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Subprocesses of Stationary Markov Processes

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1. Introduction

1.1 The so-called killing operation plays an important role in the theory of Markov processes and its applications. Given a Markov process X in a space E, this operation enables us to construct a new Markov process $K(X)$ on any rather "good" part D of this space. But stationarity of the process is lost under such a transformation. In this paper we shall study another operation Q, which transforms a stationary Markov process X (a Markov process under a stationary distribution and with time parameter set $]-\infty, +\infty[$) in the space E into a process of the same type with a state space $D\subset E$. But unlike the initial process X, the process $Q(X)$ has random birth and death times, and the corresponding measure in the space of paths can be infinite. To distinguish such processes from the traditional stationary processes, we call the latter "conservative processes". The transition probabilities of the process $Q(X)$ are equal to those of $K(X)$; and the one-dimensional distributions of $Q(X)$ and X are equal on D. But in contrast to K, the operation Q is invariant under time reversal. We are interested only in the case in which the one-dimensional distributions of X are concentrated on D and, therefore, are equal to those of $Q(X)$.

The main part of the paper is devoted to the inverse problem: for a given stationary Markov process Y in the space D to construct a conservative stationary Markov process in a space $E\supset D$ such that $Q(X)=Y$. It is obvious that for the possibility of such a construction, it is necessary for the one-dimensional distributions of Y to be probability measures. We show that this condition is also sufficient. We also give a sufficient condition for X to be uniquely determined by Y (we do not distinguish two processes having the same finite dimensional distributions).

We always use the same letter for measure and integral with respect to this measure. Thus, $\mathbf{P}\{\xi\}, \mathbf{P}$ being a probability measure and ξ being a random variable, denotes the mathematical expectation of ξ.

By the expression "a function on X", X being a measurable space, we mean a measurable bounded nonnegative function.

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We denote by $\mathscr{B}(X)$ the collection of all measurable subsets of a Borel space X. Writing $\Gamma \subset X$ means the same as $\Gamma \in \mathscr{B}(X)$

1.2 Let $(x_t(\omega), \overline{\mathbf{P}})(\omega \in \Omega, t \in T =]-\infty, +\infty[)$ be a Markov process in a space $E = D \cup V$. Suppose that the set $M = \{t: x_t \in V\}$ is closed a.s. $\overline{\mathbf{P}}$, and M is local measurable, that is

1.2.α For each $s<t$ the set $M \cap]s,t[$ is $\mathscr{B}]s,t[\times \mathscr{F}_{]s,t[}$-measurable, where $\mathscr{F}_{]s,t[}$ is the completion with respect to the measure $\overline{\mathbf{P}}$ of $\sigma(x_u, s<u<t)$.

The complement of M is a union of a countable number of open intervals $]\gamma, \delta[$. Let us denote by W the set of all paths in D defined on all open intervals $]\alpha, \beta[$. We associate with every ω and every $]\gamma, \delta[$ an element $w_\delta^\gamma(\omega)$ of W defined by the formula $w_\delta^\gamma(t) = x_t, \gamma < t < \delta$. Set $G = \sigma(w(s), s \in T)$ and for every $A \in G$ set

$$\mathbf{P}\{A\} = \overline{\mathbf{P}} \sum_\gamma 1_A(w_\delta^\gamma). \tag{1.2.1}$$

We denote the process $(w(s), \mathbf{P})$ by $\underline{Q}(x_t, \overline{\mathbf{P}})$ and we say that $(w(s), \mathbf{P})$ is a subprocess of $(x_t, \overline{\mathbf{P}})$ in D and that $(x_t, \overline{\mathbf{P}})$ is a covering process for $(w(s), \mathbf{P})$. The following expression for finite dimensional distributions of $\mathbf{P}$ follows from (1.2.1).

$$\begin{aligned}&\mathbf{P}\{w(s_1) \in \Gamma_1, \ldots, w(s_n) \in \Gamma_n\}\\ &\quad = \overline{\mathbf{P}}\{x_{s_1} \in \Gamma_1, \ldots, x_{s_n} \in \Gamma_n, [s_1, s_n] \cap M = \emptyset\}.\end{aligned} \tag{1.2.2}$$

Formula (1.2.2) implies that if $(x_t, \overline{\mathbf{P}})$ is Markovian or stationary then so is $(w(s), \mathbf{P})$. If the one-dimensional distributions of $(x_t, \overline{\mathbf{P}})$ are concentrated on D then the process $(w(s), \mathbf{P})$ satisfies the following relation.

1.2.A *For each* s $\mathbf{P}\{w(s) \in D\} = 1$.

In this paper we deal with (general) Markov processes with random birth and death times and it is worthwhile to give a precise definition of such processes. Let $(\Omega, \mathscr{F})$ be a measurable space and $\mathbf{P}$ be a σ-finite measure on $\mathscr{F}$. Suppose that two measurable functions $\alpha(\omega)$ and $\beta(\omega)$ $(\alpha(\omega) < \beta(\omega))$ are given; and suppose that for each $t \in T$, $x_t(\omega)$ is a measurable mapping of the set $\{\alpha(\omega) < t < \beta(\omega)\}$ into a Borel space E. We say that $(x_t, \mathbf{P})$ is a (homogeneous) Markov process if the measure $v_t(\Gamma) = \mathbf{P}\{x_t \in \Gamma\}$ is σ-finite and there exists a transition function p such that

$$\begin{aligned}&\mathbf{P}\{x_{t_1} \in dx_1, x_{t_2} \in dx_2, \ldots, x_{t_n} \in dx_n, \alpha < t_1, \beta > t_n\}\\ &\quad = v_{t_1}(dx_1) p(t_2 - t_1, x_1; dx_2) \ldots p(t_n - t_{n-1}, x_{n-1}; dx_n).\end{aligned}$$

If v_t does not depend on t then the process $(x_t, \mathbf{P})$ is stationary.

The main results of the present paper are given by Theorems 1 and 2.

Theorem 1. *Any stationary Markov process subject to* 1.2.A *is a subprocess of a conservative stationary Markov process.*

Let p be a transition function on D and let $\mathbf{R}$ be a measure on W. We denote by G_u the minimal σ-algebra in W generated by all sets $\{w: w(s) \in B,\ s \leqq u,$

$B \in \mathscr{B}(D)\}$. We put $\mathbf{R} \in S(p)$ if $(w(s), \mathbf{R})$ is a stationary process and

$$\mathbf{R}\{w(s) \in B | G_u\} = p(s-u, w(u); B), \qquad s>u, \; B \subset D.$$

A measure $\mathbf{R} \in S(p)$ is called a minimal element of $S(p)$ if for every $\mathbf{R}_1, \mathbf{R}_2 \in S(p)$ such that $\mathbf{R} = \mathbf{R}_1 + \mathbf{R}_2, \mathbf{R}_1$ and $\mathbf{R}_2$ are proportional to $\mathbf{R}$.

Theorem 2. *If* $\mathbf{P}$ *is a minimal element of* $S(p)$ *and if* $(w(s), \mathbf{P})$ *satisfies* 1.2.A, *then there exists only one conservative process covering* $(w(s), \mathbf{P})$.

All the theory is invariant with respect to time reversal. Therefore, the theorem dual to Theorem 2 is also valid where the class $S(p)$ is replaced by a class of processes having a fixed backward transition funtion.

1.3 Now we give an example of a family of Markov processes with identical subprocesses in D. Let $E = T$, $V = \{0\}$ and $D = T \setminus V$. We start from a diffusion process X_t^0 on E which has an invariant distribution v and transition function p' such that $p'(t, x; \Gamma) = p'(t, -x; -\Gamma)$ (e.g., Ornstein-Uhlenbeck process). Suppose that a mirror is placed at point 0 at time s. We consider a process X_t which coincides with X_t^0 for $t<s$ and is X_t^0 reflected in the mirrow for $t \geqq s$. Denote the corresponding transition probabilities by $\mathbf{P}_{t,x}^{(s)}$. Note that $\mathbf{P}_{t,x}^{(s)}$ does not depend on t iff $s = -\infty$, or $s = +\infty$. The symmetry principle shows that v is invariant for $\mathbf{P}_{t,x}^{(s)}$ for all s. Consider the family of Markov processes $(x_t, \mathbf{P}^{(s)})$ with transition probabilities $\mathbf{P}_{t,x}^{(s)}$ and one-dimensional distributions v. Let p be the transition function of the process X_t^0 killed at the first hitting time of 0; and let $\mathbf{P}$ be the Markov measure with the transition function p and the one-dimensional distribution v. It is easy to see that the equality (1.2.2) holds for $\overline{\mathbf{P}} = \mathbf{P}^{(s)}$; and as a result, the right hand side of (1.2.2) does not depend on s. Therefore, the subprocess in D of $(x_t, \mathbf{P}^{(s)})$ does not depend on s.

Let v_1 be the restriction of v on $]0, \infty[$ and $v_2 = v - v_1$. (Note that both v_1 and v_2 are excessive with respect to p.) Let $\mathbf{P}_i$, $i = 1, 2$, be the Markov measure with the transition function p and the one-dimensional distribution v_i. The measure $\mathbf{P}$ in our example is the sum of $\mathbf{P}_1$ and $\mathbf{P}_2$, and (in the case of Ornstein-Uhlenbeck's process) both $\mathbf{P}_1$ and $\mathbf{P}_2$ are minimal elements of $S(p)$.

2. Reduction to the Case of Finite α and β

2.1 We consider a measure $\mathbf{P} \in S(p)$, subject to 1.2.A and we try to find a covering process for $(w(s), \mathbf{P})$.

Each $\mathbf{P} \in S(p)$ is a barycenter of a probability measure concentrated on the minimal elements of $S(p)$. For each minimal element $\mathbf{R}$ either

$$\alpha = -\infty \quad \text{a.e. } \mathbf{R},$$

or

$$\alpha > -\infty \quad \text{a.e. } \mathbf{R}.$$

(See [1].) Thus $\mathbf{P}$ can be represented in the form

$$\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2, \tag{2.1.1}$$

where $\mathbf{P}_1, \mathbf{P}_2 \in S(p)$ and

$$\mathbf{P}_1\{\alpha \neq -\infty\} = 0, \tag{2.1.2}$$

$$\mathbf{P}_2\{\alpha = -\infty\} = 0. \tag{2.1.3}$$

Put

$$\begin{aligned} v^i(\Gamma) &= \mathbf{P}_i\{w(t) \in \Gamma\}, \qquad i=1,2; \\ \lambda_i &= v^i(D), \qquad\qquad\quad\; i=1,2. \end{aligned}$$

Lemma 2.1.1. *The measures v^1 and v^2 are singular.*

Proof. 1^0. For the stationary measure $\mathbf{P}_1$

$$\mathbf{P}_1\{w(t) \in D\} = \mathbf{P}_1\{\alpha < t < \beta\} = \mathbf{P}_1\{t < \beta\} \tag{2.1.4}$$

does not depend on t. Thus

$$\mathbf{P}_1\{\beta \neq \infty\} = 0. \tag{2.1.5}$$

The relation dual to (2.1.4) shows that (2.1.5) implies (2.1.2) and hence (2.1.2) and (2.1.5) are equivalent. Therefore

$$\mathbf{P}_2\{\beta = \infty\} = 0. \tag{2.1.6}$$

(If (2.1.6) is not true, then the measure $\mathbf{P}_2'\{A\} = \mathbf{P}_2\{A; \beta = \infty\}$ is a stationary one, which satisfies (2.1.5) and therefore satisfies (2.1.2); and we come to a contradiction with (2.1.3).)

2^0. The formula (2.1.5) implies

$$\mathbf{P}_1\{w(s) \in D\} = \mathbf{P}_1\{w(s) \in D, w(s+t) \in D\} = \int_D v^1(dx)\, p(t,x;D) = v^1(D).$$

Thus

$$p(t,x;D) = 1 \quad \text{a.e. } v^1. \tag{2.1.7}$$

On the other hand (2.1.6) implies

$$\begin{aligned} &\mathbf{P}_2\{w(s) \in D, w(s+t) \in D\} = \mathbf{P}_2\{\alpha < s, \beta \neq s+t\} \\ &\quad = \int v^2(dx)\, p(t,x;D) \to 0 \qquad \text{as } t \to \infty. \end{aligned}$$

Thus

$$p(t,x;D) \to 0 \qquad \text{as } t \to \infty \text{ a.e. } v^2. \tag{2.1.8}$$

Comparing (2.1.7) and (2.1.8), we obtain the statement of the lemma.

2.2 Consider the measures $\lambda_1^{-1}\mathbf{P}_1$ and $\lambda_2^{-1}\mathbf{P}_2$. They both belong to $S(p)$ and satisfy 1.2.A.

The process $X_1 = (w(s), \lambda_1^{-1}\mathbf{P}_1)$ is a covering process for X_1. Suppose we construct a covering process X_2 for $(w(s), \lambda_2^{-1}\mathbf{P}_2)$. The one-dimensional distributions of X_1 and X_2 are respectively $\lambda_1^{-1} v^1$ and $\lambda_2^{-1} v^2$, which are singular. The mixture of X_1 and X_2 with the coefficients λ_1 and λ_2 is a stationary Markov process (as a mixture of two stationary Markov processes with singular one-dimensional distributions). It is easy to see that this mixture is a covering process fo $(w(s), \mathbf{P})$.

In the sequel we shall consider only measures $\mathbf{P}$ for which $\mathbf{P}\{\beta=\infty\}=\mathbf{P}\{\alpha=-\infty\}=0$.

3. Construction of a Covering Process

3.1 In this section we construct a process $(x_t, \overline{\mathbf{P}})$, given its subprocess $(w(s), \mathbf{P})$. The state space for $(x_t, \overline{\mathbf{P}})$ is a union of D and a one point set V.

Suppose now that the process $(x_t, \overline{\mathbf{P}})$ is constructed. Let

$$M(\omega)=\{t: x_t(\omega)=V\}.$$

Applying (1.2.1) to the function $g(w)=f(\alpha(w), \beta(w))$, f being a function on $T\times T$, we get

$$\overline{\mathbf{P}}\sum_{\gamma} f(\gamma, \delta)=\mathbf{P}\{f(\alpha, \beta)\}. \tag{3.1.1}$$

Denote by $I(t)=]L_t, \tau_t[$ the interval contiguous to M which contains the point t. The set M is translation invariant, i.e. for each finite set $t_1, t_2, \ldots, t_n$ and for each t the joint distribution of $I(t_1+t), I(t_2+t), \ldots, I(t_n+t)$ coincides with that of $I(t_1)$, $I(t_2), \ldots, I(t_n)$.

Suppose that the strong Markov property holds for $(x_t, \mathbf{P})$ at least for all stopping times τ_t. Denote by ${}^t w={}^t w(\omega)$ the part of the path $x_t(\omega)$ over the interval $I(t)$. The paths ${}^s w$ and ${}^t w$ are conditionally independent on the set ${}^s w\neq{}^t w$ given (τ_t, x_{τ_t}). But $x_{\tau_t}=V$; therefore for $s>t$ ${}^t w$ and ${}^s w$ are conditionally independent given τ_t. Inasmuch as $\tau_s=\beta({}^s w)$ and $L_t=\alpha({}^t w)$; the conditional independence of ${}^s w$ and ${}^t w$ holds when τ_s, L_s, τ_t and L_t are all fixed, $s\neq t$.

There exists a function $m(x, y; A)$, $x<y\in T$, $A\in G$, such that

$$\overline{\mathbf{P}}\{{}^t w\in A | L_t, \tau_t\}=m(L_t, \tau_t; A) \text{ a.s. } \mathbf{P}$$

(m can be chosen independently from t because ${}^t w={}^s w$ on the set $\{\tau_s>t\}\cup\{L_t<s\}, s>t$). It follows from (1.2.1) that

$$\mathbf{P}\{w\in A | \alpha, \beta\}=m(\alpha, \beta; A).$$

That gives us a clue to constructing $(x_t, \overline{\mathbf{P}})$. First we construct a translation invariant Markov set M, satisfying (3.1.1) and then we "plug" into its contiguous intervals $]\gamma, \delta[$ trajectories w_δ^γ in such a way that M being fixed, they are all conditionally independent with distribution equal to $m(\gamma, \delta; -)$.

3.2 To construct the required set M, consider the one-dimensional distribution of our process

$$\nu(\Gamma)=\mathbf{P}\{w(s)\in\Gamma\}.$$

The measure ν is a probability measure on D and it is p-null excessive (p being the transition function of $(w(s), \mathbf{P})$). It was proved in [1] that ν can be represented in the form

$$\nu(\Gamma)=\int_0^\infty \nu^s(\Gamma)\,ds,$$

where ν^s is an entrance law for p. Denote by $\mathbf{P}^*$ a Markov measure on G with the transition function p and the one-dimensional distributions ν^s (we put $\nu^s\equiv 0$ for $s\leqq 0$, so $\alpha=0$ $\mathbf{P}^*$-a.e.). Denote by $\mathbf{P}_t^*$ the t-shift of measure $\mathbf{P}^*$, that is

$$\mathbf{P}_t^*\{w(s_1)\in\Gamma_1,\dots,w(s_n)\in\Gamma_n\}=\mathbf{P}^*\{w(s_1-t)\in\Gamma_1,\dots,w(s_n-t)\in\Gamma_n\}. \tag{3.2.1}$$

For every $A\in G$ we can write

$$\mathbf{P}\{A\}=\int_{-\infty}^{\infty}\mathbf{P}_t^*\{A\}\,dt. \tag{3.2.2}$$

Now put

$$\Pi(\Gamma)=\mathbf{P}^*\{\beta\in\Gamma\}.$$

In view of (3.2.1) $\mathbf{P}_t^*\{\beta>s\}=\mathbf{P}^*\{\beta>s-t\}$ and we have

$$\begin{aligned}1=\nu(D)=\mathbf{P}\{w(0)\in D\}&=\int_{-\infty}^{\infty}\mathbf{P}_t^*\{w(0)\in D\}\,dt=\int_{-\infty}^{\infty}\mathbf{P}_t^*\{\beta>0\}\,dt\\&=\int_{-\infty}^{\infty}\mathbf{P}^*\{\beta>-t\}\,dt=\int_0^\infty\mathbf{P}^*\{\beta>u\}\,du=\int_0^\infty\Pi(]u,\infty[)\,du.\end{aligned} \tag{3.2.3}$$

The relation (3.2.3) is equivalent to

$$\int_0^\infty x\,\Pi(dx)=1. \tag{3.2.4}$$

Therefore Π satisfies the conditions of Theorem 1 in [2] and we can construct a $(0,\Pi)$-generated translation invariant closed Markov set M (for definitions and properties see [2]). Let $(\tilde\Omega,\tilde{\mathscr{F}},\tilde{\mathbf{P}})$ be a sample space for M and let $]\gamma,\delta[$ denote as usual the intervals contiguous to M.

Lemma 3.2.1. *For any function f in $T\times T$*

$$\tilde{\mathbf{P}}\sum_\gamma f(\gamma,\delta)=\mathbf{P}\{f(\alpha,\beta)\}. \tag{3.2.5}$$

Proof. Due to (3.2.2) the right side of (3.2.5) may be rewritten as

$$\int_{-\infty}^{\infty}\mathbf{P}_t^*\{f(\alpha,\beta)\}\,dt=\int_{-\infty}^{\infty}\mathbf{P}_t^*\{f(t,\beta)\}\,dt. \tag{3.2.6}$$

In view of (3.2.1)

$$\mathbf{P}_t^*\{g(\beta)\}=\mathbf{P}^*\{g(\beta+t)\}=\int g(x+t)\,\Pi(dx);$$

and the right side of (3.2.6) is equal to

$$\int_{-\infty}^{\infty}\left\{\int_0^\infty f(t,t+y)\,\Pi(dy)\right\}dt. \tag{3.2.7}$$

Theorem 1 in [2] yields

$$\tilde{\mathbf{P}}\sum_{\gamma} f(\gamma,\delta)=c\int_{-\infty}^{\infty}\left\{\int_{0}^{\infty} f(t,t+y)\,\Pi(dy)\right\}dt, \tag{3.2.8}$$

where c is given by (1.5) in [2]. By virtue of (3.2.4) $c=1$ and (3.2.7) is equal to (3.2.8).

3.3 Unfortunately the function $m(x,y;-)$, which represents the conditional distribution of w_δ^γ given $\gamma=x$, $\delta=y$, cannot be obtained as a kernel from $T\times T$ into W, but only as a quasi kernel (as defined below). That is why to justify the definition of measure $\overline{\mathbf{P}}$ given by (3.3.3) we need Theorem 3.3.1.

Let $(X,\mathscr{A})$ and $(Y,\mathscr{B})$ be two measurable spaces and $\mathbf{Q}$ be a measure on $\mathscr{A}$. We say that $n(x;\Gamma)$, $x\in X$, $\Gamma\in\mathscr{B}$ is a stochastic $\mathbf{Q}$-quasi kernel from X into Y if the following conditions are satisfied:

3.3.α *for any* $\Gamma\in\mathscr{B}$ $n(\cdot\,;\Gamma)$ *is* $\mathscr{A}$*-measurable;*
3.3.β *for* $\mathbf{Q}$*-almost all* $x\in X$ $\quad n(x;Y)=1$;

3.3.γ *If* Γ_i *is a sequence of disjoint sets then*

$$n\Big(x;\bigcup_k \Gamma_k\Big)=\sum_k n(x;\Gamma_k)$$

for $\mathbf{Q}$*-almost all* $x\in X$.

Note that if $\overline{\mathbf{Q}}$ is any measure on the product $X\times Y$ and a σ-finite measure $\mathbf{Q}$ on X is a projection of $\overline{\mathbf{Q}}$ on X then the function $n(x;A)$ which is a Radon-Nikodym derivative of $\overline{\mathbf{Q}}(dx\times A)$ with respect to $\mathbf{Q}(dx)$ is a stochastic $\mathbf{Q}$-quasi kernel from X into Y.

Lemma 3.3.1. *Suppose that* X_1, X_2 *and* Y *are measurable spaces and* $\mathbf{Q}_i$ *is a measure on* X_i. *If* ξ *is a mapping of* X_1 *into* X_2 *such that* $\xi^{-1}(\mathbf{Q}_2)$ *is absolutely continuous with respect to* $\mathbf{Q}_1$, *then for every stochastic* $\mathbf{Q}_2$*-quasi kernel* n *from* X_2 *to* Y *the function* $n(\xi(x);A)$ *is a stochastic quasi kernel from* X_1 *into* Y.

The proof of this lemma is trivial.

We need the following theorem. (The writing $(Y^\infty,\mathscr{B}^\infty)$ means the countable product of the space $(Y,\mathscr{B})$).

Theorem 3.3.1. *Let* $(X,\mathscr{A})$ *and* $(Y,\mathscr{B})$ *be two measurable spaces and* $\mathbf{Q}$ *be a finite measure on* $\mathscr{A}$. *If* $n_1,n_2,\ldots$ *is a sequence of stochastic* $\mathbf{Q}$*-quasi kernels from* X *into* Y, *then there exists a measure* $\overline{\mathbf{Q}}$ *on* $(X\times Y^\infty,\mathscr{A}\times\mathscr{B}^\infty)$ *such that for any* n

$$\mathbf{Q}(\Delta\times\Gamma_1\times\ldots\times\Gamma_n\times Y\times Y\times\ldots)=\int_{\Delta} n(x;\Gamma_1)\ldots n(x;\Gamma_n)\,\mathbf{Q}(dx),\qquad \Delta\in\mathscr{A},\ \Gamma_i\in\mathscr{B}.$$

The proof of this theorem does not differ from the proof of the Kolmogorov theorem.

Consider now a measure $\overline{N}$ on $T^2\times W$

$$\overline{N}(\Gamma\times\Delta\times A)=\mathbf{P}\{\alpha\in\Gamma,\beta\in\Delta,w\in A\},\qquad \Gamma,\Delta\subset T,\ A\in G;$$

and let $N(B)=\overline{N}(B\times W)$, $B\subset T^2$. It is obvious that N is concentrated on the set $\{(x,y)\colon x,y\in T, x<y\}$ and

$$N(]-\infty,t[\times]t,\infty[)=\mathbf{P}\{\alpha<t,\beta>t\}=\mathbf{P}\{w(t)\in D\}=1;$$

so N is a σ-finite measure on T^2 and there exists a stochastic N-quasi kernel $m(x,y;A)$ which is a Radon-Nikodym derivative of $\overline{N}(dx\times dy\times A)$ with respect to $N(dx,dy)$.

Let $r_1,r_2,\dots,r_k,\dots$ be a sequence of all rational numbers. Denote

$$x(k)=x(k,\tilde\omega)=L_{r_k}(\tilde\omega);$$
$$y(k)=y(k,\tilde\omega)=\tau_{r_k}(\tilde\omega);$$
$$z(k)=z(k,\tilde\omega)=(x(k,\tilde\omega),y(k,\tilde\omega)),$$

where L_t and τ_t are defined relative to M as in Sect. 3.1.

Lemma 3.3.2. *For every* k

$$n_k(\tilde\omega;A)=m(z(k);A)$$

is a stochastic $\tilde{\mathbf{P}}$*-quasi kernel from* $\tilde\Omega$ *into* W.

Proof. By virtue of Lemma 3.3.1 it is only necessary to check that

3.3.1 For any $\Delta\subset T^2$ of N-measure zero

$$\tilde{\mathbf{P}}\{z(k)\in\Delta\}=0.$$

The formula (1.3) in [2] shows that for any $(0,\Pi)$-generated M for every t

$$\tilde{\mathbf{P}}\{t\in M\}=0; \tag{3.3.1}$$

and thus $x(k)<r_k<y(k)$ a.s. $\tilde{\mathbf{P}}$. Therefore

$$\begin{aligned}\tilde{\mathbf{P}}\{z(k)\in\Delta\}&=\tilde{\mathbf{P}}\{z(k)\in\Delta, x(k)<r_k<y(k)\}\\&=\tilde{\mathbf{P}}\sum_\gamma 1_{\gamma<r_k<\delta}1_\Delta(\gamma,\delta)\leqq\tilde{\mathbf{P}}\sum_\gamma 1_\Delta(\gamma,\delta).\end{aligned} \tag{3.3.2}$$

In view of (3.2.5) the right side of (3.3.2) is equal to

$$\mathbf{P}\{(\alpha,\beta)\in\Delta\}=N(\Delta)=0.$$

Put

$$\Omega=\tilde\Omega\times W^\infty,\qquad \mathscr{F}=\tilde{\mathscr{F}}\times G^\infty.$$

Theorem 3.3.1 provides the existence of a measure $\overline{\mathbf{P}}$ on $(\Omega,\mathscr{F})$ such that

$$\begin{aligned}&\overline{\mathbf{P}}\{A\times B_1\times B_2\times\dots\times B_k\times W\times W\times\dots\}\\&\quad=\int_A n_1(\tilde\omega,B_1)\,n_2(\tilde\omega,B_2)\dots n_k(\tilde\omega,B_k)\,\tilde{\mathbf{P}}(d\tilde\omega),\qquad A\in\tilde{\mathscr{F}},\ B_i\in G.\end{aligned} \tag{3.3.3}$$

Now we define the process $x_t(\omega)$. As it was mentioned the state space of x_t is equal to $D\cup V$, where V is a singleton. Put

$$k(t)=\inf\{m: x(m)<t<y(m)\},$$

$$x_t(\omega)=x_t(\tilde{\omega}, w_1, w_2, \ldots, w_k, \ldots)=\begin{cases} V & \text{if } t\in M(\tilde{\omega}) \\ w_{k(t)}(t) & \text{otherwise.}\end{cases}$$

In the next section we shall show that $(x_t, \overline{\mathbf{P}})$ is the desired process.

4. Proof of Theorem 1

4.1 For the proof of Theorem 1 it is sufficient to show that $(x_t, \overline{\mathbf{P}})$ constructed in the previous section is a stationary Markov process and its subprocess in D is $(w(s), \mathbf{P})$.

The following lemma shows the fundamental relation between the measure $\tilde{\mathbf{P}}$ (the distribution of the random set M) and the measure $\mathbf{P}$. The fact that $(w(s), \mathbf{P})$ is a subprocess of $(x_t, \overline{\mathbf{P}})$ is a simple consequence of this lemma.

Lemma 4.1.1. *For any functions f and g on T and any $A\in G$*

$$\tilde{\mathbf{P}}\sum_{\gamma} f(\gamma)\, g(\delta)\, m(\gamma, \delta; A)=\mathbf{P}\{f(\alpha)\, g(\beta)\, 1_A\}. \tag{4.1.1}$$

Proof. We can apply Lemma 3.2.1 to the left side of (4.1.1) and obtain

$$\begin{aligned}\tilde{\mathbf{P}}\sum_{\gamma} f(\gamma)\, g(\delta)\, m(\gamma, \delta; A)&=\int f(x)\, g(y)\, m(x, y; A)\, \mathbf{P}\{(\alpha, \beta)\in(dx, dy)\}\\ &=\int f(x)\, g(y)\, m(x, y; A)\, N(dx, dy),\end{aligned} \tag{4.1.2}$$

where N is the measure defined in Sect. 3.3. Since $m(x, y; A)$ is the Radon-Nikodym derivative of $\overline{N}$ with respect to N, the right side of (4.1.2) may be rewritten as

$$\int f(x)\, g(y)\, 1_A(w)\, \overline{N}(dx, dy, dw)=\mathbf{P}\{f(\alpha)\, g(\beta)\, 1_A\},$$

and that is equal to right side of (4.1.1).

Corollary. *The process $(x_t, \overline{\mathbf{P}})$ is a covering for $(w(s), \mathbf{P})$.*

Proof. Take $A\in G$ and calculate

$$\overline{\mathbf{P}}\sum_{\gamma} 1_A(w_\delta^\gamma)=\tilde{\mathbf{P}}\sum_{\gamma} m(\gamma, \delta; A)=\mathbf{P}(A).$$

Denote by $v_{t_1 t_2 \ldots t_n}$ the n-dimensional distributions of $(x_t, \overline{\mathbf{P}})$

$$v_{t_1 \ldots t_n}(\Gamma)=\overline{\mathbf{P}}\{(x_{t_1}, \ldots, x_{t_n})\in\Gamma\}, \qquad \Gamma\subset E^n.$$

The Markov property of $(x_t, \mathbf{P})$ follows from

Lemma 4.1.2. *Fix $t_1<t_2<\ldots<t_n$. For any $\Gamma_n\subset E$ there exists a function g on E such that for any $\Gamma_1, \Gamma_2, \ldots, \Gamma_{n-1}\subset E$*

$$\begin{aligned}&v_{t_1 \ldots t_n}(\Gamma_1\times\ldots\times\Gamma_n)\\ &\quad=\int_{\Gamma_1\times\ldots\times\Gamma_{n-1}} v_{t_1 \ldots t_{n-1}}(dx_1, \ldots, dx_{n-1})\, g(x_{n-1}).\end{aligned} \tag{4.1.3}$$

Proof. In view of (3.3.1) $\mathbf{P}\{x_t \in V\} = 0$ for any t and it is sufficient to prove (4.1.3) only for $\Gamma_i \subset D$.

Let J be a set of k two-dimensional integer-valued vectors $\{(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)\}$. We say that J is a k-partition of n if

$$1 = i_1 < i_2 < \ldots < i_k \leqq n, \qquad 1 \leqq j_1 < j_2 < \ldots < j_k = n,$$
$$i_{\ell+1} = j_\ell + 1, \qquad \ell = 1, 2, \ldots, k-1.$$

We denote by $\mathfrak{S}(n,k)$ the set of all k-partitions of n and by $\mathfrak{S}(k)$ the union of $\mathfrak{S}(n,k)$ over all n. Let $\mathfrak{U}(n,k)$ be the subset of $\mathfrak{S}(n,k)$ containing all $J \in \mathfrak{S}(n,k)$ whose k-th vector is equal to (n,n) and $\mathfrak{V}(n,k)$ be the compliment of $\mathfrak{U}(n,k)$ in $\mathfrak{S}(n,k)$. It is obvious that

$$\mathfrak{U}(n,k) = \{J \cup \{(n,n)\} \colon J \in \mathfrak{S}(n-1, k-1)\} \tag{4.1.4}$$

Denote

$$\bar{A}_i = \{\omega \colon x_{t_i}(\omega) \in \Gamma_i\}, \qquad A_i = \{w \colon w(t_i) \in \Gamma_i\}.$$

Let $\sum^k$ stand for the sum taken over all $\gamma_1, \gamma_2, \ldots, \gamma_k$ such that $\gamma_1 < \gamma_2 < \ldots < \gamma_k$. We denote by $\sum_{(n,k)}$ the sum over all $J = \{(i_1, j_1), \ldots, (i_k, j_k)\} \in \mathfrak{S}(n,k)$. The symbol $\sum'_{(n,k)}$ stands for the sum over all $J \in \mathfrak{U}(n,k)$ and $\sum''_{(n,k)}$ stands for the sum over all $J \in \mathfrak{V}(n,k)$.

We have

$$\begin{aligned}
v_{t_1 t_2 \ldots t_n}&(\Gamma_1 \times \Gamma_2 \times \ldots \times \Gamma_n) = \overline{\mathbf{P}}\{\bar{A}_1 \bar{A}_2 \ldots \bar{A}_n\} \\
&= \sum_{k \leqq n} \sum\nolimits_{(n,k)} \sum\nolimits^k \overline{\mathbf{P}}\{\bar{A}_1 \bar{A}_2 \ldots \bar{A}_n, \gamma_1 < t_1 < t_2 < \ldots < t_{j_1} < \delta_1, \\
&\quad \gamma_2 < t_{i_2} < \ldots < t_{j_2} < \delta_2, \ldots, \gamma_k < t_{i_k} < \ldots < t_n < \delta_k\} \\
&= \tilde{\mathbf{P}}\left\{\sum_{k \leqq n} \sum\nolimits_{(n,k)} \sum\nolimits^k \sum_{\ell=1}^{k} m(\gamma_\ell, \delta_\ell; A_{i_\ell} A_{i_\ell+1} \ldots A_{j_\ell})\right\} \\
&= \tilde{\mathbf{P}}\{\sum_{k \leqq n} \sum\nolimits'_{(n,k)} \sum\nolimits^k \ldots\} + \tilde{P}\{\sum_{k \leqq n} \sum\nolimits''_{(n,k)} \sum\nolimits^k \ldots\}.
\end{aligned} \tag{4.1.5}$$

Denote the first and the second summand in the right side of (4.1.5) by Z_1 and Z_2 respectively. Let $\tilde{\mathbf{P}}_y$ be the transition probabilities of a $(0, \Pi)$-process y_t and $\tilde{\mathbf{P}}^y$ be the transition probabilities of the process y_t^* which is equal to $-y_t$. Let Θ denote the set of the discontinuities of a process with independent increments. Put

$$\psi(x) = 1_{x < t_n} \tilde{\mathbf{P}}_x\{\sum_{t \in \Theta} \mathbf{1}_{y_{t-} < t_n < y_t} m(y_{t-}, y_t; A_n)\}.$$

For $J = \{(i_1, j_1), (i_2, j_2), \ldots, (i_q, j_q)\} \in \mathfrak{S}(q)$ put $a(J) = t_{j_q}$; and

$$g_\ell^J(x,y) = m(x, y; A_{i_\ell} A_{i_\ell+1} \ldots A_{j_\ell}), \qquad x, y \in T;$$
$$\varphi^J(x) = 1_{x > a(J)} \tilde{\mathbf{P}}^x\{\sum g_1^J(y_{s_1}^*, y_{s_1-}^*)\, g_2^J(y_{s_2}^*, y_{s_2-}^*) \ldots g_q^J(y_{s_q}^*, y_{s_q-}^*)\},$$

where the sum under $\tilde{\mathbf{P}}^x$ is taken over all sequences $s_1, s_2, \ldots, s_q \in \Theta$ such that $s_1 > s_2 > \ldots > s_q$. (We put $\mathfrak{S}(0) = \emptyset$ and $\varphi^J(x) \equiv 1$ for $J \in \mathfrak{S}(0)$.)

Applying successively (4.1.1) and Lemma 6.8 in [2] we get

$$
\begin{aligned}
Z_1 &= \tilde{\mathbf{P}}\left\{\sum_{k\leq n} \sum\nolimits'_{(n,k)} \sum\nolimits^k \prod_{\ell=1}^{k} g^J_\ell(\gamma_\ell, \delta_\ell)\right\} \\
&= \tilde{\mathbf{P}}\left\{\sum_{k\leq n-1} \sum\nolimits_{(n-1,k)} \sum\nolimits^k \sum_{\gamma>\gamma_k} m(\gamma,\delta; A_n) \prod_{\ell=1}^{k} g^J_\ell(\gamma_\ell, \delta_\ell)\right\} \\
&= \tilde{\mathbf{P}}\left\{\sum_{k\leq n-1} \sum\nolimits_{(n-1,k)} \sum\nolimits^k \psi(\delta_k) \prod_{\ell=1}^{k} g^J_\ell(\gamma_\ell, \delta_\ell)\right\} \\
&= \tilde{\mathbf{P}}\left\{\sum_{m=1}^{n-1} \sum_{k\leq m-1} \sum\nolimits_{(m-1,k)} \sum\nolimits^k \sum_{\gamma>\gamma_k} m(\gamma,\delta; A_m A_{n+1} \dots A_{n-1}) \psi(\delta) \cdot \prod_{\ell=1}^{k} g^J_\ell(\gamma_\ell, \delta_\ell)\right\} \\
&= \tilde{\mathbf{P}}\left\{\sum_{m=1}^{n-1} \sum_{k\leq m-1} \sum\nolimits_{(m-1,k)} \sum_{\gamma} \varphi^J(\gamma)\, m(\gamma,\delta; A_m \dots A_{n-1}) \psi(\delta)\right\}.
\end{aligned} \tag{4.1.6}
$$

Lemma 4.1.1 and the Markov property of $(w(s), \mathbf{P})$ provide that the right side of (4.1.6) is equal to

$$\sum_{m=1}^{n-1} \sum_{k\leq m-1} \sum\nolimits_{(m-1,k)} \mathbf{P}\{\varphi^J(\alpha)\, 1_{A_m A_{m+1} \dots A_{n-1}}\, g_1(w(t_{n-1}))\} \tag{4.1.7}$$

where

$$g_1(w(t_{n-1})) = \mathbf{P}\{\psi(\beta)\, 1_{\beta > t_{n-1}} | w(t_{n-1})\}.$$

A similar computation yields

$$Z_2 = \sum_{m=1}^{n-1} \sum_{k\leq m-1} \sum\nolimits_{(m-1,k)} \mathbf{P}\{\varphi^J(\alpha)\, 1_{A_m A_{m+1} \dots A_{n-1}}\, g_2(w(t_{n-1}))\}, \tag{4.1.8}$$

where $g_2(w(t_{n-1})) = \mathbf{P}\{w(t_n) \in \Gamma_n | w(t_{n-1})\}$. Adding (4.1.7) and (4.1.8) we obtain (4.1.3) with $g = g_1 + g_2$.

4.2 Now we prove that $(x_t, \overline{\mathbf{P}})$ is a stationary process. Because $(x_t, \overline{\mathbf{P}})$ is Markov it is sufficient to prove that $(x_t, \overline{\mathbf{P}})$ has stationary two-dimensional distributions.

Consider

$$
\begin{aligned}
\overline{\mathbf{P}}\{x_s \in \Gamma, x_t \in \Delta\} &= \overline{\mathbf{P}}\{\sum_{\gamma} 1_{\gamma<s<t<\delta}\, 1_\Gamma(x_s)\, 1_\Delta(x_t)\} \\
&\quad + \overline{\mathbf{P}}\{\sum_{\gamma_1<\gamma_2} 1_{\gamma_1<s<\delta_1<t<\delta_2}\, 1_\Gamma(x_s)\, 1_\Delta(x_t)\}.
\end{aligned} \tag{4.2.1}
$$

In view of Lemma 4.1.1 and the stationarity of $\mathbf{P}$ the first term in the right side of (4.2.1) is equal to

$$\mathbf{P}\{w(s) \in \Gamma, w(t) \in \Delta\} = \mathbf{P}\{w(s+a) \in \Gamma, w(t+a) \in \Delta\}. \tag{4.2.2}$$

Denote

$$
\begin{aligned}
A_1 &= \{w\colon w(s) \in \Gamma\}, \quad & B_1 &= \{w\colon w(s+a) \in \Gamma\}; \\
A_2 &= \{w\colon w(t) \in \Delta\}, \quad & B_2 &= \{w\colon w(t+a) \in \Delta\}, \quad a \in T.
\end{aligned}
$$

The second term in the right sife of (4.2.1) is equal to

$$\tilde{\mathbf{P}}\{m(z(s); A_1)\, m(z(t); A_2); z(s) \neq z(t)\}.$$

Here $z(t)$ is a two-dimensional vector (L_t, τ_t). Since $\mathbf{P}$ is a stationary measure

$$m(x,y;A_1)=m(x+a,y+a;B_1) \quad \text{for} \quad N \text{ a.e. } (x,y)$$

In view of 3.3.A

$$m(z(s);A_1)=m(z(s)+a;B_1) \quad \text{a.s. } \tilde{\mathbf{P}}.$$

(The writing $z+a$ for $z=(x,y)$ means $(x+a, y+a)$.) Similarly for $m(z(t);A_2)$. Owing to the fact that the set M is translation invariant, we get

$$\begin{aligned}&\tilde{\mathbf{P}}\{m(z(s);A_1)\,m(z(t);A_2);\, z(s)\neq z(t)\}\\&\quad=\tilde{\mathbf{P}}\{m(z(s)+a;B_1)\,m(z(t)+a;B_2);\, z(t)\neq z(s)\}\\&\quad=\tilde{\mathbf{P}}\{m(z(s+a);B_1)\,m(z(t+a);B_2);\, z(s+a)\neq z(t+a)\}\end{aligned} \tag{4.2.3}$$

Combining (4.2.3) and (4.2.2) we obtain the stationarity of the left side of (4.2.1).

Remark. All the proofs remain valid if W is not the set of all paths in D with random birth and death times, but if W is some subset of this set (say the set of all right-continuous, continuous, etc. paths). The construction of x_t shows that the trajectories of a covering process may be obtained from the trajectories of its subprocesses.

5. Theorem of Uniqueness

5.1 The rest of the paper is devoted to the proof of Theorem 2.

We suppose that $(x_t(\omega), \overline{\mathbf{P}})$, $\omega \in \Omega$, is a Markov process with a state space $E = D \cup V$, whose subprocess in D is the process $(w(s), \mathbf{P})$ subject to the conditions of the Theorem 2. We shall prove that the two-dimensional distributions of $(x_t, \overline{\mathbf{P}})$ are uniquely determined by $(w(s), \mathbf{P})$. Since $(x_t, \overline{\mathbf{P}})$ is Markov, all its finite-dimensional distributions can be calculated from the two-dimensional ones and are uniquely determined by $(w(s), \mathbf{P})$. Let $\mathbf{P}^*$ be the measure defined in Sect. 3.2. In Sect. 5.3 we prove Theorem 2 for the case of finite $\mathbf{P}^*$. The rest of the paper is devoted to the case of infinite $\mathbf{P}^*$.

In Sect. 5.4 we investigate the properties of a local time ξ_s of the process x_t corresponding to the set V. Then we evaluate the expression (1.2.1) in terms of the local time ξ_s and the shifts of measure $\mathbf{P}^*$ (Lemma 5.5.2). This expression is similar to the main result of [3]. Using this formula, we prove that the inverse function y_t for the function ξ_s is a process with independent increments; and we calculate the characteristics of this process from $\mathbf{P}^*$ (Sect. 5.6). Using this fact, we find the expression for the two-dimensional distributions of $(x_t, \overline{\mathbf{P}})$, which involves only the measure $\mathbf{P}$, shifts of the measure $\mathbf{P}^*$ and the transition probabilities of y_t; therefore this expresison ultimately depends only on $(w(s), \mathbf{P})$.

By $G_{\alpha+}$ we denote the σ-field in W of all sets A such that $A \cap \{\alpha < t\} \in G_t$. Denote

$$\mathscr{F}_t = \bar{\sigma}(x_s, s \leqq t), \qquad \mathscr{F}^t = \bar{\sigma}(x_u, u \geqq t),$$

$$\mathscr{A}_t = \bigwedge_{u>t} \mathscr{F}_u, \qquad \mathscr{A}^t = \bigwedge_{s>t} \mathscr{F}^s.$$

(The bar over σ means the completion of the corresponding σ-field with respect to the measure $\overline{\mathbf{P}}$.)

We don't suppose that x_t has any regularity properties but we assume that

5.1.A *The set* $M=\{(t,\omega):x_t(\omega)\in V\}$ *in* $T\times\Omega$ *is progressively measurable with respect to the filtration* $\mathscr{A}_t$ *and* $\mathscr{A}^t$.

(Without an assumption of such type the relation (1.2.1) can be senseless. Note also that for a closed set M 5.1.A is a consequence of 1.2.α.) We also don't assume that the process $(x_t,\overline{\mathbf{P}})$ has a transition function. Nevertheless for f a function on D we write $\bar{p}(s,x;t,f)$ for a function on D such that

$$\bar{p}(s,x_s;t,f)=\overline{\mathbf{P}}\{f(x_t)|x_s\} \quad \text{a.s. } \overline{\mathbf{P}}. \tag{5.1.1}$$

(Since for each $s\,\bar{p}(s,x;t,f)$ is defined by (5.1.1) only up to the measure v we may not define $\bar{p}$ for $x\in V$; therefore the definition of $\bar{p}$ is meaningful.)

The process $(w(s),\mathbf{P})$ has a homogeneous transition function p and we can construct transition probabilities of $\mathbf{P}$, that is the family of probability Markov measures $\mathbf{P}_{s,x}$ on $G^s=\sigma(w(t),t>s)$ such that $\mathbf{P}_{s,x}\{w(t)\in\Gamma\}=p(t-s,x;\Gamma)$. Put

$$\sigma_t=\tau_{t+}=\lim_{u\downarrow t}\tau_u=\inf\{u\colon u>t,x_u\in V\}. \tag{5.1.2}$$

Note that

$$\mathbf{P}_{t,w(t)}\{g(\beta)\}=\mathbf{P}\{g(\beta)\,1_{\beta>t}|w(t)\} \quad \text{a.e. } \mathbf{P};$$

and

$$\mathbf{P}_{t,x_t}\{g(\beta)\}=\overline{\mathbf{P}}\{g(\sigma_t)|x_t\} \quad \text{a.s. } \overline{\mathbf{P}}. \tag{5.1.3}$$

A real-valued process $\xi_t(\omega),t\in T,\ \omega\in\Omega$ is called well measurable if it is measurable with respect to the σ-field in $T\times\Omega$ generated by right-continuous processes $\eta_t(\omega)$ adapted to $\mathscr{A}_t$ (see [4] for detailes).

By R-$\lim g(s)$, g being a function on T we mean the limit of g over the set of rational numbers. The letter m will denote the Lebesgue measure on T.

5.2 If $\mathbf{P}$ is a minimal element of $S(p)$ then either

$$\mathbf{P}\{\alpha\neq-\infty\}=\mathbf{P}\{\beta\neq+\infty\}=0, \tag{5.2.1}$$

or

$$\mathbf{P}\{\alpha=-\infty\}=\mathbf{P}\{\beta=\infty\}=0. \tag{5.2.2}$$

If $\mathbf{P}$ satisfies (5.2.1) then Theorem 2 is trivial. The measure $\mathbf{P}$ is a probability one and the only covering process for $(w(s),\mathbf{P})$ is $(w(s),\mathbf{P})$ itself.

So, we shall consider the case in which $\mathbf{P}$ is subject to (5.2.2). In this case $\mathbf{P}$ is represented in the form (3.2.2) with $\mathbf{P}_t^*$ given by (3.2.1). By [1] $\mathbf{P}_t^*$ is an extreme measure in the class of all Markov measures with the transition function p.

The following lemma is crucial in he proof of Theorem 2.

Lemma 5.2.1. *Let* $g(t,x)$ *be a function on* D *for each* $t\in T$. *The following three conditions are equivalent.*

5.2.A $R\text{-}\lim_{r\downarrow\gamma} g(r,x_r(\omega))$ *exists for all* γ a.s. $\overline{\mathbf{P}}$.

5.2.B $R\text{-}\lim_{r\downarrow\alpha} g(r, w(r))$ *exists* a.e. $\mathbf{P}$.

5.2.C *For m-almost all t,*

$$R\text{-}\lim_{r\downarrow t} g(r, w(r)) \quad \textit{exists} \text{ a.e. } \mathbf{P}_t^*.$$

Moreover if 5.2.A *holds, then there exists a function* $h^g(s)$ *on* T *such that*

$$R\text{-}\lim_{r\downarrow\gamma} g(r, x_r) = h^g(\gamma) \quad \textit{for all } \gamma \text{ a.s. } \mathbf{P}. \tag{5.2.3}$$

The function h^g *is determined uniquely up to the measure m by the process* $(w(s), \mathbf{P})$.

Proof. Set

$$A = \{w\colon R\text{-}\lim_{r\downarrow\gamma} g(r, w(r)) \text{ does not exist}\}.$$

By (1.2.1)

$$\begin{aligned}\mathbf{P}\{A\} &= \overline{\mathbf{P}} \sum_{\gamma} 1_A(w_\delta^\gamma) \\ &= \overline{\mathbf{P}}\{\# \gamma\colon R\text{-}\lim_{r\downarrow\gamma} g(r, x_r) \text{ does not exist}\}.\end{aligned} \tag{5.2.4}$$

The expression (5.2.4) shows the equivalence of 5.2.A and 5.2.B. Owing to the fact that $\mathbf{P}$ satisfies (3.2.2) and that $\mathbf{P}_t^*\{\alpha \neq t\} = 0$, the equivalence of 5.2.B and 5.2.C holds.

The function

$$\xi(w) = (1 - 1_A(w))\, R\text{-}\lim_{r\downarrow\alpha} g(r, w(r))$$

is $G_{\alpha+}$-measurable. Since the measure $\mathbf{P}_t^*$ is extreme, ξ is a constant $\mathbf{P}_t^*$-almost everywhere (see [1]). Denote this constant by $h^g(t)$. By 5.2.C $\mathbf{P}_t^*\{A\} = 0$ for m-almost all t and we can write

$$\begin{aligned}&\overline{\mathbf{P}}\{\#\gamma\colon R\text{-}\lim_{r\downarrow\gamma} g(r, x_r) \neq h^g(\gamma)\} = \mathbf{P}\{R\text{-}\lim_{r\downarrow\alpha} g(r, w(r)) \neq h^g(\alpha)\} \\ &\quad = \int \mathbf{P}_t^*\{R\text{-}\lim_{r\downarrow t} g(r, w(r)) \neq h^g(t)\}\, dt = 0.\end{aligned}$$

That proves (5.2.3).

5.3 We consider the case in which

$$\mathbf{P}^*\{W\} < \infty. \tag{5.3.1}$$

We prove that the set M in this case is a.s. discrete (Lemma 5.3.1). Lemmas 5.3.2, 5.3.3 and 5.3.4 show that the strong Markov property holds for the stopping times τ which belongs to M and that the corresponding conditional distributions can be computed from the measures $\mathbf{P}_u^*$. Lemma 5.5 gives us an expression for the two-dimensional distributions of $(x_t, \overline{\mathbf{P}})$, which depends only on $\mathbf{P}$.

Lemma 5.3.1. *If* (5.3.1) *holds then for any finite interval* $]s, t[$ *the number* γ *such that* $\gamma \in]s, t[$ *is finite* a.s. $\overline{\mathbf{P}}$, the set $M = \{t\colon x_t \in V\}$ *is* a.s. $\overline{\mathbf{P}}$ discrete. Moreover

$$M = \{t\colon t = \gamma\} = \{t\colon t = \delta\} \quad \text{a.s. } \overline{\mathbf{P}}. \tag{5.3.2}$$

Proof. The second and the third statements of the lemma follow trivially from the first one. To prove the first consider

$$\overline{\mathbf{P}}\{\#\gamma\colon \gamma\in(s,t)\}=\overline{\mathbf{P}}\sum_{s<\gamma<t}1_W(w_0^\gamma)=\mathbf{P}\{s<\alpha<t\}$$
$$=\int_s^t \mathbf{P}_u^*\{W\}\,du=(t-s)\,\mathbf{P}^*\{W\}<\infty.$$

Lemma 5.3.2. *If f is function on T then there exists a function η^f on T such that*

$$R\text{-}\lim_{r\downarrow\gamma}\mathbf{P}_{r,x_r}\{f(\beta)\}=\eta^f(\gamma)\quad \textit{for all } \gamma \text{ a.s. } \overline{\mathbf{P}}. \tag{5.3.3}$$

If ℓ is a function on D then for any $t\in T$ there exists a function $\zeta_{t,\ell}$ on T such that

$$R\text{-}\lim_{r\downarrow\gamma}p(r,x_r;t,\ell)=\zeta_{t,\ell}(\gamma). \text{ a.s. } \overline{\mathbf{P}} \tag{5.3.4}$$

The functions η^f and $\zeta_{t,\ell}$ are determined uniquely up to the measure m by the process $(w(s),\mathbf{P})$.

Proof. The function

$$g(s,x)=\mathbf{P}_{s,x}\{f(\beta)\},\qquad s\in T,\ x\in D \tag{5.3.5}$$

is p-excessive. In addition, for any u

$$\sup_t \mathbf{P}_u^*\{g(t,w(t))\}\leqq\mathbf{P}_u^*\{f(\beta)\}\leqq[\sup_{s\in T}f(s)]\,\mathbf{P}_u^*\{W\}<\infty.$$

Therefore

$$R\text{-}\lim_{r\downarrow\alpha}g(r,w(r))\qquad\text{exists a.e. } \mathbf{P}_u^*.$$

Applying Lemma 5.2.1 we obtain (5.3.3) with η^f equal to h^g, g given by (5.3.5).
The proof of (5.3.4) is similar.
Put

$$\sigma_s^1=\sigma_s,\qquad \sigma_s^{m+1}=\sigma_{\sigma_s^m}=\inf\{u\colon u>\sigma_s^m, u\in V\},\ m=2,3,\ldots. \tag{5.3.6}$$

Lemma 5.3.3. *For any $s\in T$ and any function g on D and any function f on T*

$$\overline{\mathbf{P}}\{g(x_s)f(\sigma_s^{m+1})\}=\overline{\mathbf{P}}\{g(x_s)\,\eta^f(\sigma_s^m)\}. \tag{5.3.7}$$

Proof. Let

$$\tau=\sigma_s^{m+1};$$
$$\tau^n=k2^{-n},\qquad \text{if } (k-1)2^{-n}\leqq\sigma_s^m<k2^{-n}. \tag{5.3.8}$$

The assumption 5.1.A provides that $\sigma_s,\sigma_s^m,\tau^n$ are stopping times with respect to $\mathscr{A}_t$. The stopping time τ^n takes only rational values. So we can write

$$\overline{\mathbf{P}}\{g(x_s)f(\sigma_s^{m+1})\}=\lim_{n\to\infty}\overline{\mathbf{P}}\{g(x_s)\,f(\tau)\,1_{\tau^n<\tau}\}$$
$$=\lim_{n\to\infty}\overline{\mathbf{P}}\{g(x_s)\,1_{\tau^n<\tau}\,\xi(\tau^n,x_{\tau^n})\}, \tag{5.3.9}$$

where $\xi(s,x)$ is a function on $T\times D$ defined for each s up to measure ν such that

$$\xi(s,x_s)=\overline{\mathbf{P}}\{f(\sigma_s)|x_s\} \quad \text{a.s. } \overline{\mathbf{P}}.$$

By (5.1.3)

$$\xi(s,x)=\mathbf{P}_{s,x}\{f(\beta)\} \qquad \text{for } \nu\text{-almost all } \ x.$$

Since $\tau^n\downarrow\sigma_s^m$ and σ_s^m coincides with some γ, we can apply (5.3.3) to (5.3.9) and obtain

$$\overline{\mathbf{P}}\{g(x_s)f(\sigma_s^{m+1})\}=\lim_{n\to\infty}\overline{\mathbf{P}}\{g(x_s)\mathbf{P}_{\tau^n,x_{\tau^n}}\{f(\beta)\}\}=\overline{\mathbf{P}}\{g(x_s)\eta^f(\sigma_s^m)\}.$$

The following lemma completes the proof of the Theorem 2 in the case when (5.3.1) holds.

Lemma 5.3.4. *Fix $s,t\in T$. Define*

$$\begin{aligned}\theta_0(x;\Delta)&=p(s,x;t,\Delta),\\ \theta_n(x;\Delta)&=\mathbf{P}_{s,x}\{\pi_n(\beta)\}, \qquad x\in D,\ \Delta\subset D,\\ \pi_1(u)&=\zeta_{t,1_\Delta}(u),\\ \pi_n(u)&=\eta^{\pi_{n-1}}(u), \qquad u\in T,\end{aligned}$$

Then

$$\overline{\mathbf{P}}\{x_s\in\Gamma,x_t\in\Delta\}=\sum_{k=0}^{\infty}\int_\Gamma \nu(dx)\,\theta_k(x;\Delta). \tag{5.3.10}$$

Proof. Since the set M is discrete

$$\overline{\mathbf{P}}\{x_s\in\Gamma,x_t\in\Delta\}=\sum_{k=0}^{\infty}\overline{\mathbf{P}}\{x_s\in\Gamma,x_t\in\Delta,\ \#(M\cap]s,t[)=k\}. \tag{5.3.11}$$

Prove that k-th additive in (5.3.11) is equal to that of (5.3.10). For $k=0$ it is obvious. Let $k>0$ and let σ^m and τ^n be given by (5.3.6) and (5.3.8) respectively. Then

$$\begin{aligned}&\overline{\mathbf{P}}\{x_s\in\Gamma,x_t\in\Delta,\ \#(M\cap]s,t[)=k\}\\ &\quad=\overline{\mathbf{P}}\{x_s\in\Gamma,x_t\in\Delta,\sigma_s^k<t<\sigma_s^{k+1}\}\\ &\quad=\lim_{n\to\infty}\overline{\mathbf{P}}\{x_s\in\Gamma,x_t\in\Delta,\tau^n<t,\sigma_{\tau^n}>t\}\\ &\quad=\lim_{n\to\infty}\overline{\mathbf{P}}\{1_\Gamma(x_s)\,p(\tau^n,x_{\tau^n};t,\Delta)\}.\end{aligned} \tag{5.3.12}$$

Applying Lemma 5.3.2 we see that (5.3.12) is equal to

$$\overline{\mathbf{P}}\{1_\Gamma(x_s)\,\zeta_{t,1_\Delta}(\sigma_s^k)\}. \tag{5.3.13}$$

If $k=1$ then (5.3.13) is equal to

$$\overline{\mathbf{P}}\{1_\Gamma(x_s)\,\mathbf{P}_{s,x_s}\{\zeta_{t,1_\Delta}(\beta)\}\}=\overline{\mathbf{P}}\{1_\Gamma(x_s)\,\theta_1(x_s,\Delta)\}.$$

For $k>1$ we must apply $k-1$ times (5.3.12) and we obtain that (5.3.13) is equal to

$$\overline{\mathbf{P}}\{1_\Gamma(x_s)\,\theta_k(x_s;\Delta)\}=\int_\Gamma \nu(dx)\,\theta_k(x;\Delta).$$

5.4 From now on we consider the case

$$\mathbf{P}^*\{W\} = \infty. \tag{5.4.1}$$

Lemma 5.4.1. *If* (5.4.1) *holds then, for each s,*

$$R\text{-}\lim_{r \downarrow s} \mathbf{P}_{r,w(r)}\{\beta - r\} = 0 \quad \text{a.e. } \mathbf{P}_s^*. \tag{5.4.2}$$

Proof. Let $s=0$. The function

$$h(x) = h(r,x) = \mathbf{P}_{r,x}\{\beta - r\} \tag{5.4.3}$$

is p-excessive. In addition

$$\sup_{u>0} \mathbf{P}^*\{h(u,w(u))\} = \sup_{u>0} \mathbf{P}^*\{(\beta-u)\, 1_{\beta>u}\} \leqq \mathbf{P}^*\{\beta\} = 1.$$

The last equality is due to (3.2.4). Therefore

$$\xi = R\text{-}\lim_{r \downarrow 0} h(r, w(r))$$

exists a.e. $\mathbf{P}^*$. Since ξ is $G_{\alpha+}$-measurable and $\mathbf{P}^*$ is an extreme Markov measure then $\xi = \varepsilon =$ constant for $\mathbf{P}^*$ a.e. w (see [1]). Suppose $\varepsilon > 0$. In view of (5.4.1) there exists $r>0$ such that $\mathbf{P}\{h(r,w(r)) > \varepsilon/2\} > 4/\varepsilon$ we have

$$\mathbf{P}^*\{\beta\} \geqq \mathbf{P}^*\{(\beta - r)\, 1_{\beta>r}\} = \mathbf{P}^*\{h(r,w(r))\} \geqq \varepsilon/2 \cdot 4/\varepsilon = 2;$$

and we come to a contradiction with $\mathbf{P}^*\{\beta\} = 1$.

Put

$$\begin{aligned} M^{\leftarrow} = {} & \{t : x_t \in V, \text{ and for some } \varepsilon > 0,\ x_s \bar{\in} V \text{ for all } s \in]t, t+\varepsilon]\} \\ & = \{t : t = \gamma \text{ for some } \gamma\} \end{aligned}$$

Lemma 5.4.2. *For each stopping time η with respect to $\mathscr{A}_t$*

$$\overline{\mathbf{P}}\{\eta \in M^{\leftarrow}\} = 0.$$

Proof. For $s \in T$ put

$$\ell_n(s) = k2^{-n}, \quad \text{if } (k-1)\,2^{-n} \leqq s < k2^{-n}. \tag{5.4.4}$$

Put $A = \{\eta \in M^{\leftarrow}\}$ and $\eta_n = \ell_n(\eta)$. Applying (5.1.3), Lemma 5.4.1 and Lemma 5.2.1 we get

$$\begin{aligned} \overline{\mathbf{P}}\{(\sigma_\eta - \eta)\, 1_A\} &= \lim_{n\to\infty} \overline{\mathbf{P}}\{1_A \mathbf{P}\{\sigma_{\eta_n} - \eta_n | \mathscr{A}_\eta\} \\ &= \lim_{n\to\infty} \overline{\mathbf{P}}\{1_A \lim_{n\to\infty} h(\eta_n, x_{\eta_n})\} = 0. \end{aligned}$$

Since $\sigma_\eta - \eta > 0$ on A, $\overline{\mathbf{P}}\{A\} = 0$.

Corollary. *The set M is nowhere dense and does not contain isolated points a.s.* $\overline{\mathbf{P}}$.

Proof. By [4], Ch. VI, T9 the set of isolated points is a countable union of graphs of stopping times. Since $\overline{\mathbf{P}}\{t\in M\}=\overline{\mathbf{P}}\{x_t\in V\}=0$, by virtue of Fubini's theorem $m(M)=0$ a.s. $\overline{\mathbf{P}}$.

Put
$$\hat{\xi}_s=\sum_{0<\gamma\leqq s}(\delta-\gamma), \qquad s\geqq 0.$$

By [4], Ch. 5 there exists a dual well measurable projection ξ_s of $\hat{\xi}_s$ with respect to $\mathscr{A}_t$.

Lemma 5.4.3. *The process $\xi_s, s\geqq 0$ is a continuous process which increases iff $s\in M$.*

Proof. 1°. By the construction ξ_s is right-continuous. By T30, Ch. V in [4] for any well measurable ζ_t

$$\overline{\mathbf{P}}\left\{\int_0^\infty \zeta_t\,d\xi_t\right\}=\overline{\mathbf{P}}\left\{\int_0^\infty \zeta_t\,d\hat{\xi}_t\right\}=\overline{\mathbf{P}}\{\sum_\gamma \zeta_\gamma(\delta-\gamma)\}. \tag{5.4.5}$$

Let η be an arbitrary stopping time. By Lemma 5.4.2 $\overline{\mathbf{P}}\{\eta=\gamma$ for some $\gamma\}=0$. Applying (5.4.5) to $\zeta_t=1_{t=\eta}$, we get $\overline{\mathbf{P}}\{\xi_\eta-\xi_{\eta-}\}=0$. By T30, Ch. IV in [4] ξ is a continuous increasing process.

2°. By virtue of 5.1.A and T4, Ch. VI in [4] M is a well measurable set. Take $\zeta_t=1_{T\setminus M}(t)$ and apply (5.4.5):

$$\overline{\mathbf{P}}\left\{\int_0^\infty 1_{T\setminus M}(t)\,d\xi_t\right\}=\overline{\mathbf{P}}\{\sum_\gamma 1_{T\setminus M}(\gamma)(\delta-\gamma)\}=0.$$

Therefore, ξ_t does not increase on $T\setminus M$. The set N of increasing point of ξ (called in [4] the support of ξ) is a well measurable set. To prove that N contains M we must prove that for each stopping time η and for each $\varepsilon>0$ $\xi_{\eta+\varepsilon}-\xi_\eta>0$ a.s. on $\{\eta\in M\}$. Put $\varphi=\inf\{t\colon t>\eta,\ \xi_t>\xi_\eta\}$, $A=\{\eta\in M,\ \varphi>\eta\}$; set $\zeta_t=1_A\,1_{\eta<t<\varphi}$ and apply (5.4.5).

$$0=\overline{\mathbf{P}}\left\{1_A\int_\eta^\varphi d\xi_t\right\}=\overline{\mathbf{P}}\{1_A\sum_{\eta<\gamma<\varphi}(\delta-\gamma)\}. \tag{5.4.6}$$

By Corollary of Lemma 5.4.2, $m(M)=0$ a.s.; therefore on the set A

$$\sum_{\eta<\gamma<\varphi}(\delta-\gamma)\geqq\varphi-\eta>0 \quad \text{a.s. } \overline{\mathbf{P}}. \tag{5.4.7}$$

Comparing (5.4.6) and (5.4.7), we see that $\overline{\mathbf{P}}\{A\}=0$.

5.5 Here we express (1.2.1) in terms of the functional ξ and the measures $\mathbf{P}_t^*$.

Consider $h(x)$ defined by (5.4.3). Note that since $\beta>s$ a.e. $\mathbf{P}_{s,x}$ then $h(x)>0$ for all $x\in D$.

Lemma 5.5.1. *For each function f on D such that f/h is bounded and for all γ*

$$R\text{-}\lim_{r\downarrow\gamma} p(t+r,x_r;f)/h(x_r)=\mathbf{P}_\gamma^*\{f(w(t))\} \text{ a.s. } \overline{\mathbf{P}}.$$

Proof. Put $g(u,x)=p(u,x;f)/h(x)$. By Lemma 5.2.1 it is enough to prove that for all s

$$R\text{-}\lim_{r\downarrow s} g(t-r, w(r)) = \mathbf{P}_s^*\{f(w(t))\} \quad \text{a.s. } \mathbf{P}_s^*. \tag{5.5.1}$$

Let $s=0$. Put $p^h(s,x;dy)=h(x)^{-1}\,p(s,x;dy)\,h(y)$; put $\hat{v}_u(\Gamma)=\mathbf{P}^*\{1_\Gamma(w(u))\,h(w(u))\}$. Let $\mathbf{Q}$ be a Markov measure with transition function p^h and one-dimensional distributions $\hat{v}_u$. By [1] $\mathbf{Q}$ is an extreme measure; $\mathbf{Q}\{W\}=\mathbf{P}^*\{\beta\}=1$; and

$$R\text{-}\lim_{r\downarrow 0} p^h(t-r, w(r); f/h) = \mathbf{Q}\{f/h(w(t))\} \quad \text{a.s. } \mathbf{Q}. \tag{5.5.2}$$

Note that for any $A\in G_\varepsilon$, $\mathbf{Q}\{A, \beta>\varepsilon\}=\mathbf{P}^*\{1_A 1_{\beta>\varepsilon} h(w(\varepsilon))\}$. Therefore the relation (5.5.2) is true a.s. $\mathbf{P}^*$. Since $\mathbf{Q}\{f/h(w(t))\}=\mathbf{P}^*\{f(w(t))\}$ and $p^h(u,x;f/h)=g(u,x)$, we get (5.5.1).

Now recall that G^0 is a σ-field in W generated by the sets $\{w(s)\in\Gamma\}$, $s>0$.

Lemma 5.5.2. *For any G^0-measurable function $F(w)$ and any process $\zeta_t(\omega)$, well measurable with respect to $\mathscr{A}_t$*

$$\overline{\mathbf{P}}\{\sum_{\gamma>0} \zeta_\gamma F(w_\delta^\gamma)\} = \overline{\mathbf{P}}\left\{\int_0^\infty \zeta_t\, \mathbf{P}_t^*\{F\}\, d\xi_t\right\}. \tag{5.5.3}$$

Proof. It suffices to prove (5.5.3) for

$$F(w)=f_1(w(t_1))\,f_2(w(t_2))\dots f_k(w(t_k)),$$

where $0<t_1<\dots<t_k$, f_i is a function on D such that f_i/h is bounded. Put $A=\{x_s\bar{\in} V \text{ for all } s\in[t_1,t_k]\}$ and

$$\begin{aligned} g(x) &= \overline{\mathbf{P}}\{f_1(x_{t_1})\dots f_k(x_{t_k})\,1_A | x_{t_1}=x\} \\ &= \mathbf{P}\{f_1(w(t_1)\dots f_k(w(t_k)) | w(t_1)=x\}. \end{aligned}$$

Set $\gamma_n=\ell_n(\gamma)$. Then

$$\begin{aligned} \overline{\mathbf{P}}\{\sum_{\gamma>0} \zeta_\gamma F(w_\delta^\gamma)\} &= \overline{\mathbf{P}}\{\sum_{\gamma>0} \zeta_\gamma 1_{\gamma<t_1<t_k<\delta} F(w_\delta^\gamma)\} \\ &= \lim_{n\to\infty} \overline{\mathbf{P}}\{\sum_{\gamma>0} \zeta_\gamma 1_{\gamma_n<t_1<t_k<\delta}\, h(x_{\gamma_n})^{-1}(\delta-\gamma_n)\, p(t_1-\gamma_n, x_{\gamma_n}; g)\}. \end{aligned} \tag{5.5.4}$$

Applying (5.5.1), we see that the limit in (5.5.4) is equal to

$$\overline{\mathbf{P}}\{\sum_{\gamma>0} \zeta_\gamma \mathbf{P}_\gamma^*(g(w(t_1))\}\,(\delta-\gamma)\} = \overline{\mathbf{P}}\{\sum_{\gamma>0} \zeta_\gamma \mathbf{P}_\gamma^*\{F\}\,(\delta-\gamma)\}. \tag{5.5.5}$$

Since $\tilde{\zeta}_t(\omega)=\mathbf{P}_t^*\{F\}\,\zeta_t(\omega)$ is a function well measurable with respect to $\mathscr{A}_t$, we can apply the formula (5.4.5) to $\tilde{\zeta}_t$. Doing so, we get that (5.5.5) is equal to (5.5.3).

5.6 In this section we consider the inverse to the process ξ_t and prove that this inverse is a process with independent increments.

Denote

$$y_u = \inf\{t\colon \xi_t > u\}.$$

Lemma 5.6.1. *For any function $F(t)$ on T*

$$\int_a^b F(t)\,d\xi_t = \int_{-\infty}^{\infty} 1_{a<y_s<b} F(y_s)\,ds.$$

To prove this lemma it is sufficient to change variables in the Lebesgue integral.

Lemma 5.6.2. *The process y_s is a right-continuous strictly increasing process of pure jump type, i.e.,*

$$y_u - y_s = \sum_{s<t\leqq u} (y_t - y_{t-}). \tag{5.6.1}$$

Proof. The right-continuity of y_s follows immediately from its definition. By virtue of Lemma 5.4.3 ξ_s is a continuous increasing process; hence for all u $\xi_{y_u} = u$. Consequently $y_u > y_t$ if $u > t$. Since y_s is an increasing process, for any $u > s \geqq 0$

$$y_u - y_s \geqq \sum_{s<t\leqq u} (y_t - y_{t-}). \tag{5.6.2}$$

Take $a(U) = \xi_{\sigma_U}$. By Lemma 5.4.3 $y_{a(U)} = \sigma_U$. Since ξ_s does not increase for $s \in T \setminus M$; we have

$$\sum_{0<\gamma<\sigma_U} (\delta - \gamma) \geqq \sum_{t \leqq a(U)} (y_t - y_{t-}). \tag{5.6.3}$$

We know (see Lemma 5.4.2, section 2°) that $m(M) = 0$; therefore

$$y_{a(U)} - y_0 = \sigma_U - \sigma_0 = \sum_{0<\gamma<\sigma_U} (\delta - \gamma). \tag{5.6.4}$$

The relations (5.6.4) and (5.6.3) yield

$$y_{a(U)} - y_0 \leq \sum_{t \leqq a(U)} (y_t - y_{t-}). \tag{5.6.5}$$

Comparing (5.6.2) and (5.6.5), we see that (5.6.1) holds for all $s < u \leqq a(U)$. Since $a(U) \to \infty$ as $U \to \infty$, we get (5.6.1) for all s and u.

Lemma 5.6.3. *The process $(y_s, \overline{\mathbf{P}})$ is a $(0, \Pi)$-process with $\Pi(\Gamma) = \mathbf{P}^*\{\beta \in \Gamma\}$. Moreover $y_t - y_s$ is independent on $\mathscr{A}_{y_s}$.*

Proof. Let f be a function on T and let $A_f(s, u)$ stand for the sum $f(y_t - y_{t-})$ taken over all $t \in]s, u]$ such that $y_t > y_{t-}$. By Lemma 5.6.2

$$A_f(s, u) = \sum_{\gamma} f(\delta - \gamma)\, 1_{y_s \leqq \gamma < y_u}. \tag{5.6.6}$$

Take $A \in \mathscr{A}_{y_s}$, $\zeta_t = 1_A\, 1_{y_s \leqq t < y_u}$, $F(w_\delta^\gamma) = f(\delta - \gamma)$. Applying successively (5.6.6) and Lemma 5.5.2, we get

$$\begin{aligned}
\overline{\mathbf{P}}\{1_A A_f(s, u)\} &= \overline{\mathbf{P}}\{\sum_{\gamma} \zeta_\gamma F(w_\delta^\gamma)\} \\
&= \overline{\mathbf{P}}\{\int \zeta_t \mathbf{P}_t^*\{F\}\, d\zeta_t = \overline{\mathbf{P}}\{1_A \int 1_{y_s \leqq t < y_u} \mathbf{P}_t^*\{f(\beta - \alpha)\}\, d\zeta_t\} \\
&= \overline{\mathbf{P}}\left\{1_A \Pi(f) \int_{y_s}^{y_u} d\zeta_t\right\} = (u - s)\, \Pi(f)\, \overline{\mathbf{P}}\{A\}.
\end{aligned} \tag{5.6.7}$$

Formula (5.6.7) shows that for each f $A_f(s,u)$ is independent of $\mathscr{A}_{y_s}$ and $\overline{\mathbf{P}}\{A_f(s,u)\}=(u-s)\Pi(f)$. This implies the statement of the lemma.

5.7 In this section we derive the formula for two-dimensional distributions of $(x_t, \overline{\mathbf{P}})$. We show that they can be calculated from the characteristics of $(w(s), \mathbf{P})$.

Lemma 5.7.1. *Let* $\tilde{\mathbf{P}}_y$ *stand for the transition probabilities of* $(0, \Pi)$*-process. If* $f(t)$ *is a function on t and* $A \in \mathscr{A}_{\sigma_0}$, *then*

$$\overline{\mathbf{P}}\left\{1_A \int_0^\infty f(t)\, d\zeta_t\right\} = \overline{\mathbf{P}}\{1_A \varphi(\sigma_0)\}, \tag{5.7.1}$$

where

$$\varphi(y) = \tilde{\mathbf{P}}_y\left\{\int_0^\infty f(y_t)\, dt\right\}. \tag{5.7.2}$$

The proof of this lemma follows from Lemmas 5.6.1 and 5.6.3 and the equality $\sigma_0 = y_0$.

Lemma 5.7.2. *Put* $f(t)=1_{t<u}\mathbf{P}_t^*\{w(u)\in\Delta\}$, $\Delta\subset D$. *Let* $\varphi(y)$ *be defined by* (5.7.2). *Then for* $\Gamma\subset D$

$$\overline{\mathbf{P}}\{x_s\in\Gamma, x_u\in\Delta\} = \mathbf{P}\{w(s)\in\Gamma, w(u)\in\Delta\} + \mathbf{P}\{1_\Gamma(x_s)\,\varphi(\beta)\}. \tag{5.7.3}$$

Proof. We prove (5.7.3) for $s=0$ and $u>0$.

$$\begin{aligned}\overline{\mathbf{P}}\{x_0\in\Gamma, x_u\in\Delta\} &= \overline{\mathbf{P}}\{x_0\in\Gamma, x_u\in\Delta, [0,u]\cap M=\emptyset\}\\ &\quad+\overline{\mathbf{P}}\{x_0\in\Gamma, x_u\in\Delta, [0,u]\cap M\neq\emptyset\}.\end{aligned} \tag{5.7.4}$$

The first summand in the right side of (5.7.4) is equal to

$$\overline{\mathbf{P}}\{\sum_\gamma 1_{\gamma<0} 1_{\delta>u} 1_\Gamma(w(0))\, 1_\Delta(w(t))\} = \mathbf{P}\{w(0)\in\Gamma, w(u)\in\Delta\}. \tag{5.7.5}$$

Put $A=\{x_0\in\Gamma\}$, $\zeta_t=1_{t<u}$. The second summand in the right side of (5.7.4) can be written as

$$\overline{\mathbf{P}}\{1_A \sum_{\gamma>0} 1_\Delta(w_\delta^\gamma(u))\, 1_{\gamma<u<\delta}\} = \overline{\mathbf{P}}\{1_A \sum_{\gamma>0} \zeta_\gamma 1_\Delta(w_\delta^\gamma(u))\}. \tag{5.7.6}$$

Applying successively (5.5.3) and (5.7.1) we get that (5.7.6) is equal to

$$\begin{aligned}\overline{\mathbf{P}}\{1_\Gamma(x_0)\mathbf{P}_{0,x_0}\{\varphi(\beta)\}\} &= \int_\Gamma \nu(dx)\,\mathbf{P}_{0,x}(\varphi(\beta))\\ =\mathbf{P}\{1_\Gamma(w(0))\,\mathbf{P}_{0,x_0}\{\varphi(\beta)\}\} &= \mathbf{P}\{1_\Gamma(w(0))\,\varphi(\beta)\}.\end{aligned} \tag{5.7.7}$$

Adding (5.7.5) to (5.7.7) we get (5.7.3).

Lemma 5.7.2 implies easily Theorem 2.

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