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On the Stationary Measures of Critical Branching Processes

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Abstract. Let Z_n $(n=0, 1, \ldots)$ be an aperiodic critical Galton-Watson process and let σ^2 be the (possibly infinite) variance of Z_1. Let η_k $(k=1, 2, \ldots)$ denote the stationary measure of the process. Kesten, Ney and Spritzer proved in 1966 that

$$\eta_k \to 2/\sigma^2 \quad \text{as} \quad k \to \infty \tag{$*$}$$

under the additional assumption that

$$EZ_1^2 \log Z_1 < \infty. \tag{$**$}$$

In the present paper, $(*)$ is proved without the assumption $(**)$. The proof uses complex function theory.

1. Introduction

Let Z_n $(n=0, 1, \ldots)$ be a *Galton-Watson branching process* starting from $Z_0=1$. In the standard interpretation, Z_n is the random number of individuals in the n-th generation; these individuals propagate independently and the probability that an individual has k direct descendents $(k=0, 1, \ldots)$ is always p_k. This process is called *critical* if the average number of direct descendents is 1, that is if $p_1+2p_2+3p_3 \ldots =1$; we assume $p_0>0$ because otherwise we have the trivial case $p_1=1$, $p_k=0$ $(k\geqq 2)$. The process is called *aperiodic* if there does not exist $m\geqq 2$ such that $p_k=0$ for $k \neq \mu m$ $(\mu=0, 1, \ldots)$. We refer to the books of Harris [4] and Athreya and Ney [1] for the theory of branching processes.

We describe now this process in analytical terms. The generating function of Z_1,

$$f(z)=\sum_{k=0}^{\infty} p_k z^k \qquad (p_k \geqq 0 \text{ aperiodic, } p_0>0), \tag{1.1}$$

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is analytic in the unit disk $\mathbb{D}$ and satisfies

$$f(1)=\sum_{k=0}^{\infty} p_k=1, \quad f'(1)=\sum_{k=1}^{\infty} k p_k=1. \tag{1.2}$$

The variance of Z_1 is given by

$$\sigma^2=f''(1)=\sum_{k=1}^{\infty} k(k-1) p_k \leqq +\infty. \tag{1.3}$$

The iterates $f_n=f\circ\ldots\circ f$ are the generating functions of Z_n. The limit

$$h(z)\equiv \sum_{k=1}^{\infty} \eta_k z^k = \lim_{n\to\infty} \frac{f_n(z)-f_n(0)}{f_{n+1}(0)-f_n(0)} \tag{1.4}$$

exists locally uniformly in $\mathbb{D}$ and satisfies

$$h(f(z))=h(z)+1 \qquad (z\in\mathbb{D}); \tag{1.5}$$

see [9], [1, Th. I8.2]. Similar results hold for the iteration of power series with complex coefficients that satisfy $f(\mathbb{D})\subset\mathbb{D}$; see [10, Chapt. VI], [8, 2, 3].

It follows from (1.5) that η_k $(k=1, 2, \ldots)$ is the stationary measure of the process, that is

$$\eta_k=\sum_{j=1}^{\infty} P(Z_{n+1}=k\,|\,Z_n=j)\,\eta_j \qquad (k=1, 2, \ldots;\ n=1, 2, \ldots). \tag{1.6}$$

It is unique (up to a multiplicative constant) in the critical case under consideration [1, Sect. II2].

Theorem. *For every aperiodic critical Galton-Watson process with variance $\sigma^2\leqq +\infty$, the stationary measure satisfies*

$$\eta_k\to 2/\sigma^2 \quad \textit{as} \quad k\to\infty. \tag{1.7}$$

This limit relation was proved by Kesten, Ney and Spitzer [6] under the additional assumption

$$E(Z_1^2 \log^+ Z_1)\equiv \sum_{k=1}^{\infty} k^2 p_k \log k<\infty. \tag{1.8}$$

We do not even assume that the variance is finite.

Our proof is complex analytic and is modeled after Hayman's proof of the asymptotic form of the Bieberbach conjecture for univalent functions [5, Th. 5.7]. The new feature is the unrestricted limit relation

$$(1-z)\,h(z)\to 2/\sigma^2 \quad \text{as} \quad z\to 1,\ z\in\mathbb{D}.$$

Its proof uses strongly the assumption that f has non-negative coefficients.

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2. Some Auxiliary Results

Let (1.1) and (1.2) be satisfied and let h be defined by (1.4). An analytic function g is called *univalent* in a domain G if $g(z_1) \neq g(z_2)$ for distinct $z_1, z_2 \in G$.

Lemma 1. *There exists $\rho > 0$ such that f and h are univalent in*

$$G = \mathbb{D} \cap \{|z-1| < \rho\}. \tag{2.1}$$

Proof. By (1.1) and (1.2), the derivative f' is continuous in the closed unit disk $\overline{\mathbb{D}}$ and $f'(1)=1$. Hence we can determine $\rho > 0$ such that $\operatorname{Re} f'(z) > 0$ for $z \in G$. Thus

$$\operatorname{Re} \frac{f(z_2)-f(z_1)}{z_2 - z_1} = \int_0^1 \operatorname{Re} f'(z_1 + (z_2 - z_1)t)\,dt > 0$$

for distinct $z_1, z_2 \in G$ and f is therefore univalent in G.

We obtain from $|f'(z)| \leqq 1$ by integration that

$$|1-f(z)| \leqq |1-z| \quad \text{for} \quad z \in \mathbb{D}. \tag{2.2}$$

It follows that $f(G) \subset G$ and thus that $f_n(G) \subset G$ for all n. Hence f_n is univalent in G and therefore also h, by (1.4).

Lemma 2. *The derivative h' has a continuous extension to $\overline{\mathbb{D}} \setminus \{1\}$.*

Proof. If $z \in \overline{\mathbb{D}} \setminus \{1\}$ then, by (1.2),

$$|f(z)| \doteq \left| \sum_{k=0}^{\infty} p_k z^k \right| < \sum_{k=0}^{\infty} p_k = 1; \tag{2.3}$$

equality cannot hold because (p_k) is aperiodic and $p_0 > 0$. We obtain from (1.5) by differentiation that

$$h'(z) = h'(f(z)) f'(z) \qquad (z \in \mathbb{D}). \tag{2.4}$$

Since f' is continuous in $\overline{\mathbb{D}}$ and since $f(\overline{\mathbb{D}} \setminus \{1\}) \subset \mathbb{D}$ by (2.3), it follows that h' is continuous in $\overline{\mathbb{D}} \setminus \{1\}$.

Lemma 3. *Let $\sigma \leqq +\infty$ and $c = 2/\sigma^2$. Then*

$$(1-x)\,h(x) \to c \quad \text{as} \quad x \to 1-0, \; x \in \mathbb{R}. \tag{2.5}$$

Proof (See [1, p. 88]). It follows from (1.5) that

$$h(f_n(z)) = h(z) + n. \tag{2.6}$$

Hence we obtain that, for $0 < x < 1$ and $n \to \infty$,

$$(1-f_n(x))\,h(f_n(x)) = (1-f_n(x))\,h(x) + (1-f_n(x))\,n \to c$$

because $(1-f_n(x))\,n \to c$ [1, p. 19]. It is easy to see that this implies (2.5).

We come now to our main lemma.

Lemma 4. *Let $\sigma<\infty$ and $c=2/\sigma^2$. Then*

$$(1-z)\,h(z)\to c \quad as \quad z\to 1,\ z\in\mathbb{D}. \tag{2.7}$$

Proof. (a) We show first that $(1-z)\,h(z)$ has the *angular* limit c at 1. If ψ is univalent and $\neq 0$ in $\mathbb{D}$ then [5, Th. 5.1]

$$\left|\frac{\psi'(s)}{\psi(s)}\right|\leqq\frac{4}{1-|s|^2} \qquad (s\in\mathbb{D}). \tag{2.8}$$

Let the univalent function φ map $\mathbb{D}$ onto G such that $\varphi(1)=1$ and $\varphi(\xi)$ is real for real ξ. Then $1-\varphi(s)$ and $\psi(s)=h(\varphi(s))$ are univalent and $\neq 0$ in $\mathbb{D}$ (if ρ is sufficiently small). Hence $g(s)=(1-\varphi(s))\,\psi(s)$ $(s\in\mathbb{D})$ satisfies

$$(1-|s|^2)\left|\frac{g'(s)}{g(s)}\right|\leqq\frac{(1-|s|^2)\,|\varphi'(s)|}{|1-\varphi(s)|}+\frac{(1-|s|^2)\,|\psi'(s)|}{|\psi(s)|}\leqq 8$$

by (2.8). We conclude that g is a normal function [7, Sect. 9.1].

Since $g(\xi)=(1-\varphi(\xi))\,h(\varphi(\xi))\to c$ as $\xi\to 1-0$ by Lemma 3, it follows from a result on normal functions [7, Th. 9.3] that g has the angular limit c at 1. Since φ is analytic at 1 and $\varphi'(1)\neq 0$ (by the reflection principle), we conclude that

$$(1-z)\,h(z)\to c \quad \text{as} \quad z\to 1,\ |\arg(1-z)|<\alpha \tag{2.9}$$

for each $\alpha<\pi/2$.

(b) We set

$$w\equiv u+iv=\frac{1}{1-z}, \qquad F(w)=\frac{1}{1-f(z)}, \qquad H(w)=h(z). \tag{2.10}$$

Then F and H are analytic in the halfplane $\{u+iv: u>\frac{1}{2}\}$. The iterates F_n are given by $F_n(w)=1/(1-f_n(z))$. Hence it follows from (2.2) that

$$|F_{n+1}(w)|=|F(F_n(w))|\geqq|F_n(w)| \qquad (u>\tfrac{1}{2}). \tag{2.11}$$

Integration by parts shows that, for $z\in\mathbb{D}$,

$$\frac{f(z)-z}{(1-z)^2}=\int_0^1(1-t)\,f''(1-(1-z)\,t)\,dt. \tag{2.12}$$

Since $\sigma<\infty$ we see from (1.3) that f'' is continuous in $\overline{\mathbb{D}}$. Hence it follows from (2.12) that

$$\frac{f(z)-z}{(1-z)^2}\to\frac{1}{2}f''(1)=\frac{\sigma^2}{2}=\frac{1}{c} \qquad (z\to 1) \tag{2.13}$$

and thus, by (2.10),

$$F(w)-w=\frac{f(z)-z}{(1-z)^2}\,\frac{1-z}{1-f(z)}\to\frac{1}{c} \qquad (|w|\to\infty,\ u>\tfrac{1}{2}).$$

Hence we conclude from (2.11) that

$$\frac{1}{n}(F_n(w)-w)=\frac{1}{n}\sum_{\nu=0}^{n-1}[F(F_\nu(w))-F_\nu(w)]\to\frac{1}{c} \qquad (|w|\to\infty) \tag{2.14}$$

uniformly in n. Hence there exists R such that $|F_n(w)-w-n/c|<n/(2c)$ for $|w|>R$, $u>1/2$, $n\in\mathbb{N}$ and thus

$$\frac{|\operatorname{Im} F_n(w)|}{\operatorname{Re} F_n(w)}<\frac{|v|+n/(2c)}{n/(2c)}=2c\,\frac{|v|}{n}+1. \tag{2.15}$$

(c) We assume now $|w|>R$ and $|v|\geqq u\geqq 1$. Setting $n=[|v|]$ we obtain from (2.15) that

$$\frac{|\operatorname{Im} F_n(w)|}{\operatorname{Re} F_n(w)}<2c\,\frac{n+1}{n}+1\leqq 4c+1. \tag{2.16}$$

It follows from (2.10) and (2.9) that, in particular,

$$\frac{H(w)}{w}\to c \quad \text{as} \quad |w|\to\infty, \qquad \frac{|v|}{u}\leqq 4c+1. \tag{2.17}$$

Since $H(w)=H(F_n(w))-n$ by (2.10) and (1.5), we obtain

$$\frac{H(w)}{w}=\left(\frac{H(F_n(w))}{F_n(w)}\,\frac{F_n(w)-w}{n}-1\right)\frac{n}{w}+\frac{H(F_n(w))}{F_n(w)}.$$

We let now $|w|\to\infty$; note that n depends on w. It follows from (2.16) and (2.17) that $H(F_n(w))/F_n(w)\to c$ and from (2.14) that $(F_n(w)-w)/n\to 1/c$. Since $|n/w|\leqq(|v|+1)/|w|\leqq 2$ we conclude that

$$H(w)/w\to c \quad \text{as} \quad |w|\to\infty \tag{2.18}$$

if $|v|\geqq u$. The case $|v|\leqq u$ follows from (2.17). Thus (2.18) always holds, and by (2.10) this is equivalent to the assertion (2.7).

3. Proof of the Theorem

(a) Let first $\sigma<\infty$ and thus $c=2/\sigma^2>0$. We set

$$g(z)=h(z)-\frac{c}{1-z} \qquad (z\in\mathbb{D}). \tag{3.1}$$

Since $g'(z)=\sum_{k=0}^{\infty}(k+1)(\eta_{k+1}-c)z^k$ we see that, for $0<r<1$,

$$(k+1)(\eta_{k+1}-c)=\frac{1}{2\pi r^k}\int_{-\pi}^{\pi} g'(re^{it})\,e^{-ikt}\,dt \qquad (k=0,1,\ldots). \tag{3.2}$$

We set

$$r_k = \frac{k-1}{k+1}, \qquad r'_k = \frac{k}{k+1} \qquad (k=2, 3, \ldots). \tag{3.3}$$

Since $r_k^{k-1} \geqq e^{-2}$ we obtain from (3.2) that

$$|\eta_{k+1} - c| \leqq 2 \int_{r_k}^{r'_k} \int_{-\pi}^{\pi} |g'(re^{it})| \, r \, dt \, dr. \tag{3.4}$$

(b) Let $0 < \varepsilon < 1$ be given and let $\lambda = 1/\varepsilon^2$. We denote by $K_1, K_2, \ldots$ constants independent of ε and k. Let ρ be as in Lemma 1. Then

$$G_k = \{z : r_k < |z| < r'_k, \ \lambda/k < |\arg z| < \rho\} \subset G \tag{3.5}$$

for $k > k_1(\varepsilon)$. It follows from Lemma 4 that

$$|h(z)| \leqq \frac{K_1}{|1-z|} \leqq \frac{K_2 k}{\lambda} \qquad (z \in G_k). \tag{3.6}$$

Let $k > k_1$. We obtain from the Schwarz inequality that

$$\Big(\iint_{G_k} |h'(z)| \, dx \, dy\Big)^2 \leqq \iint_{G_k} |h(z)|^{-\frac{3}{2}} |h'(z)|^2 \, dx \, dy \iint_{G_k} |h(z)|^{\frac{3}{2}} \, dx \, dy, \tag{3.7}$$

and since $G_k \subset G$ the univalent substitution $w = h(z)$ shows that the first right-hand integral is (see (3.6))

$$= \iint_{h(G_k)} |w|^{-\frac{3}{2}} \, du \, dv \leqq \iint_{|w| \leqq K_2 k/\lambda} |w|^{-\frac{3}{2}} \, du \, dv = K_3 (k/\lambda)^{\frac{1}{2}}.$$

Since $|1 - re^{it}| \geqq \sqrt{r} \, |1 - e^{it}|$, the second right-hand integral in (3.7) is, by (3.6),

$$\leqq K_4 \int_{r_k}^{r'_k} \int_{\lambda/k}^{\rho} |1 - e^{it}|^{-\frac{3}{2}} \, dt \, dr \leqq \frac{K_5}{k+1} \left(\frac{k}{\lambda}\right)^{\frac{1}{2}} < K_5 (k\lambda)^{-\frac{1}{2}}$$

because of (3.3). Hence we obtain from (3.7) that

$$\int_{r_k}^{r'_k} \int_{\lambda/k}^{\rho} |h'(re^{it})| \, r \, dt \, dr \leqq K_6 \lambda^{-\frac{1}{2}} = K_6 \varepsilon. \tag{3.8}$$

The integral over $[\rho, \pi]$ instead of $[\lambda/k, \rho]$ is $\leqq K_7 k^{-1}$ by Lemma 2 and (3.3). Furthermore

$$\int_{r_k}^{r'_k} \int_{\lambda/k}^{\pi} \frac{c}{|1 - re^{it}|^2} \, r \, dt \, dr \leqq \frac{c}{k+1} \int_{\lambda/k}^{\pi} \frac{1}{|1 - e^{it}|^2} \, dt \leqq \frac{K_8}{\lambda} \leqq K_8 \varepsilon.$$

Hence we see from (3.1) and (3.8) that, for $k > k_1(\varepsilon)$,

$$\int_{r_k}^{r'_k} \int_{\lambda/k}^{2\pi - \lambda/k} |g'(re^{it})| \, r \, dt \, dr \leqq K_9 \varepsilon + K_{10} k^{-1} \tag{3.9}$$

because the range $[\pi, 2\pi - \lambda/k]$ can be treated in a similar way.

(c) Since $(1-z)g(z)\to 0$ as $z\to 1$ by (3.1) and Lemma 4, we can find $k_2 = k_2(\varepsilon) > k_1$ such that

$$|(1-z)\,g(z)| < \varepsilon^3 \quad \text{for } |\arg z| \leqq \frac{\lambda}{k}, \quad r_k \leqq |z| \leqq r'_k.$$

Hence it follows from the Cauchy integral formula for the derivative that

$$\left|\frac{d}{dz}\,[(1-z)\,g(z)]\right| < K_{10}\,\varepsilon^3/(1-r'_k) = K_{10}(k+1)\,\varepsilon^3$$

for $|\arg z| \leqq \lambda/k$ and $r_k \leqq |z| \leqq r'_k$ hence

$$|(1-z)^2\,g'(z)| = \left|(1-z)\,g(z) + (1-z)\,\frac{d}{dz}\,[(1-z)\,g(z)]\right| < \varepsilon^3$$

$$+ K_{11}\,\frac{\lambda}{k}\,(k+1)\,\varepsilon^3 < K_{12}\,\varepsilon$$

because $\lambda = \varepsilon^{-2}$. Therefore we see that, for $k > k_2$,

$$\int_{r_k}^{r'_k} \int_{-\lambda/k}^{\lambda/k} |g'(re^{it})|\, r\,dt\,dr \leqq \int_{r_k}^{r'_k} \int_{-\pi}^{\pi} \frac{K_{12}\,\varepsilon}{|1-re^{it}|^2}\,dt\,dr$$

$$= K_{12} \int_{r_k}^{r'_k} \frac{2\pi\varepsilon}{1-r^2}\,dr \leqq 2\pi K_{12}\,\varepsilon.$$

Hence we conclude from (3.4) and (3.9) that

$$|\eta_{k+1} - c| \leqq K_{13}\,\varepsilon + 2K_{10}\,k^{-1} \qquad (k > k_2(\varepsilon)).$$

This proves $\eta_k \to c$ $(k\to\infty)$ for the case $\sigma < \infty$.

(d) Let now $\sigma = \infty$ and thus $c = 0$. Since $\eta_k \geqq 0$ by (1.1) and (1.4), we obtain from Lemma 3 that

$$|h(z)| \leqq h(|z|) = o\left(\frac{1}{1-|z|}\right) \qquad (|z|\to 1). \tag{3.10}$$

It is well-known that (3.10) implies

$$\eta_k \to 0 \quad \text{as} \quad k\to\infty \tag{3.11}$$

in the case that h is univalent in $\mathbb{D}$; compare [7, Th. 5.3].

In our case we use the standard proof of (3.11) in the domain $G = \mathbb{D} \cap \{|z-1| < \rho\}$ where h is univalent, and we make trivial estimates in $\mathbb{D} \setminus G$ where h' is bounded, by Lemma 2.

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