μ and ν are themselves σ-finite. Let $A = \{Uh < \infty\}$. Then A is stable in the sense that $\forall x \in A$, $P^x(X_t \notin A \cup \{\partial\}$ for some $t \geqq 0) = 0$. Because $0 < Uh < \infty$ on the stable set A, if f is excessive then ([4, III, 5.1] or [10, IX, T64]) there is a sequence (f_n) in $\mathscr{E}^*_+$ such that $Uf_n \uparrow f$ on A. Thus if $\mu U \geqq \nu U$ then $\langle \mu, f\rangle \geqq \langle \nu, f\rangle$ for any excessive function f, since μ and ν live on A. Now suppose $\mu U = \nu U$. We claim $\mu = \nu$. As $Uh>0$ on E, it suffices to show $\mu(K) = \nu(K)$ when $K \in \mathscr{E}^*$ such that Uh is bounded below, say by $\varepsilon > 0$, on K. Since μ and ν are σ-finite, they are inner regular with respect to compacts so we may suppose in addition that K is compact. Then there is a sequence (φ_n) of continuous functions on E such that $0 \leqq \varphi_n \leqq \varepsilon 1$ and $\varphi_n \downarrow \varepsilon 1_K$. Letting $g_n = \varphi_n \wedge Uh$ we have $g_n \downarrow \varepsilon 1_K$ since $Uh \geqq \varepsilon$ on K. Then $\langle \mu, g_n\rangle \downarrow \varepsilon \mu(K)$ and $\langle \nu, g_n\rangle \downarrow \varepsilon \nu(K)$ by dominated convergence. Since $\mu U = \nu U$ are σ-finite, the resolvent equation implies $\mu U^\alpha = \nu U^\alpha$ for all α. Since g_n is bounded and $(g_n(X_t))$ is a.s. right continuous, $\alpha U^\alpha g_n(x) \to E^x[g_n(X_0)] = g_n(x)$ as $\alpha \to \infty$, for all x. Since $g_n \leqq Uh$, $\alpha U^\alpha g_n \leqq \alpha U^\alpha Uh \leqq Uh$. Thus, using dominated convergence, $\langle \mu, g_n\rangle = \lim_{\alpha\to\infty} \langle \mu U^\alpha, \alpha g_n\rangle = \lim_{\alpha\to\infty} \langle \nu U^\alpha, \alpha g_n\rangle = \langle \nu, g_n\rangle$.

Therefore $\mu(K) = \nu(K)$ and so $\mu = \nu$. Of course the observations of this paragraph did not use duality. Neither does the proof of:

Lemma 1. *Let ν be a measure on $\mathscr{E}^*$ such that νU is σ-finite. Suppose $Z \subseteq E$ is nearly Borel, $\nu(Z)=0$, and Z is ν-polar (i.e., $P^\nu(T_Z<\infty)=0$). Then there exist a decreasing sequence (G_n) of finely open nearly Borel subsets of E containing Z and an excessive function f such that $\langle \nu, P_{G_n} 1\rangle \downarrow 0$, $f=+\infty$ on $\bigcap_n G_n \supseteq Z$, $\langle \nu, f\rangle < \infty$, and $\langle \nu, P_{G_n} f\rangle \downarrow 0$.*

Proof. Let $h \in \mathscr{E}^*_+$ such that $h>0$ on E and $\langle \nu, Uh\rangle < \infty$. First suppose Uh is bounded below on Z, say by ε. Now $\nu(Z)=0$ so certainly $\nu(Z \setminus Z^r)=0$. Therefore ([4, III, 6.1] or [8, 12.10]) there exists a decrasing sequence (B_n) of finely open nearly Borel subsets of E containing Z such that $T_{B_n} \uparrow T_Z$ P^ν-a.s. Then $T_{B_n} \uparrow \infty P^\nu$-a.s. Replacing B_n by $B_n \cap \{Uh > \varepsilon\}$ if need be, we may assume that $Uh > \varepsilon$ on B_n. This having been done, $Uh(X(T_{B_n})) \geqq \varepsilon$ a.s. on $\{T_{B_n} < \infty\}$ since $(Uh(X_t))$ is a.s. right continuous. Thus

$$\begin{aligned}\langle \nu, P_{B_n} 1\rangle &= P^\nu(T_{B_n}<\infty) \leqq \varepsilon^{-1} E^\nu[Uh(X_{T_{B_n}})]\\ &= \varepsilon^{-1} E^\nu\left[\int_{T_{B_n}}^{\infty} h(X_t)\,dt\right] \downarrow 0\end{aligned}$$

by dominated convergence. $\left(E^\nu\left[\int_0^\infty h(X_t)\,dt\right] = \langle \nu, Uh\rangle < \infty.\right)$ Now consider the general case. Since $h>0$ on E, $Uh>0$ on E so we may write Z as $\bigcup_m Z_m$ where for each m, Z_m is nearly Borel and Uh is bounded below on Z_m. By the preceeding argument, for each m we may choose a decreasing sequence $(B_{mn})_n$ of finely open nearly Borel subsets of E containing Z_m such that for each n, $\langle \nu, P_{B_{mn}} 1\rangle \leqq 2^{-(m+n)}$. Let $G_n = \bigcup_m B_{mn}$. Since

$$P_{G_n} 1(x) = P^x(T_{G_n}<\infty) = P^x(\inf_m T_{B_{mn}} < \infty) \leqq \sum_m P_{B_{mn}} 1(x),$$

we have $\langle \nu, P_{G_n} 1\rangle \leqq \sum_m 2^{-(m+n)} = 2^{-n+1}$. Each G_n is finely open and nearly Borel and $G_n \supseteq G_{n+1} \supseteq Z$. Let $f = \sum_n P_{G_n} 1$. Then f is excessive and $\langle \nu, f\rangle \leqq \sum_n 2^{-n+1} < \infty$. Since $P_{G_n} 1 = 1$ on G_n, $f=+\infty$ on $\bigcap_n G_n \supseteq Z$. If $n \leqq m$ then since $P_{G_n} 1 = 1$ on $G_n \supseteq G_m$ and $(P_{G_n} 1(X_t))$ is a.s. right continuous, $P_{G_n} 1(X(T_{G_m})) = 1$ a.s. on $\{T_{G_m} < \infty\}$ so $P_{G_m} P_{G_n} 1 = P_{G_m} 1$. Thus

$$P_{G_m} f = \sum_{n \leqq m} P_{G_m} P_{G_n} 1 + \sum_{n>m} P_{G_m} P_{G_n} 1 \leqq (m+1) P_{G_m} 1 + \sum_{n>m} P_{G_n} 1$$

so

$$\langle \nu, P_{G_m} f\rangle \leqq (m+1)\, 2^{-m+1} + \sum_{n>m} 2^{-n+1} \to 0 \quad \text{as} \quad m \to \infty. \quad \square$$

Now let us recall the following result, whose proof we give for completeness:

Lemma 2 *(Domination principle). Let λ be a σ-finite measure on $\mathscr{E}^*$ which does not charge semipolar sets. Let f be coexcessive and suppose $f \geqq \lambda \hat{U}$ λ-a.e. Then $f \geqq \lambda \hat{U}$ everywhere.*

Proof. This is an elementary consequence of Hunt's switching identity [4, VI, 1.16] which states that if $A \subseteq E$ is nearly Borel then for all $x, y \in E$,

$$\int P_A(x, dz)\, u(z, y) = \int u(x, z)\, \hat{P}_A(dz, y).$$

(Note [4, VI, 1.20] that a subset of E is nearly Borel iff it is co-nearly Borel so P_A and $\hat{P}_A$ both make sense.) Recall that

$$\lambda \hat{U}(y) = \int \lambda(dx)\, u(x, y) \qquad (y \in E).$$

With the aid of the switching identity and an interchange in order of integration which is valid because λ is σ-finite, we find that $\lambda P_A \hat{U} = \lambda \hat{U} \hat{P}_A$ (i.e., $(\lambda P_A)\hat{U} = (\lambda \hat{U})\hat{P}_A$). Let us take $A = \{f \geqq \lambda \hat{U}\}$ which is nearly Borel. By assumption, $\lambda(A^c) = 0$. Since $A \setminus A^r$ is semipolar and since λ does not charge semipolar sets, λ lives on A^r. Thus $P^\lambda(T_A > 0) = 0$ so $\lambda P_A = \lambda$. Now $f(\hat{X}(\hat{T}_A)) \geqq \lambda \hat{U}(\hat{X}(\hat{T}_A))$ a.s. on $\{\hat{T}_A < \infty\}$ since $(f(\hat{X}_t))$ and $(\lambda \hat{U}(\hat{X}_t))$ are a.s. right continuous. Therefore $f \geqq f \hat{P}_A \geqq \lambda \hat{U} \hat{P}_A = \lambda P_A \hat{U} = \lambda \hat{U}$. □

Proof of Theorem 1. ($\Leftarrow$). Suppose T is an $(\mathscr{F}_t)$ stopping time such that $\mu P_T = \nu$. By the strong Markov property, for any $h \in \mathscr{E}^*_+$ we have

$$\begin{aligned} \langle \nu U, h \rangle &= \langle \mu, P_T U h \rangle = E^\mu[U h(X_T)] = E^\mu\left\{E^{X(T)}\left[\int_0^\infty h(X_s)\, ds\right]\right\} \\ &= E^\mu\left[\int_T^\infty h(X_s)\, ds\right] \leqq E^\mu\left[\int_0^\infty h(X_s)\, ds\right] = \langle \mu U, h \rangle \end{aligned}$$

so $\nu U \leqq \mu U$. Let $C = \{x \in E\colon P^x(T=0) = 1\}$. Then $C \in \mathscr{E}^*$ and, by the Blumenthal $0-1$ law, $\forall x \in E \setminus C$, $P^x(T > 0) = 1$. Thus if $Z \in \mathscr{E}^*$ is polar then $\nu(Z) = P^\mu(X_T \in Z)$ $= P^\mu(X_T \in Z, T = 0) = P^\mu(X_0 \in Z, T = 0) = \mu(Z \cap C)$.

($\Rightarrow$). *Step 1.* Consider the case where μ does not charge polar sets. Let $\mathscr{T}$ be the set of $(\mathscr{F}_t)$-stopping times T such that $\mu P_T U \geqq \nu U$. Then $0 \in \mathscr{T}$ so $\mathscr{T} \neq \emptyset$. If (T_n) is a sequence in $\mathscr{T}$ which increases to T then for every $h \in \mathscr{E}^*_+$ such that $\langle \mu U, h \rangle < \infty$ we have

$$\begin{aligned} \langle \mu P_{T_n} U, h \rangle &= E^\mu\left[\int_{T_n}^\infty h(X_t)\, dt\right] \\ &\downarrow E^\mu\left[\int_T^\infty h(X_t)\, dt\right] = \langle \mu P_T U, h \rangle \end{aligned}$$

by dominated convergence so, as μU is σ-finite, $\mu P_T U \geqq \nu U$, whence $T \in \mathscr{T}$. That is, the pointwise limit of any increasing sequence in $\mathscr{T}$ belongs to $\mathscr{T}$. Since P^μ is σ-finite, $\mathscr{T}$ has P^μ-essentially maximal elements; let T be one of them. Let $\lambda = \mu P_T$. We claim $\lambda = \nu$. Suppose not. Now λ does not charge polar sets since μ already did not. Since we are supposing that semipolar sets are polar, λ does not charge semipolar sets. Next, $\lambda U \leqq \mu U$ so λU is σ-finite, whence λ itself is σ-finite. Consider the nearly Borel cofinely open set A $= \{\lambda \hat{U} > \nu \hat{U}\}$. If $\lambda(A) = 0$ then $\nu \hat{U} \geqq \lambda \hat{U}$ λ-a.e. so by the domination principle (Lemma 2), $\nu \hat{U} \geqq \lambda \hat{U}$ everywhere. As $\nu \hat{U}$ is the density of the measure νU with respect to ξ and similarly for $\lambda\hat{U}$, $\nu U \geqq \lambda U$. Then $\nu U = \lambda U$ so $\nu = \lambda$, contradicting the supposition that $\lambda \neq \nu$. Therefore $\lambda(A) > 0$. Now [4, VI, 1.19] a set is semipolar iff it is cosemipolar. Thus λ does not charge cosemipolar sets. Since $\hat{X}$ has a reference measure (namely ξ; see [4, VI, 1.13]) it follows that [4, V, 1.18] every cofinely open set is λ-measurable and [4, V, 1.21] there is a smallest cofinely closed set F such that $\lambda(E \setminus F) = 0$. F is called the cofine support of λ. If $B \subseteq E$ is cofinely open then $\lambda(B) > 0$ iff $B \cap F \neq \emptyset$. Thus $A \cap F \neq \emptyset$. Let $x_0 \in A \cap F$. Let $\hat{R} = \hat{T}_{\{x_0\}^c}$. Then $\hat{P}^{x_0}(\hat{R} > 0) = 0$ or 1.

Sub-step (a). Suppose $\hat{P}^{x_0}(\hat{R} > 0) = 0$. Let (V_n) be a decreasing sequence of open neighbourhoods of x_0 such that $\bigcap_n V_n = \{x_0\}$. Then $\hat{T}_{V_n^c} \downarrow \hat{R}$. Hence $\hat{T}_{V_n^c} \downarrow 0$ $\hat{P}^{x_0}$-a.s. Now

$$\begin{aligned} \nu\hat{U}(x_0) < \lambda\hat{U}(x_0) &= \hat{E}^{x_0}[\lambda\hat{U}(\hat{X}_0)] \\ &\leqq \liminf_{n\to\infty} \hat{E}^{x_0}[\lambda\hat{U}(\hat{X}(\hat{T}_{V_n^c}))] \end{aligned}$$

where the last step follows from the a.s. right continuity of $(\lambda\hat{U}(\hat{X}_t))$ by Fatou's lemma. Thus there exists n such that for $V = V_n$ we have $\nu\hat{U}(x_0) < \hat{E}^{x_0}[\lambda\hat{U}(\hat{X}(\hat{T}_{V^c}))]$. Let $W = V \cap \{\nu\hat{U} < \lambda\hat{U}\hat{P}_{V^c}\}$. Then $x_0 \in W \cap F$ and W is cofinely open so $\lambda(W) \neq 0$. Also W is nearly Borel. Let $S = T_{W^c}$, $\hat{S} = \hat{T}_{W^c}$. For $x \in W$, $\lambda P_S \hat{U}(x) = \lambda\hat{U}\hat{P}_{\hat{S}}(x) \geqq \lambda\hat{U}\hat{P}_{V^c}(x) > \nu\hat{U}(x)$. Hence if $h \in \mathscr{E}_+^*$ and $h_1 = h 1_W$, $h_2 = h 1_{W^c}$, then

$$\langle \lambda P_S U, h_1 \rangle = \int \lambda P_S \hat{U}(y)\, h_1(y)\, \xi(dy) \geqq \int \nu\hat{U}(y)\, h_1(y)\, \xi(dy) = \langle \nu U, h_1 \rangle.$$

Now [4, II, 1.3] $P_{W^c} U h_2 = U h_2$ because h_2 vanishes off W^c. Thus $\langle \lambda P_S U, h_2 \rangle = \langle \lambda, P_{W^c} U h_2 \rangle = \langle \lambda, U h_2 \rangle = \langle \lambda U, h_2 \rangle \geqq \langle \nu U, h_2 \rangle$. Hence $\langle \lambda P_S U, h \rangle \geqq \langle \nu U, h \rangle$. Thus $\lambda P_S U \geqq \nu U$. Let $T' = T + S \circ \theta_T$. Then $\mu P_{T'} = \lambda P_S$ by the strong Markov property. Thus $T' \in \mathscr{T}$. Now [4, VI, 1.25] for any set $B \subseteq E$, the fine interior and the cofine interior of B differ at most by a semipolar set. Let W' be the fine interior of W. Then $\lambda(W \setminus W') = 0$ so $\lambda(W') > 0$. Now [4, I, 11.4 and II, 4.9] $X_S \in$ fine closure (W^c) a.s. on $\{S < \infty\}$ so $\lambda P_S(W') = 0$. Thus $\mu P_T \neq \mu P_{T'}$ so $P^\mu(T \neq T') > 0$. But $T \leqq T'$ so this contradicts the P^μ-essential maximality of T in $\mathscr{T}$.

Sub-step (b). Suppose on the other hand that $\hat{P}^{x_0}(\hat{R} > 0) = 1$. That is, suppose $\{x_0\}$ is cofinely open. Then $\{x_0\} \setminus$ fine int$\{x_0\}$ is semipolar so if $\{x_0\}$ is not finely open it is semipolar, hence cosemipolar, and hence (as it consists of just one point) co-thin; i.e., ${}^r\{x_0\}$ is empty. But $x_0 \in {}^r\{x_0\}$ as $\{x_0\}$ is cofinely open. Thus $\{x_0\}$ must be finely open. That is, $P^{x_0}(R > 0) = 1$ where $R = T_{\{x_0\}^c}$. Now let $S = R \wedge t$ where $t \in (0, \infty)$ is to be chosen. If $h \in \mathscr{E}_+^*$ and $h_1 = h 1_{\{x_0\}^c}$, $g = 1_{\{x_0\}}$ then $P_R U h_1 = U h_1$ since h_1 vanishes outside $\{x_0\}^c$ so

$$\langle \lambda P_S U, h_1 \rangle \geqq \langle \lambda P_R U, h_1 \rangle = \langle \lambda, P_R U h_1 \rangle = \langle \lambda, U h_1 \rangle = \langle \lambda U, h_1 \rangle.$$

Next,

$$\langle \lambda P_S U, g \rangle = E^\lambda \left[\int_{R\wedge t}^{\infty} g(X_s)\, ds \right] \uparrow E^\lambda \left[\int_0^\infty g(X_s)\, ds \right] = \langle \lambda U, g \rangle$$

as $t \downarrow 0$ while $\xi(\{x_0\}) > 0$ as $\{x_0\}$ is finely open so

$$\langle \lambda U, g\rangle = \int \lambda \hat{U}(y)\, g(y)\, \xi(dy) = \lambda \hat{U}(x_0)\, \xi(\{x_0\}) > \nu \hat{U}(x_0)\, \xi(\{x_0\}) = \langle \nu U, g\rangle.$$

Thus we may choose $t \in (0, \infty)$ so that with $S = R \wedge t$, we have $\langle \lambda P_S U, g\rangle > \langle \nu U, g\rangle$. Then $\langle \lambda P_S U, h\rangle = \langle \lambda P_S U, h_1\rangle + h(x_0)\langle \lambda P_S U, g\rangle \geqq \langle \nu U, h_1\rangle + h(x_0)\langle \nu U, g\rangle = \langle \nu U, h\rangle$. Note that t does not depend on h. Again let $T' = T + S \circ \theta_T$. As before $T \leqq T' \in \mathcal{T}$. Now

$$\begin{aligned} P^\mu(T < T') &= P^\mu(T < \infty, S \circ \theta_T > 0) \\ &\geqq P^\mu(X_T = x_0, S \circ \theta_T > 0) \\ &= P^\lambda(X_0 = x_0, S > 0) \\ &= \lambda(\{x_0\})\, P^{x_0}(S > 0) = \lambda(\{x_0\}). \end{aligned}$$

Now $\lambda(\{x_0\}) > 0$ since $x_0 \in F$ and $\{x_0\}$ is cofinely open. Thus again we have a contradiction of the P^μ-maximality of T in $\mathcal{T}$. Thus it must be that $\lambda = \nu$ after all. That is, $\mu P_T = \nu$. We have now shown that the theorem is true whenever μ does not charge polar sets.

Step 2. Consider the case where ν does not charge polar sets. The measure μ is σ-finite and the collection of nearly Borel polar sets is closed under countable unions so there exists a μ-essentially largest nearly Borel polar set $Z \subseteq E$. If A is a polar set then by definition A is contained in a nearly Borel polar set B; if $A \cap Z = \emptyset$ then we may take $B \cap Z = \emptyset$ and then $\mu(B) = 0$ so A is μ-measurable and $\mu(A) = 0$. By Lemma 1 there exist an excessive function f and a decreasing sequence (G_n) of finely open nearly Borel subsets of E containing Z such that $f = +\infty$ on $\bigcap_n G_n \supseteq Z$, $\langle \nu, f\rangle < \infty$, and $\langle \nu, P_{G_n} f\rangle \leqq 2^{-n}$. Let $H_0 = G_0^c$ and for $n \geqq 1$, let $H_n = G_{n-1} \cap G_n^c$. Let $\mu_n = \mu P_{H_n}$, $\nu_n = \nu P_{H_n}$. Then $\mu_n U \leqq \mu U$ so $\mu_n U$ is σ-finite. Now for any excessive function g, $\langle \mu, g\rangle \geqq \langle \nu, g\rangle$. Hence for any $h \in \mathscr{E}^*_+$, $\langle \mu_n U, h\rangle = \langle \mu, P_{H_n} U h\rangle \geqq \langle \nu, P_{H_n} U h\rangle = \langle \nu_n U, h\rangle$ so $\mu_n U \geqq \nu_n U$. If $A \subseteq E$ is polar, let $B \subseteq E$ be nearly Borel and polar with $A \subseteq B$. Then

$$\begin{aligned} \mu_n(B) &= P^\mu(X(T_{H_n}) \in B) = P^\mu(X(T_{H_n}) \in B,\, T_{H_n} = 0) \\ &\leqq P^\mu(X_0 \in B,\, T_{H_n} = 0,\, X_0 \in G_n^c) \leqq \mu(B \cap G_n^c) = 0 \end{aligned}$$

so A is μ_n-measurable and $\mu_n(A) = 0$. Thus μ_n does not charge polar sets. Thus by Step 1, for each n there exists an $(\mathcal{F}_t)$-stopping time R_n such that $\mu_n P_{R_n} = \nu_n$. Let $S_n = T_{H_n} + R_n \circ \theta(T_{H_n})$. Then $\mu P_{S_n} = \nu_n$. Let $S = \inf_n S_n$ and for each n, let $Q_n = S_0 \wedge \ldots \wedge S_n$ so that $Q_n \downarrow S$. Then

$$\begin{aligned} \langle \mu P_S, f\rangle &= E^\mu[f(X_S)] \\ &\leqq \liminf_{n \to \infty} E^\mu[f(X_{Q_n})] \\ &\leqq \liminf_{n \to \infty} \sum_{k=0}^{n} E^\mu[f(X_{S_k}), Q_n = S_k] \\ &\leqq \sum_n E^\mu[f(X_{S_n})] \\ &= \sum_n \langle \mu P_{S_n}, f\rangle \end{aligned}$$

$$
\begin{aligned}
&=\sum_n \langle \nu_n, f\rangle \\
&=\sum_n \langle \nu, P_{Hn} f\rangle \\
&\leqq \langle \nu, f\rangle + \sum_n \langle \nu, P_{Gn} f\rangle \\
&\leqq \langle \nu, f\rangle + \sum_n 2^{-n} < \infty
\end{aligned}
$$

where the second step follows from the a.s. right continuity of $(f(X_t))$ by Fatou's lemma. Since $f=+\infty$ on Z, $\mu P_S(Z)$ must equal 0. If $B\subseteq E\setminus Z$ is polar and nearly Borel then $\mu P_S(B)=0$ since already $\mu(B)=0$. Thus μP_S does not charge polar sets. Since $\mu U \geqq \mu P_S U$, $\mu P_S U$ is σ-finite. Consider $h\in\mathscr{E}_+^*$. Then $h=g+\sum_n h_n$ where $h_n=0$ on H_n^c and $g=0$ on $\bigcup_n H_n=(\bigcap_n G_n)^c \supseteq \{f=\infty\}^c$. Then $P_{H_n} U h_n = U h_n$ so

$$\langle \mu P_S U, h_n\rangle \geqq \langle \mu P_{S_n} U, h_n\rangle = \langle \nu_n U, h_n\rangle = \langle \nu, P_{H_n} U h_n\rangle = \langle \nu, U h_n\rangle = \langle \nu U, h_n\rangle.$$

Also $\{f=\infty\}$ is ν-polar since $\langle \nu, f\rangle < \infty$ so $0=\langle \nu U, g\rangle \leqq \langle \mu P_S U, g\rangle$. Thus $\langle \mu P_S U, h\rangle \geqq \langle \nu U, h\rangle$ so $\mu P_S U \geqq \nu U$. Therefore by Step 1 again, this time with μ replaced by μP_S, there exists an $(\mathscr{F}_t)$-stopping time T' such that $\mu P_S P_{T'}=\nu$. Then $\mu P_T=\nu$ where $T=S+T'\circ\theta_S$.

Step 3. Consider the general case. Then there is a trivial reduction to the case where ν does not charge polar sets. For let $C\in\mathscr{E}^*$ such that for every polar set $Z\in\mathscr{E}^*$, $\nu(Z)=\mu(Z\cap C)$. Let M be a ν-essentially largest $\mathscr{E}^*$-measurable polar set. Since $\nu(M\setminus(M\cap C))=\mu(M\cap C^c\cap C)=0$, we may and we do assume that $M\subseteq C$. Then if $Z\in\mathscr{E}^*$ is polar, $\nu(Z)=\nu(Z\cap M)=\mu(Z\cap M\cap C)=\mu(Z\cap M)$. Hence if we let $\mu'(A)=\mu(Z\setminus M)$, $\nu'(A)=\nu(Z\setminus M)$ and $\gamma(A)=\mu(A\cap M)(=\nu(A\cap M))$ for $A\in\mathscr{E}^*$ then μ', ν', and γ are measures on $\mathscr{E}^*$, $\mu'+\gamma=\mu$, $\nu'+\gamma=\nu$, and ν' does not charge polar sets. As μU is σ-finite and $\mu U\geqq \nu U$, $\mu' U$ is σ-finite and $\mu' U \geqq \nu' U$. Thus, by Step 2, there is an $(\mathscr{F}_t)$-stopping time T' such that $\mu' P_{T'}=\nu'$. Let

$$T=\begin{cases} T' & \text{on } \{X_0\notin M\} \\ 0 & \text{on } \{X_0\in M\}. \end{cases}$$

Then for $A\in\mathscr{E}^*$,

$$
\begin{aligned}
\mu P_T(A) &= P^\mu(X_T\in A) = P^\mu(X_{T'}\in A, X_0\notin M) + P^\mu(X_0\in A, X_0\in M) \\
&= P^{\mu'}(X_{T'}\in A) + \mu(A\cap M) = \mu' P_{T'}(A) + \gamma(A) = \nu'(A) + \gamma(A) = \nu(A).
\end{aligned}
$$

Thus $\mu P_T=\nu$. The theorem is now completely proved. □

In order to establish Corollary 1, we have need of the following result whose proof does not use duality:

Lemma 3. *Let μ and ν be measures on $\mathscr{E}^*$ such that μU is σ-finite and $\mu U\geqq \nu U$. Let $Z\subseteq E$ be nearly Borel and μ-polar. Then Z is ν-polar and $\mu(Z)\geqq \nu(Z)$.*

Proof. For any excessive function g, $\langle \mu, g\rangle \geqq \langle \nu, g\rangle$. Thus $\langle \nu, P_Z 1\rangle \leqq \langle \mu, P_Z 1\rangle = 0$ so Z is ν-polar. Next, by Lemma 1, there exists a sequence (g_n) of excessive functions such that each $g_n=1$ on Z and $\langle\gamma, g_n\rangle\downarrow 0$ where γ is the measure on $\mathscr{E}^*$ defined by $\gamma(A)=\mu(A\backslash Z)$. (In the notation of that lemma, take $g_n=P_{G_n}1$.) Then $v(Z)\leqq\langle v, g_n\rangle\leqq\langle\mu, g_n\rangle=\mu(Z)+\langle\gamma, g_n\rangle\downarrow\mu(Z)$. $\square$

Proof of Corollary 1. (b)$\Rightarrow$(a) is clear.

(a)$\Rightarrow$(b). By Lemma 3, for any polar nearly Borel set $Z\subseteq E$, $\mu(Z)\geqq v(Z)$. Now a polar set which is $\mathscr{E}^*$-measurable is actually nearly Borel. Let M be a v-essentially largest $\mathscr{E}^*$-measurable polar set. Let $\beta(A)=v(A\cap M)$, $v'(A)=v(A\backslash M)$ for $A\in\mathscr{E}^*$. Then $v'+\beta=v$ and β lives on the polar set M. Also $\beta\leqq\mu$ so as μ is σ-finite, there is a (unique) measure α on $\mathscr{E}^*$ such that $\alpha+\beta=\mu$. Now $\mu U=\alpha U+\beta U\geqq v' U+\beta U=v U$. Since μU is σ-finite, $\alpha U\geqq v' U$. Note that v' does not charge polar sets. Thus, by Theorem 1, there exists an $(\mathscr{F}_t)$-stopping time T such that $\alpha P_T=v'$. Then $\alpha P_T+\beta=v$. $\square$

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