Z. Wahrscheinlichkeitstheorie verw. Gebiete
55, 255–259 (1981)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Weak Convergence to the Law of Two-Parameter Continuous Processes

David Nualart

Universitat de Barcelona, Facultat de Matemàtiques, Departament d'Estadistica,
Gran Via 585, Barcelona 7, Espanya

Introduction

Let $C(S)$ be the space of real valued continuous functions on a compact metric space (S, d), and let X be a $C(S)$-valued random variable. Sufficient conditions for the central limit theorem to hold for X can be given in terms of the ε-entropy of S with respect to some continuous distances associated to X; see Giné [4], Dudley [2] and Jain and Marcus [6].

If X and Y are independent $C(S)$-valued random variables in some $(\Omega, \mathscr{F}, P)$, we are going to use these results to state sufficient conditions for the central limit theorem to hold for the $C(S^2)$-valued random variable corresponding to the process $\{X(s)Y(t); (s,t)\in S^2\}$. This variable will be denoted by $X * Y$.

As an application we discuss in Sect. 2 the convergence to the law of a two-parameter Wiener process.

Section 1

In a metric space S with metric ρ, we denote by $N_\rho(S, \varepsilon)$ the minimal number of balls or radius ε which cover S, and the ε-entropy of S is defined as $H_\rho(S, \varepsilon) = \log N_\rho(S, \varepsilon)$.

If X is a zero mean $C(S)$-valued random variable such that $\sup_{s\in S} E(X^2(s)) < \infty$ we can consider the metric τ on S given by $\tau(s,t) = [E(|X(s) - X(t)|^2)]^{1/2}$ and we will say that X has subgaussian increments (see [6ᵗ) if there exists a positive constant A such that

$$E[\exp(\lambda(X(s) - X(t))] \leqq \exp(A\lambda^2 \tau(s,t)^2), \tag{1.1}$$

for all $s, t\in S$, $\lambda \in R$.

Using basically the results of Jain and Marcus [6] we can state the following

0044-3719/81/0055/0255/$01.00

Theorem 1.1. *Let (S, d) be a compact metric space and let X be a zero mean square integrable $C(S)$-valued random variable. Consider the following conditions on X:*

(a) *There exists a nonnegative square integrable random variable M_1 and a continuous metric ρ_1 on S such that, given $s, t \in S$, $\omega \in \Omega$*

$$|X(s, \omega) - X(t, \omega)| \leqq M_1(\omega)\, \rho_1(s, t) \tag{1.2}$$

$$\int_0^1 H_{\rho_1}^{1/2}(S, u)\, du < \infty. \tag{1.3}$$

(b) *X has subgaussian increments and the metric τ_1 associated to the covariance of X is continuous and verifies*

$$\int_0^1 H_{\tau_1}^{1/2}(S, u)\, du < \infty. \tag{1.4}$$

Then, if X and Y are zero mean, square integrable $C(S)$-valued independent random variables satisfying one of these properties, $X * Y$ verifies the central limit theorem on $C(S^2)$.

Proof. Denote by ρ_2, τ_2 the metrics and by M_2 the random variable corresponding to Y in conditions (a) and (b).

If (a) holds for X and Y, the theorem is an immediate consequence of Theorem 1 of [6]. Indeed, $X * Y$ also verifies condition (a) on $C(S^2)$, with metric $(\rho_1 \vee \rho_2)((s, t), (s', t')) = \max\{\rho_1(s, s'), \rho_2(t, t')\}$ and nonnegative random variable $\sup_s |X(s)| \cdot M_2(\omega) + \sup_s |Y(s)| \cdot M_1(\omega)$.

Under condition (b) the proof follows from Lemma 1 of [6] using an argument similar to the proof of Theorem 2 of [6].

In fact, let $(\Omega_1 \times \Omega_2, \mathscr{F}_1 \times \mathscr{F}_2, P_1 \times P_2)$ be a product probability space such that $\{X^n, n \geqq 1\}$, a sequence of independent copies of X, is defined on $(\Omega_1, \mathscr{F}_1, P_1)$, and a sequence $\{Y^n, n \geqq 1\}$ of independent copies of Y is defined on $(\Omega_2, \mathscr{F}_2, P_2)$. Denote by E_1 and E_2 the expectations with respect to P_1 and P_2 respectively.

The random variables $\|X\| = \sup_s |X(s)|$ and $\|Y\| = \sup_s |Y(s)|$ are square integrable (see Sect. 6.1 of [3] adapted to random variables with subgaussian increments and satisfying (1.4)), and we can suppose without loss of generality that $E(\|X\|^2) = 1$ and $E(\|Y\|^2) = 1$.

Let $\eta > 0$ given. By the strong law of large numbers there exists n_0, $\Omega_{1\eta} \subset \Omega_1$ with $P_1(\Omega_{1\eta}) \geqq 1 - \eta$ and $\Omega_{2\eta} \subset \Omega_2$ with $P_2(\Omega_{2\eta}) \geqq 1 - \eta$ such that for all $\omega_1 \in \Omega_{1\eta}$, $\omega_2 \in \Omega_{2\eta}$ and $n \geqq n_0$

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n \|X^i(\omega_1)\|^2 \leqq 2, \qquad \frac{1}{\sqrt{n}} \sum_{i=1}^n \|Y^i(\omega_2)\|^2 \leqq 2. \tag{1.5}$$

Set $\Omega_\eta = \Omega_{1\eta} \times \Omega_{2\eta} \subset \Omega$. For each $\lambda > 0$, $n \geqq n_0$, and $z = (s, t)$, $z' = (s', t')$ in S^2, we have

$$
\begin{aligned}
&P\left\{\Omega_\eta \cap \left(\frac{1}{\sqrt{n}}\left|\sum_{i=1}^n X^i(s)Y^i(t)-X^i(s')Y^i(t')\right|>6A\lambda\right)\right\}\\
&\leqq P\left\{\Omega_\eta \cap \left(\frac{1}{\sqrt{n}}\left|\sum_{i=1}^n X^i(s)(Y^i(t)-Y^i(t'))\right|>3A\lambda\right)\right\}\\
&\quad +P\left\{\Omega_\eta \cap \left(\frac{1}{\sqrt{n}}\left|\sum_{i=1}^n Y^i(t')(X^i(s)-X^i(s'))\right|>3A\lambda\right)\right\}\\
&\leqq 2\int_{\Omega_{1\eta}} e^{-3A\lambda\beta} E_2\left(\exp\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n X^i(s)(Y^i(t)-Y^i(t'))\,\beta\right)\right) dP_1\\
&\quad +2\int_{\Omega_{2\eta}} e^{-3A\lambda\beta} E_1\left(\exp\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n Y^i(t')(X^i(s)-X^i(s'))\,\beta\right)\right) dP_2\\
&\leqq 2e^{-3A\lambda\beta}(\exp(2A\beta^2\tau_2(t,t')^2)+\exp(2A\beta^2\tau_1(s,s')^2))\\
&\leqq 4e^{-3A\lambda\beta}\exp(2A\beta^2(\tau_1\vee\tau_2)(z,z')^2)\\
&\leqq 4\exp(-\lambda^2/(\tau_1\vee\tau_2)(z,z')^2),
\end{aligned}
$$

taking $\beta=\lambda/(\tau_1\vee\tau_2)(z,z')^2$.

Now the function $\phi(x)=4e^{-x^2/36A^2}$ verifies on Ω_η the hypotheses of Lemma 1 of [6] applied to the process $\frac{1}{\sqrt{n}}\sum_{i=1}^n X^i(s)Y^i(t)$, $(s,t)\in S^2$ and the metric $\tau_1\vee\tau_2$ on S^2.

Thus, given $\varepsilon>0$, $\eta>0$, there exists n_0, $\delta>0$ such that

$$
P\left\{\sup_{\substack{d(s,s')\leqq\delta\\ d(t,t')\leqq\delta}} \frac{1}{\sqrt{n}}\left|\sum_{i=1}^n X^i(s)Y^i(t)-X^i(s')Y^i(t')\right|>\varepsilon\right\}<\eta+P(\Omega_\eta^c)\leqq 3\eta
$$

for every $n\geqq n_0$, and the proof of the theorem follows easily. □

Note. In Theorem 1.1 the condition

$$
\int_0^1 H_\tau^{1/2}(S,u)\,du<\infty \tag{1.6}
$$

where τ represents one of the metrics ρ_1, ρ_2, τ_1, or τ_2, can be replaced by the following hypothesis:

There exists a probability measure λ on S (provided with the τ-Borel sigma-field) such that

$$
\lim_{\varepsilon\downarrow 0}\sup_{s\in S}\int_0^\varepsilon \sqrt{\log\frac{1}{\lambda\{t:\rho(s,t)<u\}}}\,du=0. \tag{1.7}
$$

This condition implies property (1.6) of the metric entropy (see [3], Corollary 6.2.4).

Under hypothesis (1.7) the proof of Theorem 1.1 would follow the same lines that the preceding arguments, but using Proposition 1.1 of [5] instead of Lemma 1 of [6].

Section 2

A two-parameter Wiener process $\{W(s,t);\ (s,t)\in[0,1]^2\}$ is a Gaussian, zero mean process with the covariance function

$$E[W(s_1,t_1)\,W(s_2,t_2)]=(s_1\wedge s_2)(t_1\wedge t_2).$$

It is well-known (see [1]) that there exists a version of W with continuous paths. Then, the results of Sect. 1 lead to the following convergence to the law of W:

Proposition 2.1. *Let $\{X^n(t);\, t\in[0,1],\ n\in N\}$ and $\{Y^n(t);\, t\in[0,1],\ n\in N\}$ be two independent infinite dimensional Brownian motions. Then the sequence of two-parameter continuous processes*

$$Z_n(s,t)=\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X^i(s)\,Y^i(t)$$

converges weakly to a two-parameter Wiener process.

This result can be used to provide some estimates for the distribution of $F(W)$ where F is a continuous functional on $C([0,1]^2)$.

For instance, the well-known upper bound for the tail distribution of the maximum of $W(s,t)$ (see [7]) can be obtained by the limit of the following sequence of inequalities.

Proposition 2.2.

$$P\{\sup_{s,t\in[0,1]} Z_n(s,t)>\lambda\}\leqq 4P\{Z_n(1,1)>\lambda\}.$$

Proof. As before let $(\Omega_1\times\Omega_2,\ \mathscr{F}_1\times\mathscr{F}_2,\ P_1\times P_2)$ be a product probability space such that two independent infinite dimensional Brownian motions $\{X^n(t);$ $t\in[0,1],\ n\in N\}$ and $\{Y^n(t);\ t\in[0,1],\ n\in N\}$ are defined on $(\Omega_1,\mathscr{F}_1,P_1)$ and $(\Omega_2,$ $\mathscr{F}_2,P_2)$ respectively.

Define

$$S(\omega_1,\omega_2)=\inf\{s\geqq 0\colon \text{there exists } t\in[0,1] \text{ such that } Z_n(s,t;\ \omega_1,\omega_2)>\lambda\},$$

and

$$T(\omega_1,\omega_2)=\inf\{t\geqq 0\colon Z_n(S(\omega_1,\omega_2),t;\ \omega_1,\omega_2)>\lambda\},$$

where $S=1$ and $T=1$ if the above sets are empty.

S and T are stopping times with respect to the increasing family $\{\mathscr{F}_1(s)\times\mathscr{F}_2;$ $s\geqq 0\}$, where $\{\mathscr{F}_1(s);\ s\geqq 0\}$ is the natural right continuous filtration associated to the n-dimensional Brownian motion $\{X^1(t),\dots,X^n(t);\ t\in[0,1]\}$.

Then, using the classical reflection principle, we have

$$\begin{aligned}&P\{\sup_{s,t\in[0,1]} Z_n(s,t)>\lambda\}\\&\quad=E_2(P_1\{Z_n(1,T)>\lambda\}+P_1\{\sup_{s,t\in[0,1]} Z_n(s,t)>\lambda,\ Z_n(1,T)\leqq\lambda\})\end{aligned}$$

$$
\begin{aligned}
&= E_2(P_1\{Z_n(1,T)>\lambda\}+P_1\{Z_n(S,T)\geqq\lambda, Z_n(1,T)\leqq\lambda\}) \\
&= 2E_2(P_1\{Z_n(1,T)>\lambda\})\leqq 2E_2(P_1\{\sup_{t\in[0,1]} Z_n(1,t)>\lambda\}) \\
&= 2E_1(P_2\{\sup_{t\in[0,1]} Z_n(1,t)>\lambda\})=4E_1(P_2\{Z_n(1,1)>\lambda\}) \\
&= 4P\{Z_n(1,1)>\lambda\}. \quad \square
\end{aligned}
$$

Finally we can state the following version of the functional law of the iterated logarithm which is implied by the central limit theorem in the case of $C(S)$-valued square integrable random variables (see [8]).

Proposition 2.3. *Let $\{X^n(t);\ t\in[0,1],\ n\in N\}$ and $\{Y^n(t);\ t\in[0,1],\ n\in N\}$ be two independent infinite dimensional Brownian motions. Then, for almost every $\omega\in\Omega$ the set of limit points of*

$$\sum_{i=1}^{n} \frac{X^i(s,\omega)\, Y^i(t,\omega)}{\sqrt{2n\log\log n}}$$

is equicontinuous in $C([0,1]^2)$ and coincides with the family of all absolutely continuous functions h defined on $[0,1]^2$ which vanishes along the axes and satisfy

$$\int_0^1\int_0^1 \left(\frac{\partial^2 h}{\partial s\,\partial t}\right)^2 ds\,dt \leqq 1.$$

In fact, this family of functions is the unit closed ball of the reproducing Kernel Hilbert space of a two-parameter Wiener process.

Acknowledgments. I would like express my gratitude to B. Heinkel for his valuable remarks concerning the use of hypothesis (1.7) in Theorem 1.1. I am also grateful to E. Giné for his helpful advice.

References

1. Dudley, R.M.: Sample functions of the Gaussian processes. Ann. Probab. **1**, 66–103 (1973)
2. Dudley, R.M.: Metric entropy and the Central Limit Theorem in $C(S)$. Ann. Inst. Fourier, **24**, 49–60 (1974)
3. Fernique, X.M.: Regularité des trajectoires des fonctions aléatoires Gaussiennes. Lec. Notes in Math. **480**, 1–96. Berlin-Heidelberg-New York: Springer 1974
4. Giné, E.: On the Central Limit Theorem for sample continuous processes. Ann. Probability, **2**, 629–641 (1974)
5. Heinkel, B.: Mesures majorantes et théorème de la limite centrale dans $C(S)$. Z. Wahrscheinlichkeitstheorie verw. Gebiete **38**, 339–351 (1977)
6. Jain, N.C., Marcus, M.B.: Central Limit Theorems for $C(S)$-valued random variables. Journal of Functional Analysis, **19**, 216–231 (1975)
7. Orey, S., Pruitt, W.E.: Sample functions of the N-parameter Wiener process. Ann. Probability, **1**, 138–163 (1973)
8. Pisier, G.: Le théorème de la limite centrale et la loi du logarithme itéré dans les espaces de Banach. Séminaire Maurey-Schwarz. Ecole Politechnique, Palaiseau 1975

Received March 3, 1980; in revised form August 5, 1980