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Limiting Point Processes in the Branching Random Walk

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Summary. Take the nth generation of a supercritical branching random walk (a spatially homogeneous branching process) as a process of cluster centres and take independent copies of some simple point process Y as the clusters. Let the resulting point process be Y_n. For a given sequence of real numbers $\{x_n\}$ let Y_n be centred on x_n. Under certain conditions, when an appropriate scale change is made, the resulting point process converges in distribution to a non-trivial limit.

1. Introduction

In this paper we will consider the supercritical branching random walk (i.e. the spatially homogeneous supercritical branching process) on the real line $\mathbb{R}$ [1, 6]. We will start the process from a single initial ancestor at the origin, though this is not crucial. Let $Z^{(n)}$ be the point process of the positions of the nth generation people. We obtain $Z^{(n+1)}$ from $Z^{(n)}$ by clustering independent copies of $Z^{(1)}$ on each point of $Z^{(n)}$.

More formally, let the translation operator T_y be defined on the continuous functions of compact support, C_0, by

$$T_y f(x) = f(x+y).$$

Let the operator induced by T_y on the locally finite measures on $\mathbb{R}$, $\mathfrak{M}$, also be denoted by T_y so that for any $\mu \in \mathfrak{M}$

$$(T_y \mu) f = \mu(T_y f) \qquad \text{for all } f \in C_0.$$

Let the positions of the nth generation people be $\{z_{n,r}: r\}$ and let $Z_{n,r}$ be the independent copy of $Z^{(1)}$ associated with the person at $z_{n,r}$. Then

$$Z^{(n+1)} = \sum_r T_{z_{n,r}} Z_{n,r}.$$

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We will also associate with each person an independent copy of a simple point process Y, $Y_{n,r}$ being the one associated with $z_{n,r}$. Now define the point process Y_n by

$$Y_n = \sum_r T_{z_{n,r}} Y_{n,r} \tag{1.1}$$

so that, when $Z^{(1)}$ is simple, we can take $Z^{(1)} = Y$ and then $Z^{(n+1)} = Y_n$. The introduction of these $Y_{n,r}$ is similar to the notion of a random characteristic in the age-dependent branching process, [3]. Let $\mathscr{F}^{(n)}$ be the σ-field generated by $\{\{(Z_{m,r}, Y_{m,r}):r\}: m = 1, 2, \ldots, n-1\}$, the σ-field containing all information about the first n generations.

We are concerned with certain limiting properties of the point process Y_n. From (1.1) we can see that, given $\mathscr{F}^{(n)}$, Y_n is the superposition of a number, which increases with n, of independent simple point processes. It is certainly plausible then, because of Grigelionis' theorem [2], that some suitably scaled version of Y_n should converge to a Poisson process. We will have to expand the scale of Y_n and so we introduce the scale operator S_y defined on C_0 by

$$S_y f(x) = f(xy)$$

and on $\mathfrak{M}$ by

$$(S_y \mu) f = \mu(S_y f) \quad \text{for all } f \in C_0.$$

We are going to suppose that a sequence of real numbers $\{x_n\}$ is given and examine $T_{x_n} Y_n$. Specifically we will seek $\{K_n\}$ such that

$$S_{K_n} T_{x_n} Y_n$$

converges in distribution to a non-trivial limit.

In the next section some more notation is introduced together with a heuristic argument leading to a formulation of the results to be proved. The final section contains the proofs.

2. The Results

As we will be expanding the scale of Y_n it is not surprising that a smoothness condition on the intensity measure of Y is required; in fact we will assume that

(A1) $\mathbf{E}Y = \nu$ is a finite measure with a bounded continuous density function g (with respect to Lebesgue measure).

Let $\mathscr{B}$ be the bounded Borel subsets of $\mathbb{R}$, and let $\mathscr{I}$ be the bounded intervals. We will use A both for a set and for its indicator function I_A when no confusion will result. Let B_a be the interval $(-a, a)$, and let m be Lebesgue measure on $\mathbb{R}$. Now consider the intensity measure of $T_{x_n} Y_n$ conditional on $\mathscr{F}^{(n)}$,

$$\mathbf{E}[T_{x_n} Y_n | \mathscr{F}^{(n)}] = \sum_r T_{x_n} T_{z_{n,r}} \nu;$$

we need to arrange that, after rescaling, this intensity measure converges to a non-trivial limit. For $A \in \mathscr{B}$

$$\sum_r T_{x_n} T_{z_{n,r}} \nu A = \sum_r \int_A g(y - x_n - z_{n,r})\, m(dy) \\ = \int_A \left(\int g(y-z)\, T_{x_n} Z^{(n)}(dz)\right) m(dy). \tag{2.1}$$

If we write

$$\tilde{g}(-x) = g(x)$$

then, provided that A is a small set near the origin (2.1) should be approximated by

$$\left(\int_A m(dy)\right)(T_{x_n} Z^{(n)} \tilde{g}) = (mA)(T_{x_n} Z^{(n)} \tilde{g}).$$

Hence, approximately,

$$\mathbf{E}[S_{K_n} T_{x_n} Y_n A \,|\, \mathscr{F}^{(n)}] = (mA)\, K_n^{-1} (T_{x_n} Z^{(n)} \tilde{g}).$$

This suggests that the appropriate choice for K_n is $T_{x_n} Z^{(n)} \tilde{g}$. It also explains why a condition on $T_{x_n} Z^{(n)}$ is to be expected. We will make the following assumption.

(A 2) There exists constants $\{k_n\}$ tending to infinity such that

$$k_n^{-1}\, T_{x_n} Z^{(n)} \to \zeta \text{ a.s.} \tag{2.2}$$

where ζ is a random measure.

The convergence is with respect to the vague topology on $\mathfrak{M}$.

It is easy to deduce from Theorem 2 of [1] that, when the conditions of that theorem hold, (2.2) holds with $x_n = nb$, and the sequence k_n can be described quite precisely as can the limit measure ζ. Theorem 1 of [5] can also be reformulated to yield a result like (2.2). Hence the assumption (A 2) is certainly non-vacuous. We will also assume that

(A 3) $$k_n^{-1}\, T_{x_n} Z^{(n)} \tilde{g} \to \zeta \tilde{g} < \infty \quad \text{a.s.}$$

Of course if $\tilde{g} \in C_0$ then (A 3) is implied by (A 2). Furthermore if the convergence in (A 2) is with respect to the weak topology on $\mathfrak{M}$ then (A 3) would not be needed.

Theorem 1. *Suppose that* (A 1), (A 2) *and* (A 3) *hold. If* $h \in C_0$ *let* $K_n = T_{x_n} Z^{(n)} h$ *then, given* ζ,

$$S_{K_n} T_{x_n} Y_n \xrightarrow{d} \eta$$

on $\{\zeta h > 0\}$ *and there* η *is a Poisson process of rate* $\zeta \tilde{g} / \zeta h$.

Essentially the same proof as that of Theorem 1 yields the following two results, under the same conditions

Corollary 1. *If* $K_n = T_{x_n} Z^{(n)} \tilde{g}$ *then, given* ζ,

$$S_{K_n} T_{x_n} Y_n \xrightarrow{d} \eta$$

on $\{\zeta \tilde{g} > 0\}$ *and there* η *is a Poisson process of unit rate.*

Corollary 2

$$S_{k_n} T_{x_n} Y_n \xrightarrow{d} \eta$$

where η *is a mixed Poisson process, with random rate* $\zeta \tilde{g}$.

It is notationally more complicated but, in fact, more natural to take Y to be a marked point process on $\mathbb{R}$. We still assume that Y is a simple point process on $\mathbb{R}$ but now we also assume that each point has associated with it a mark, a label, drawn from some mark space. We will consider the mark space to be $\{1, 2, 3, \ldots\} = \mathbb{N}$ in Theorem 2. Let X_k be the point process, on $\mathbb{R}$, formed by considering only those points of Y with the mark k. Let us replace (A1) by

(A1)′ (A1) holds and, for each $k \in \mathbb{N}$, $\mathbf{E} X_k$ has a continuous density g_k on $\mathbb{R}$.

Theorem 2. *Suppose that* (A1)′, (A2) *and* (A3) *hold. Let* $K_n = T_{x_n} Z^{(n)} \tilde{g}$ *then, given* ζ,

$$(S_{k_n} T_{x_n} Y_n) \xrightarrow{d} \eta$$

where, on $\{\zeta \tilde{g} > 0\}$, η *is a Poisson process of rate one. The points of* η *have marks in* $\mathbb{N}$, *chosen independently, with a mark* k *occurring with probability* $\zeta \tilde{g}_k / \zeta \tilde{g}$.

We can consider $Z^{(1)}$ to be a marked simple point process with marks, corresponding to the multiplicity of the points, in $\mathbb{N}$. Viewed in this way it is easy to apply Theorem 2 to $S_{k_n} T_{x_n} Z^{(n+1)}$.

3. The Proofs

Notice that

$$S_{K_n} T_{x_n} Y_n = \sum_r S_{K_n} T_{x_n} T_{z_{n,r}} Y_{n,r} = \sum_r \gamma_{n,r}$$

where, given $\mathscr{F}^{(n)}$, $\{\gamma_{n,r} : r\}$ are independent simple point processes. It is fairly clear that $\{\gamma_{n,r}\}$ form a null array. To prove this observe that, for $a > 0$,

$$\begin{aligned} \sup\{\mathbf{P}(T_x Y B_a > 0) : x\} &\leqq \sup\{\mathbf{E} T_x Y B_a : x\} \\ &= \sup\{T_x v B_a : x\} \leqq 2\|g\| a \end{aligned}$$

where $\|g\| = \sup\{|g(x)| : x \in \mathbb{R}\}$. By (A2) $K_n \to \infty$ almost surely on $\{\zeta h > 0\}$ and so

$$\sup_r P(\gamma_{n,r} B_a > 0 \mid \mathscr{F}^{(n)}) \leqq 2\|g\| \frac{a}{K_n} \to 0 \text{ a.s. on } \{\zeta h > 0\}.$$

In the remainder of this section a.s. will mean a.s. on $\{\zeta h > 0\}$. By Corollary 10.10 of [4] it now suffices to show that

$$\sum_r \mathbf{P}(\gamma_{n,r} A > 0 | \mathscr{F}^{(n)}) \to \frac{\zeta \tilde{g}}{\zeta h} mA \text{ a.s.} \tag{3.1}$$

for $A \in \mathscr{I}$, and that

$$\sum_r \mathbf{E}[\gamma_{n,r} B \, I_{\{\gamma_{n,r} B > 1\}} | \mathscr{F}^{(n)}] \to 0 \text{ a.s.} \tag{3.2}$$

for $B \in \mathscr{B}$. We will establish that

$$v_n = \sum_r \mathbf{E}[\gamma_{n,r} | \mathscr{F}^{(n)}] \to \frac{\zeta \tilde{g}}{\zeta h} m \text{ a.s.} \tag{3.3}$$

(in the vague topology) but first we will show that this implies both (3.1) and (3.2) here.

Suppose that, for some $\varepsilon > 0$, no two points of Y are within ε of one another. For any Borel set $A \subset B_a$

$$\begin{aligned} \sum_r \mathbf{E}[\gamma_{n,r} A | \mathscr{F}^{(n)}] = & \sum_r \mathbf{P}(\gamma_{n,r} A > 0 | \mathscr{F}^{(n)}) \\ & + \sum_r \mathbf{E}[(\gamma_{n,r} A - 1) \, I_{\{\gamma_{n,r} A > 1\}} | \mathscr{F}^{(n)}]. \end{aligned} \tag{3.4}$$

When $K_n^{-1} a \leqq \varepsilon$ there is at most one point of $\gamma_{n,r}$ in A, and so, because $K_n \to \infty$, we can see that when (3.3) holds so does (3.2) and then, from (3.4), (3.1) holds also. A fairly simple truncation argument now shows that (3.1) and (3.2) hold in general. Let Y^ε be obtained from Y by deleting all points within ε of one another. All quantities in the process based on Y^ε rather than Y will be denoted by a superscript ε. We have for $A \in \mathscr{I}$

$$\begin{aligned} \sum_r \mathbf{E}[\gamma_{n,r} A | \mathscr{F}^{(n)}] = & \sum_r \mathbf{P}(\gamma_{n,r} A > 0 | \mathscr{F}^{(n)}) + \sum_r \mathbf{E}[(\gamma_{n,r} A - 1) \, I_{\{\gamma_{n,r} A > 1\}} | \mathscr{F}^{(n)}] \\ & \geqq \sum_r \mathbf{P}(\gamma_{n,r} A > 0 | \mathscr{F}^{(n)}) \geqq \sum_r \mathbf{P}(\gamma_{n,r}^\varepsilon A > 0 | \mathscr{F}^{(n)}) \to \frac{\zeta \tilde{g}^\varepsilon}{\zeta h} mA \text{ a.s.} \end{aligned} \tag{3.5}$$

Hence, using (3.3),

$$\begin{aligned} \frac{\zeta \tilde{g}}{\zeta h} mA & \geqq \limsup_n \sum_r \mathbf{P}(\gamma_{n,r} A > 0 | \mathscr{F}^{(n)}) \\ & \geqq \liminf_n \sum_r \mathbf{P}(\gamma_{n,r} A > 0 | \mathscr{F}^{(n)}) \geqq \frac{\zeta \tilde{g}^\varepsilon}{\zeta h} mA, \end{aligned}$$

and, letting $\varepsilon \downarrow 0$, $\zeta \tilde{g}^\varepsilon \uparrow \zeta g$ so that (3.1) holds. Now

$$\mathbf{E}[\gamma_{n,r} A \, I_{\{\gamma_{n,r} A > 1\}} | \mathscr{F}^{(n)}] \leqq 2 \mathbf{E}[(\gamma_{n,r} A - 1) \, I_{\{\gamma_{n,r} A > 1\}} | \mathscr{F}^{(n)}]$$

and so (3.3) and (3.1), together with (3.5), suffice to establish (3.2).

It only remains to establish (3.3). Let

$$\delta(\varepsilon, a) = \sup_{|x| \leqq a} \sup_{|y| \leqq \varepsilon} |\tilde{g}(x+y) - \tilde{g}(x)|.$$

As g is continuous and so uniformly continuous on compact sets we have, for any $a>0$,

$$\delta(\varepsilon, a)\downarrow 0 \text{ as } \varepsilon\downarrow 0.$$

Let A^c denote the complement of the set A.

Lemma. *For* $\mu\in\mathfrak{M}$, $|y|\leqq\varepsilon_1<\varepsilon$ *and* $a>0$

$$|T_y\mu\tilde{g}-\mu\tilde{g}|\leqq\delta(\varepsilon_1, a+\varepsilon)\,\mu B_{a+\varepsilon}+2\mu\tilde{g}B_a^c.$$

Proof.

$$\begin{aligned}|T_y\mu\tilde{g}-\mu\tilde{g}|&=|\textstyle\int(\tilde{g}(x+y)-\tilde{g}(x))\,\mu(dx)|\leqq\int|\tilde{g}(x+y)-\tilde{g}(x)|\mu(dx)\\&\leqq\int_{B_{a+\varepsilon}}|\tilde{g}(x+y)-\tilde{g}(x)|\mu(dx)+\int_{B^c_{a+\varepsilon}}(\tilde{g}(x+y)+\tilde{g}(x))\,\mu(dx)\\&\leqq\delta(\varepsilon_1, a+\varepsilon)\,\mu B_{a+\varepsilon}+2\mu\tilde{g}B_a^c.\end{aligned}$$

Now for $f\in C_0$

$$\begin{aligned}v_n f&=\mathbf{E}[\textstyle\sum_r\gamma_{n,r}f\,|\,\mathscr{F}^{(n)}]\\&=\textstyle\sum_r\int f((x_n+z_{n,r}+x)\,K_n)\,g(x)\,m(dx)\\&=\textstyle\iint f((x_n+z+x)\,K_n)\,g(x)\,m(dx)\,Z^{(n)}(dz)\\&=\textstyle\iint f(K_n y)\,g(y-x_n-z)\,Z^{(n)}(dz)\,m(dy)\\&=\textstyle\int f(K_n y)(T_{-y}\,T_{x_n}Z^{(n)}\tilde{g})\,m(dy).\end{aligned}\tag{3.6}$$

We will show that, for any $f\in C_0$,

$$|v_n f-\frac{\zeta\tilde{g}}{\zeta h}mf|\to 0 \text{ a.s.}\tag{3.7}$$

as $n\to\infty$, which is equivalent to (3.3). Note first that

$$\left|v_n f-\frac{\zeta\tilde{g}}{\zeta h}mf\right|\leqq\left|v_n f-\frac{T_{x_n}Z^{(n)}\tilde{g}}{T_{x_n}Z^{(n)}h}mf\right|+|mf|\left|\frac{T_{x_n}Z^{(n)}\tilde{g}}{T_{x_n}Z^{(n)}h}-\frac{\zeta\tilde{g}}{\zeta h}\right|$$

where the final term tends to zero as n tends to infinity; using (3.6) and the definition of K_n the other term on the right of this inequality can be written as

$$|\textstyle\int f(K_n y)(T_{-y}\,T_{x_n}Z^{(n)}\tilde{g}-T_{x_n}Z^{(n)}\tilde{g})\,m(dy)|.\tag{3.8}$$

Let a be sufficiently large that $|f|\leqq\|f\|B_a$. Now fix $\varepsilon>0$. For n sufficiently large $|y|\leqq a/K_n<\varepsilon$ and so, using the lemma (3.8) is less than

$$\begin{aligned}&\textstyle\int|f(K_n y)|\{\delta(\frac{a}{K_n}, a+\varepsilon)\,T_{x_n}Z^{(n)}B_{a+\varepsilon}+2\,T_{x_n}Z^{(n)}\tilde{g}B_a^c\}\,m(dy)\\&\leqq\frac{m|f|}{K_n}\{\delta\left(\frac{a}{K_n}, a+\varepsilon\right)T_{x_n}Z^{(n)}B_{a+\varepsilon}+2\,T_{x_n}Z^{(n)}\tilde{g}B_a^c\}\\&\to m|f|\left\{0.\frac{\zeta B_{a+\varepsilon}}{\zeta h}+\frac{2\zeta\tilde{g}B_a^c}{\zeta h}\right\}.\end{aligned}$$

Here a is arbitrary and $\zeta \tilde{g} B_a^c \to 0$ as $a \to \infty$. This completes the proof of (3.7) and hence of Theorem 1.

Only obvious modifications are needed in this proof to prove Corollaries 1 and 2.

Essentially the same proof works in proving Theorem 2. Here Y, and also $\{\gamma_{n,r}\}$, must be regarded as a point process on $\mathbb{R} \times \mathbb{N}$, rather than $\mathbb{R}$. The proof above that it suffices to establish (3.3) is essentially the same; it depends only on the assumptions that Y is simple as a point process on $\mathbb{R}$ and that g is bounded and continuous. Notice that (3.3) involves measures not on $\mathbb{R}$ but on $\mathbb{R} \times \mathbb{N}$. If we write functions and measures on $\mathbb{R} \times \mathbb{N}$ in co-ordinate form we must prove that

$$\sum_{i=1}^{k} (v_n)_i\, f_i \to \sum_{i=1}^{k} \frac{\zeta \tilde{g}_i}{\zeta \tilde{g}} m f_i$$

for $k \in N$ and $f_i \in C_0$. This follows in much the same way as (3.7) did. This proves that $S_{K_n} T_{x_n} Y_n$ converges to a Poisson process on $\mathbb{R} \times \mathbb{N}$ with its rate on $\mathbb{R} \times \{i\}$ given by $\zeta \tilde{g}_i / \zeta \tilde{g}$, which is equivalent to the assertion of the theorem.

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