Z. Wahrscheinlichkeitstheorie verw. Gebiete
67, 399–432 (1984)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Duality Theorems for Marginal Problems

Hans G. Kellerer

Universität München, Mathematisches Institut, Theresienstraße 39, D-8000 München 2,
Federal Republic of Germany

Summary. Given topological spaces $X_1, \ldots, X_n$ with product space X, probability measures μ_i on X_i together with a real function h on X define a marginal problem as well as a dual problem. Using an extended version of Choquet's theorem on capacities, an analogue of the classical duality theorem of linear programming is established, imposing only weak conditions on the topology of the spaces X_i and the measurability resp. boundedness of the function h. Applications concern, among others, measures with given support, stochastic order and general marginal problems.

Introduction

An extension of the well-known transportation problem to an infinite number of origins resp. destinations (to some extent considered as early as in 1781 by Monge [17]), combined with a change of sign in the cost function and a generalization from $n=2$ to $n \in \mathbf{N}$, leads to the following measure theoretic version:

Given probability distributions μ_i on spaces X_i, $1 \leqq i \leqq n$, and a real function h on their product $X = X_1 \times \ldots \times X_n$, consider the "marginal problem"

(1) $$\text{maximize} \int_X h \, d\mu,$$

where the distribution μ on X is subject to the restriction

$$\mu(\{x \in X\colon\ x_i \in B_i\}) = \mu_i(B_i) \quad \text{for } 1 \leqq i \leqq n.$$

As in classical linear programming, this gives rise to a "dual problem"

(2) $$\text{minimize} \sum_{1 \leqq i \leqq n} \int_{X_i} h_i \, d\mu_i,$$

where the real functions h_i on X_i are subject to the restriction

$$\sum_{1 \leqq i \leqq n} h_i(x_i) \geqq h(x_1, \ldots, x_n) \quad \text{for all } x_i.$$

Then, disregarding for the moment the question whether the maximum in (1) resp. the minimum in (2) is really attained, the main problem to be settled is the validity of the "duality theorem" max(1) = min(2).

To mention just one important example, consider the case $n=2$ with spaces X_i both equal to some space Y with metric d. Choosing $h=-d$, up to a change of sign, the marginal problem provides the Wasseršteín distance (see [25]) of two probability measures μ_1, μ_2 on Y, while the dual problem leads to their Kantorovitch-Rubinštein distance (see [10]). In this case the duality theorem amounts exactly to the coincidence of both metrics - and in fact holds true for compact spaces Y. This result was extended to separable metric spaces by Dudley [3], but his proof contained a gap, which was filled only by Fernique [5] for polish spaces and by the author [12] under weaker assumptions.

Returning to the general case, but restricting the spaces X_i to be compact metric or at least polish and the function h to be bounded continuous, the duality theorem essentially results from an application of the Hahn-Banach theorem, combined with the Riesz representation theorem. This fact, however, remains hidden in the papers of Rüschendorf [19] and Gaffke-Rüschendorf [6] from 1981, which are devoted roughly to this situation and are until now the only relatively general treatments of related duality theorems.

Unfortunately, the methods there as well as the procedure mentioned above fail to settle more general situations as encountered in the present paper. The main idea in this treatment consists in regarding the measures μ_i as fixed and the function h as variable and thus introducing $S(h)$ as the supremum in (1) and $I(h)$ as the infimum in (2).

Each of the functionals S and I - and this is the crucial point - can be shown to enjoy three of four possible continuity properties. First, both S and I are τ-continuous upwards ((1.23), (1.29)) and downwards ((1.26), (1.30)), from which the validity of the duality theorem can be deduced for lower as well as upper semicontinuous functions ((2.2) and (2.6)), the first result being more surprising than the second one. Moreover, both functionals - though failing to be σ-continuous downwards - are at least σ-continuous upwards ((1.21), (1.28)). By a functional version of Choquet's capacitability theorem (stated in (2.11)) these continuity properties, put together, lead to the central result (2.14): without any special topological assumptions, the equation $S(h)=I(h)$ holds for all Suslin functions h (defined in (2.12)), provided they are bounded below in some weaker sense - a condition, which can be shown to be essential.

To end these introductory remarks, the different sections making up the three parts of the paper will be briefly summarized:

After providing in Sect. 1.1 the more or less elementary tools needed permanently in the sequel, the next two sections are devoted to the continuity properties of the functionals S and I stated above. The proofs are achieved mainly by appropriate compactness concepts, which are much more involved for the n-tuples of functions h_i figuring in the definition of I than for the measures μ in that of S. Returning to the Hahn-Banach approach, Sect. 1.4 clarifies, to what extent continuity properties of the function h can be transferred to $h_1, \ldots, h_n$.

Section 2.1 starts from the duality theorem for finite spaces X_i, settles the situation for semicontinuous functions h and ends up with the case of countable spaces X_i, without any boundedness - or even finiteness - condition on h. After having collected the needed concepts and facts about Suslin functions and corresponding capacities in the next section, the step from continuity to measurability is taken in Sect. 2.3. From the key result (2.14) it is derived, for instance, that the duality theorem holds quite generally, whenever the underlying spaces X_i are metrizable or second countable (leaving the general compact case, however, as an open problem). The final section of part 2 proves the supremum $S(h)$ to be attained in the case of sufficiently smooth functions h as well as the infimum $I(h)$ in the case of suitably bounded functions h.

The first section in the applications clarifies, which functions h are integrable (possibly to the same value) for all solutions μ of a given marginal problem. Section 3.2 is limited to the case $n=2$ and devoted to measures with given support. It unifies results of Strassen, Hoffmann-Jørgensen and Edwards (in the topological setting) as well as results of Sudakov and Shortt (in a more abstract setting). The next section considers in addition an order structure, transfers monotony properties of the function h to $h_1, \ldots, h_n$ and strengthens thereby results obtained by Kamae-Krengel-O'Brien and again Hoffmann-Jørgensen and Edwards. Section 3.4 resumes a general marginal problem treated in 1964 by the author for $X_i = \mathbf{R}$ and investigated later on by Maharam and Lembcke among others. In particular, by this last application the restriction to tight measures throughout this paper can be shown to be essential.

Finally, the work of Levin-Milyutin [15] at least should be mentioned; but in spite of the title it treats an essentially different problem, having no extension to general n and even for $n=2$ making sense only in the case $X_1 = X_2$.

Notations

1. If Y is an arbitrary set, $\mathfrak{P}(Y)$ denotes its power set and $\mathscr{P}(Y)$ the set of all functions from Y to the extended real line $\tilde{\mathbf{R}}$.

A family $\mathfrak{A} \subset \mathfrak{P}(Y)$ or $\mathscr{A} \subset \mathscr{P}(Y)$ is called a lattice (σ-lattice), if it is stable with respect to finite (countable) lattice operations. The σ-lattice generated by arbitrary $\mathfrak{A}$ or $\mathscr{A}$ is denoted by $\sigma(\mathfrak{A})$ or $\sigma(\mathscr{A})$, respectively.

If $\mathscr{A}$ is any subset of $\mathscr{P}(Y)$, then (using the index b for "bounded" and f for "finite")

$$\mathscr{A}_b = \{g \in \mathscr{A} : \inf_Y g > -\infty\},$$
$$\mathscr{A}_f = \{g \in \mathscr{A} : g(y) > -\infty \textit{ for all } y \in Y\}$$

with an analogous meaning of $\mathscr{A}^b$ and $\mathscr{A}^f$.

2. Without further notice, in the sequel all topological spaces are assumed to be Hausdorff. For such a space Y,

(a) $\mathfrak{G}(Y)$ is the family of all open sets in Y,
$\mathscr{G}(Y)$ is the family of all lower semicontinuous functions in $\mathscr{P}(Y)$,

(b) $\mathfrak{F}(Y)$ is the family of all closed sets in Y,
$\mathscr{F}(Y)$ is the family of all upper semicontinuous functions in $\mathscr{P}(Y)$,
(c) $\mathfrak{B}(Y)$ is the family of all Borel sets in Y,
$\mathscr{B}(Y)$ is the family of all Borel measurable functions in $\mathscr{P}(Y)$;
(d) $\mathfrak{C}(Y)$ is the family of all Baire sets in Y,
$\mathscr{C}(Y)$ is the family of all continuous functions in $\mathscr{P}(Y)$.
Moreover, $\mathscr{E}(Y)$ denotes the family of all finite elementary functions in $\mathscr{B}(Y)$ (i.e. $g[Y]\subset\mathbf{R}$ and finite for $g\in\mathscr{E}(Y)$).

3. If ν is any measure on $\mathfrak{B}(Y)$, the space of all ν-integrable functions $g\in\mathscr{B}(Y)$ is denoted by $\mathscr{L}(\nu)$ (without identifying ν-almost equal functions) and the respective integrals are abbreviated by $\nu(g)$. Similarly, ν_* and ν^* are used for inner and outer measure as well as for lower and upper integral.

4. Suppressing again an index, $\mathsf{M}(Y)$ denotes the space of all tight (or Radon) probability measures ν on $\mathfrak{B}(Y)$; since these measures are in particular τ-continuous (i.e. continuous with respect to downwards resp. upwards directed families in $\mathfrak{F}(Y)$ resp. $\mathfrak{G}(Y)$), they have a well defined support, which is denoted by supp ν.

$\mathsf{M}(Y)$ is endowed with the weak (or narrow) topology, i.e. the (Hausdorff) topology generated by (one of) the requirements

$$\nu\mapsto\nu(g) \text{ is lower semicontinuous for all bounded } g\in\mathscr{G}(Y),$$

$$\nu\mapsto\nu(g) \text{ is upper semicontinuous for all bounded } g\in\mathscr{F}(Y).$$

5. Henceforth $X_1,\dots,X_n$ are fixed topological spaces with product space X and canonical projections π_i: $X\to X_i$.

The abbreviation

$$\bigoplus_i g_i := g_1\oplus\dots\oplus g_n=\sum_i g_i\circ\pi_i$$

is used, provided the functions g_i belong all to $\mathscr{P}_f(X_i)$ or all to $\mathscr{P}^f(X_i)$.

Moreover, $\mathscr{E}(X_1)\otimes\dots\otimes\mathscr{E}(X_n)$ denotes the tensor product of the spaces $\mathscr{E}(X_i)$, consisting of finite sums of products $\prod_i g_i\circ\pi_i$ with $g_i\in\mathscr{E}(X_i)$.

Finally, the measures μ_i are assumed to be fixed elements of $\mathsf{M}(X_i)$ in the sequel, which – though crucial – are suppressed in the definitions of Sect. 1.1.

1. The Functionals S and I

1.1. Preliminaries

To introduce the functionals in question, the basic set of measures has to be fixed first:

(1.1) *Definition.* $\mathsf{M}:=\{\mu\in\mathsf{M}(X)\colon\ \pi_i(\mu)=\mu_i \text{ for } 1\leqq i\leqq n\}$.

The essential properties of this set are summarized in:

(1.2) **Proposition.** M *is a non-void compact convex set.*

Proof. The product measure $\mu_1\otimes\dots\otimes\mu_n$, being defined only on $\mathscr{B}(X_1)\otimes\dots\otimes\mathscr{B}(X_n)$, has a unique extension $\mu\in\mathsf{M}(X)$ (see [21, p. 63]), which obviously is an element of M. Thus, the convexity being clear, only the compactness remains to be verified. To this end, let $\varepsilon>0$ be arbitrary, choose compact sets $K_i \subset X_i$ with the property $\mu_i(\complement K_i)<\frac{\varepsilon}{n}$ and consider the compact set $K:=K_1 \times \ldots \times K_n$. Evidently

$$\mu(\complement K) \leqq \sum_i \mu(\pi_i^{-1}[\complement K_i])<\varepsilon \quad \text{for all } \mu \in \mathsf{M},$$

i.e. the set M is uniformly tight and thus relatively compact (see [21, p. 379]). But in addition M is closed, since the continuity of π_i implies the continuity of the map $\mu \mapsto \pi_i(\mu)$ (see [21, p. 372]), and the proof is completed. □

Now the main definition can be given:

(1.3) *Definition.* Let $h \in \mathscr{P}(X)$ be arbitrary; then

$$S(h):=\sup\{\mu^*(h)\colon \mu \in \mathsf{M}\},$$
$$I(h):=\inf\{\sum_i \mu_i(h_i)\colon h_i \in \mathscr{L}_f(\mu_i) \text{ and } h \leqq \bigoplus_i h_i\}.$$

Here, the use of the upper integral in defining S is adequate as is seen by the case $n=1$, while in defining I the usual convention $\inf \emptyset=+\infty$ is made. Moreover, in the case $h=1_B$ the notations $S(B)$ and $I(B)$ will be used.

Now one half of the duality equation is as immediate as the other half is intricate:

(1.4) **Proposition.** *$S(h) \leqq I(h)$ for all $h \in \mathscr{P}(X)$.*

Proof. $\mu \in \mathsf{M}$ and $h_i \in \mathscr{L}_f(\mu_i)$ with $h \leqq \bigoplus_i h_i$ yield

$$\mu^*(h) \leqq \mu^*(\bigoplus_i h_i) \leqq \sum_i \mu^*(h_i \circ \pi_i)=\sum_i \mu_i(h_i). \quad \square$$

The following property will be used frequently:

(1.5) **Proposition.** *The functionals S and I are isotone.*

Proof. Obvious. □

Of course, only a very restricted additivity holds:

(1.6) **Lemma.** *If $h \in \mathscr{P}(X)$ is arbitrary, $h_i \in \mathscr{L}(\mu_i)$ is finite and $h_0:=\bigoplus_i h_i$, then:*

(a) $S(h+h_0)=S(h)+S(h_0)$ *and* $I(h+h_0)=I(h)+I(h_0)$,

(b) $S(h_0)=\sum_i \mu_i(h_i)=I(h_0)$.

Proof. (a) follows immediately, while (b) – for I – employs (1.4). □

The next result makes essential use of the convexity of M:

(1.7) **Lemma.** *If $h \in \mathscr{P}(X)$ satisfies $\mu^*(h)<\infty$ for all $\mu \in \mathsf{M}$, then $S(h)<\infty$ as well.*

Proof. Assuming $\mu^*(h)<\infty$ for all $\mu \in \mathsf{M}$ and $S(h)=\infty$, there are measures $\mu_k \in \mathsf{M}$ with $\mu_k^*(h)>2^k$ for $k \in \mathbf{N}$. According to (1.2) the measure $\mu_0:=\sum_{k \in \mathbf{N}} 2^{-k}\mu_k$ is contained in M, hence $\mu_0^*(h)<\infty$ and there is a function $f \in \mathscr{L}(\mu_0)$ with $h \leqq f$. In view of $\mathscr{L}(\mu_0)\subset\mathscr{L}(\mu_k)$, however,

$$\mu_0(f)=\sum_{k\in\mathbf{N}} 2^{-k}\mu_k(f)\geqq \sum_{k\in\mathbf{N}} 2^{-k}\mu_k^*(h)=\infty,$$

i.e. the assumption leads to a contradiction. □

The following fact, carrying over boundedness properties from h to h_i, will be used repeatedly:

(1.8) **Lemma.** *For $h\in\mathscr{P}(X)$ define $\underline{\gamma}:=\inf_X h$ and $\bar{\gamma}:=\sup_X h$; then the functions h_i appearing in the definition of $I(h)$ may be assumed to satisfy the additional requirement*

(a) $\frac{1}{n}\underline{\gamma}\leqq h_i\leqq\frac{1}{n}\underline{\gamma}+(\bar{\gamma}-\underline{\gamma})$ *in the case* $\underline{\gamma}\in\mathbf{R}$,

(b) h_i *is bounded* ▭ *in the case* $\bar{\gamma}\in\mathbf{R}$.

Proof. (a) Apparently $\underline{\gamma}=0$ may be assumed, hence $\bigoplus_i h_i\geqq h$ implies

$$\sum_i \inf_{X_i} h_i=\inf_X \bigoplus_i h_i\geqq \inf_X h=0;$$

thus there are constants $\gamma_i\leqq\inf_{X_i} h_i$ with $\sum_i\gamma_i=0$, and h_i may be replaced $h_i-\gamma_i\geqq 0$ without changing $\sum_i \mu_i(h_{h^i})$. Using now $h_i\geqq 0$, the inequality $\bigoplus_i h_i\geqq h$ remains true if h_i is replaced by $h_i\wedge\bar{\gamma}$.

(b) Because of $h_i=\inf_{k\in\mathbf{N}} h_i\vee(-k)$ the functions h_i may be assumed to be bounded below, therefore in view of $\bar{\gamma}<\infty$ to be bounded above as well. □

By the last result, the functionals S and I can be shown to be σ-subadditive:

(1.9) **Lemma.** *If $0\leqq h^k\in\mathscr{P}(X)$ for $k\in\mathbf{N}$, then*

$$S(\sum_{k\in\mathbf{N}} h^k)\leqq \sum_{k\in\mathbf{N}} S(h^k) \quad \text{and} \quad I(\sum_{k\in\mathbf{N}} h^k)\leqq \sum_{k\in\mathbf{N}} I(h^k).$$

Proof. The first inequality is a consequence of the σ-subadditivity of μ^* for all $\mu\in\mathsf{M}$. Concerning the second inequality, the functions $h_i^k\in\mathscr{L}_f(\mu_i)$ with $h^k\leqq\bigoplus_i h_i^k$ may be assumed to be non-negative in view of (1.8a); but under this restriction the assertion follows easily. □

Having collected these basic facts about the functionals S and I, it is possible to define an appropriate metric in the space $\mathscr{P}(X)$. First the relevant null sets have to be introduced:

(1.10) *Definition.* $\mathfrak{N}:=\{A\in\mathfrak{P}(X)\colon A\subset\bigcup_i \pi_i^{-1}[N_i]$ with $\mu_i(N_i)=0$ for all $i\}$.

An immediate consequence is:

(1.11) **Proposition.** *$\mathfrak{N}$ is stable with respect to the formation of subsets and countable unions.*

Proof. Obvious. □

This class gives rise to the following notions:

(1.12) *Definition.* For functions $f, g \in \mathscr{P}(X)$ the relations $f \underset{N}{=} g$ and $f \underset{N}{\leqq} g$ are defined by the requirements $\{f \neq g\} \in \mathfrak{N}$ and $\{f > g\} \in \mathfrak{N}$, respectively.

Thus for instance $h_i \underset{\mu_i}{\leqq} h'_i$ for all i implies $\underset{i}{\bigoplus} h_i \underset{N}{\leqq} \underset{i}{\bigoplus} h'_i$ (if both sides are well defined).

The main facts concerning these notions are summarized in:

(1.13) **Proposition.** $\underset{N}{=}$ *defines an equivalence relation and* $\underset{N}{\leqq}$ *a partial order in* $\mathscr{P}(X)$, *both compatible with countable lattice operations.*

Proof. See (1.11). □

Of course, (1.12) and (1.13), when specialized to indicator functions, define relations $\underset{N}{=}$ and $\underset{N}{\subset}$ in $\mathfrak{P}(X)$ with analogous properties.

After these preparations the required metric can be introduced:

(1.14) *Definition.* $d(f, g) := I(|f-g|)$ for $f, g \in \mathscr{P}(X)$.

Here, as in related situations, $f(x)-g(x)$ has to be interpreted as 0 whenever the values $f(x)$ and $g(x)$ are equal; then:

(1.15) **Proposition.** *With respect to* $\underset{N}{=}$, *d defines a (possibly infinite) metric in* $\mathscr{P}(X)$.

Proof. Since the triangle inequality follows from (1.9), the main point to be checked is the equivalence of the relations $f \underset{N}{=} g$ and $d(f, g)=0$.

(a) Given sets N_i such that

$$\{f \neq g\} \subset \bigcup_i \pi_i^{-1}[N_i] \quad \text{and} \quad \mu_i(N_i)=0,$$

the functions $h_i := \infty \cdot 1_{N_i} \in \mathscr{L}_f(\mu_i)$ satisfy

$$|f-g| \leqq \bigoplus_i h_i \quad \text{and} \quad \sum_i \mu_i(h_i)=0.$$

(b) If conversely $I(|f-g|)=0$, then (1.8a) provides functions $h_i^k \in \mathscr{L}(\mu_i)$ with $h_i^k \geqq 0$ such that

$$|f-g| \leqq \bigoplus_i h_i^k \quad \text{and} \quad \sum_i \mu_i(h_i^k) < 2^{-k} \qquad \text{for all } k \in \mathbf{N};$$

hence the functions $h_i := \limsup_{k \to \infty} h_i^k \geqq 0$ satisfy

$$|f-g| \leqq \bigoplus_i h_i \quad \text{and} \quad \sum_i \mu_i(h_i)=0,$$

and the sets $N_i := \{h_i \neq 0\}$ fulfill the requirements in (a). □

The metric d is easily seen to be appropriate:

(1.16) **Proposition.** *The functionals S and I are continuous with respect to d.*

Proof. Given functions $f, g \in \mathscr{P}(X)$ with $\delta := d(f,g) < \infty$, applying (1.4) it follows as in the proof of (1.9) that

$$S(g) \leqq S(f) + \delta \quad \text{and} \quad I(g) \leqq I(f) + \delta. \quad \square$$

It is an immediate consequence of the continuity that the functionals S and I are constant on equivalence classes with respect to $\underset{N}{=}$.

The following notation will be used in the sequel:

(1.17) *Definition.* $\bar{\mathscr{A}}$ denotes the closure of $\mathscr{A} \subset \mathscr{P}(X)$ with respect to d.

Then topology and lattice structure in $\mathscr{P}(X)$ are compatible in the following sense:

(1.18) **Lemma.** *If $\mathscr{A} \subset \mathscr{P}(X)$ is stable with respect to finite resp. countable infima or suprema, the same holds true for $\bar{\mathscr{A}}$.*

Proof. For $f := \inf_{k \in \mathbf{N}} f^k$ (or $\sup_{k \in \mathbf{N}} f^k$) and $g := \inf_{k \in \mathbf{N}} g^k$ (or $\sup_{k \in \mathbf{N}} g^k$) an application of (1.9) yields the inequality

$$d(f,g) \leqq I\Big(\sum_{k \in \mathbf{N}} |f^k - g^k|\Big) \leqq \sum_{k \in \mathbf{N}} d(f^k, g^k),$$

by which the assertion is easily established. $\square$

The final notion concerns the question whether a function $f \in \mathscr{P}(X)$ is *m*inorized or *m*ajorized by a sum $\bigoplus_i h_i$; more precisely:

(1.19) *Definition.* If $\mathscr{A} \subset \mathscr{P}(X)$ is arbitrary, then

$$\mathscr{A}_m := \{f \in \mathscr{A} :\ f \geqq \bigoplus_i h_i \text{ for suitable } h_i \in \mathscr{L}^f(\mu_i)\},$$

$$\mathscr{A}^m := \{f \in \mathscr{A} :\ f \leqq \bigoplus_i h_i \text{ for suitable } h_i \in \mathscr{L}_f(\mu_i)\}.$$

Obviously the inclusions $\mathscr{A}_b \subset \mathscr{A}_m$ and $\mathscr{A}^b \subset \mathscr{A}^m$ always hold true; but more important are the following equations:

(1.20) **Lemma.** *Let $\mathscr{A} \subset \mathscr{P}(X)$ contain the constants; then:*

$$\overline{\mathscr{A}_b} = \overline{\mathscr{A}_m} = (\bar{\mathscr{A}})_m, \quad \textit{if } \mathscr{A} \textit{ is } \vee\textit{-stable},$$

$$\overline{\mathscr{A}^b} = \overline{\mathscr{A}^m} = (\bar{\mathscr{A}})^m, \quad \textit{if } \mathscr{A} \textit{ is } \wedge\textit{-stable}.$$

Proof. 1. It suffices to prove for instance the second assertion, where the inclusion $\overline{\mathscr{A}^b} \subset \overline{\mathscr{A}^m}$ is trivial. Since $f \in \mathscr{P}^m(X)$ and $d(f,g) < \infty$ obviously implies $g \in \mathscr{P}^m(X)$, the set $\mathscr{P}^m(X)$ is closed; hence $(\bar{\mathscr{A}})^m = \bar{\mathscr{A}} \cap \mathscr{P}^m(X)$ is closed as well, which implies $\overline{\mathscr{A}^m} \subset (\bar{\mathscr{A}})^m$. Thus only the inclusion $(\bar{\mathscr{A}})^m \subset \overline{\mathscr{A}^b}$ has to be established.

2. Choosing an arbitrary $g \in (\bar{\mathscr{A}})^m$, there exist functions $g^k \in \mathscr{A}$ and $h_i \in \mathscr{L}_f(\mu_i)$ such that $d(g^k, g) \to 0$ and $g \leqq \bigoplus_i h_i$. Then, according to the assumptions on $\mathscr{A}$, the functions $f^k := g^k \wedge k$ are contained in $\mathscr{A}^b$ and satisfy

$$d(f^k, g) \leqq d(f^k, g \wedge k) + d(g \wedge k, g) \to 0$$

because of the estimates

$$\text{(a)} \qquad \begin{aligned} d(f^k, g \wedge k) &= I(|g^k \wedge k - g \wedge k|) \\ &\leqq I(|g^k - g|) \\ &= d(g^k, g), \end{aligned}$$

$$\text{(b)} \qquad \begin{aligned} d(g \wedge k, g) &= I((g-k)^+) \\ &\leqq I\left(\bigoplus_i \left(h_i - \frac{k}{n}\right)^+\right) \\ &= \sum_i \mu_i \left(\left(h_i - \frac{k}{n}\right)^+\right). \quad \square \end{aligned}$$

1.2. Continuity of S

Much more important than the metric continuity stated in (1.16) are continuity properties of the functionals S and I with respect to the natural order in $\mathscr{P}(X)$. The situation is relatively simple for S, especially in the increasing case:

(1.21) **Proposition.** *The functional S is σ-continuous upwards on $\mathscr{P}_b(X)$.*

Proof. Since μ^* is σ-continuous upwards on $\mathscr{P}_b(X)$ for all $\mu \in \mathrm{M}$, the corresponding property of S follows by interchanging two suprema. $\square$

The assumption $h \in \mathscr{P}_b(X)$ cannot be weakened to $S(h) > -\infty$, as is easily seen:

(1.22) *Example.* For $n=2$ choose $X_i = \{-1, +1\}$, let μ_i assign $\frac{1}{2}$ to each point and consider

$$h(x_1, x_2) := (x_1 + x_2) \cdot \infty \quad (\text{with } 0 \cdot \infty := 0).$$

Then $h^k := h \wedge k \uparrow h$, while obviously

$$S(h^k) = 0 \textit{ for all } k \in \mathbf{N} \quad \text{and} \quad S(h) = \infty.$$

The preceding proposition has a topological analogue:

(1.23) **Proposition.** *The functional S is τ-continuous upwards on $\mathscr{G}_b(X)$.*

Proof. The equation $\mu^*(h) = \sup_{k \in \mathbf{N}} \mu(h \wedge k)$ for $h \in \mathscr{G}_b(X)$ implies, again by interchanging two suprema, that μ^* is τ-continuous on $\mathscr{G}_b(X)$, and the assertion follows as (1.21). $\square$

The decreasing case is less pleasant:

(1.24) *Example.* For $n=2$ choose $X_i=[0,1]$, let the measures μ_i be Lebesgue measure and consider the sets

$$F^k := \left\{(x_1, x_2)\in X : x_2 = x_1 + \frac{1}{k} \,(\mathrm{mod}\ 1)\right\} \in \mathfrak{F}(X),$$

$$G^k := \left\{(x_1, x_2)\in X : x_1 < x_2 < x_1 + \frac{1}{k}\right\} \in \mathfrak{G}(X) \quad \text{for } k\in\mathbf{N}.$$

Then the sets F^k support unique measures $\mu^k \in \mathsf{M}$ and the sets G^k decrease to $G=\emptyset$, hence

$$S(G^k) \geqq \sup_{l>k} \mu^l(G^k) = \sup_{l>k}\left(1-\frac{1}{l}\right) = 1 \qquad \text{for all } k\in\mathbf{N},$$

while obviously $S(G)=0$.

By this example it is seen that S fails to be σ-continuous downwards even for indicator functions of open sets and thus (1.21) has no counterpart. To show this, however, to be true for (1.23), the following result will be used, which is essential also in Sect. 2.4:

(1.25) **Lemma.** *For $h\in\mathscr{F}^b(X)$ and $\delta<S(h)$ the set*

$$\mathsf{M}(h;\delta) := \{\mu\in\mathsf{M} : \mu^*(h) \geqq \delta\}$$

is non-void and compact.

Proof. In view of the definition of S and of (1.2) it suffices to prove $\mathsf{M}(h;\delta)$ to be closed. But the function

$$\mu \mapsto \mu^*(h) = \inf_{k\in\mathbf{N}} \mu(h \vee (-k))$$

as an infimum of upper semicontinuous functions on $\mathsf{M}(X)$ has this property as well, and the assertion follows. □

Having available this result, it is easy to show:

(1.26) **Proposition.** *The functional S is τ-continuous downwards on $\mathscr{F}^b(X)$.*

Proof. Let $(h_j)_{j\in J}$ be a decreasing net in $\mathscr{F}^b(x)$ with limit h and take an arbitrary $\delta < \inf_{j\in J} S(h^j)$. Then, in view of (1.25), the sets $\mathsf{M}(h^j;\delta)$, $j\in J$, form a net of non-void compact sets, which is obviously decreasing. Now choose any $\mu\in\bigcap_{j\in J}\mathsf{M}(h^j;\delta)$ and note that μ_* is τ-continuous downwards on $\mathscr{F}^b(X)$ in correspondence to the analogous property of μ^* stated in the proof of (1.23). Since μ_* and μ^* coincide on $\mathscr{F}^b(X)$, this yields

$$S(h) \geqq \mu^*(h) = \inf_{j\in J} \mu^*(h^j) \geqq \delta.$$

Letting δ tend to its upper bound and using (1.5), this implies

$$S(h) \geqq \inf_{j \in J} S(h^j) \geqq S(h),$$

and the assertion is proved. □

1.3. Continuity of I

The crucial point in deriving duality theorems consists in continuity properties of the functional I, which to prove is unfortunately much more involved as it was for the functional S. This has its reason mainly in the absence of an appropriate compactness concept for the set of functions $\bigoplus_i h_i$, $h_i \in \mathscr{L}_f(\mu_i)$, dominating a given function $h \in \mathscr{P}(X)$. Passage to the n-tuples $(h_1, \ldots, h_n)$ suggests to endow $\prod_i \mathscr{L}(\mu_i)$ with the product of the weak topologies of the spaces $\mathscr{L}(\mu_i)$; accordingly in this section functions h_i and h'_i with $h_i \underset{\mu_i}{=} h'_i$ are identified whenever necessary.

With these conventions a first result can be stated, which will be needed again in Sect. 2.4:

(1.27) **Lemma.** *For* $0 \leqq h \in \mathscr{P}(X)$ *and* $\delta > I(h)$ *the set* $\mathscr{L}(h; \delta)$ *consisting of all sequences* $(h_1^k, \ldots, h_n^k)_{k \in \mathbf{N}}$ *in* $\prod_i \mathscr{L}(\mu_i)$ *with*

(1) $$0 \underset{\mu_i}{\leqq} h_i^k \underset{\mu_i}{\leqq} k \quad \text{for all } i \text{ and } k,$$

(2) $$h_i^1 \underset{\mu_i}{\leqq} h_i^2 \underset{\mu_i}{\leqq} \ldots \quad \text{for all } i,$$

(3) $$\sum_i \mu_i(h_i^k) \leqq \delta \quad \text{for all } k,$$

(4) $$h \wedge k \underset{N}{\leqq} \bigoplus_i h_i^k \quad \text{for all } k$$

is a non-void subset of $(\prod_i \mathscr{L}(\mu_i))^{\mathbf{N}}$, *which is compact with respect to the product of the weak topologies in the spaces* $\mathscr{L}(\mu_i)$.

Proof. 1. First of all there are functions $h_i \in \mathscr{L}_f(\mu_i)$ satisfying the inequalities $h \leqq \bigoplus_i h_i$ and $\sum_i \mu_i(h_i) \leqq \delta$, where according to (1.8a) in addition $h_i \geqq 0$ may be assumed. Then it is easily checked that the sequence corresponding to $h_i^k := h_i \wedge k$ fulfils all requirements, and the assertion $\mathscr{L}(h; \delta) \neq \emptyset$ is verified.

2. Apparently the first three requirements define a closed subset of $(\prod_i \mathscr{L}(\mu_i))^{\mathbf{N}}$. Moreover, the set $\{h_i^k \in \mathscr{L}(\mu_i): (1)\}$ is uniformly integrable, hence relatively weakly compact for each i and k, and this carries over to the whole product. Thus it remains only to show that the last requirement defines a closed subset of $\prod_i \mathscr{L}(\mu_i)$ for each k.

3. To this end fix a function $g \in \mathscr{P}(X)$ and consider the set

$$\mathscr{L}(g) := \{(g_1, \ldots, g_n) \in \prod_i \mathscr{L}(\mu_i) : g \underset{N}{\leqq} \bigoplus_i g_i\}.$$

It is obviously convex and moreover closed with respect to the product of the norm topologies of the spaces $\mathscr{L}(\mu_i)$; for, supposing

$$g \underset{N}{\leqq} \bigoplus_i g_i^l \quad \text{and} \quad \|g_i^l - g_i\| \to 0 \qquad \text{for all } i$$

and taking a suitable subsequence, the functions g_i^l may be assumed to converge μ_i-almost everywhere as well and, using (1.13), this implies

$$g \underset{N}{\leqq} \bigoplus_i (\limsup_{l \to \infty} g_i^l) \underset{N}{=} \bigoplus_i g_i.$$

Now, since a strongly closed convex set is also weakly closed (see [20, p. 65]) and the weak topology of the product coincides with the product of the weak topologies (see [20, p. 137]), the set $\mathscr{L}(g)$ is closed in the underlying topology and the proof is completed. □

The first continuity property of I can now be established:

(1.28) **Proposition.** *The functional I is σ-continuous upwards on $\mathscr{P}_b(X)$.*

Proof. Let $(h^l)_{l \in \mathbf{N}}$ be an increasing sequence in $\mathscr{P}_b(X)$ with limit h and take an arbitrary $\delta > \sup_{l \in \mathbf{N}} I(h^l)$; moreover, formally applying (1.6a), assume $h^l \geqq 0$ for all $l \in \mathbf{N}$. Then, in view of (1.27), the sets $\mathscr{L}(h^l; \delta)$, $l \in \mathbf{N}$, form a sequence of non-void compact sets, which is obviously decreasing. Now choose any $(h_1^k, \ldots, h_n^k)_{k \in \mathbf{N}} \in \bigcap_{l \in \mathbf{N}} \mathscr{L}(h^l; \delta)$ and consider the functions $h_i := \limsup_{k \to \infty} h_i^k$, which by condition (1) may be assumed to be non-negative. By conditions (2) and (3) they belong to $\mathscr{L}(\mu_i)$ and fulfil $\sum_i \mu_i(h_i) \leqq \delta$, while condition (4) implies

$$(*) \qquad h^l \underset{N}{\leqq} \bigoplus_i h_i \quad \text{for all } l \in \mathbf{N},$$

where use of (1.13) is made. Applying (1.13) once more results in

$$(**) \qquad h \underset{N}{\leqq} \bigoplus_i h_i,$$

and, redefining the functions h_i suitably, this inequality may be assumed to hold everywhere. This yields

$$I(h) \leqq \sum_i \mu_i(h_i) \leqq \delta;$$

letting δ tend to its lower bound and using (1.5), this implies

$$I(h) \leqq \sup_{l \in \mathbf{N}} I(h^l) \leqq I(h),$$

and the assertion is proved. □

As for the functional S the assumption $h \in \mathscr{P}_b(X)$ cannot be weakened to $I(h) > -\infty$; this is seen by (1.22), checking that

$$I(h^k)=0 \text{ for all } k\in\mathbf{N} \quad \text{and} \quad I(h)=\infty.$$

Passing to the topological analogue of the last proposition leads to:

(1.29) **Proposition.** *The functional I is τ-continuous upwards on $\mathscr{G}_b(X)$.*

Proof. 1. By checking the proof of (1.28) it is seen that the only additional difficulty is contained in the passage from (*) to (**). Therefore it suffices to show that for non-negative functions $g^j \in \mathscr{G}(X)$ and $h_i \in \mathscr{L}(\mu_i)$ the inequalities

(a) $$g^j \underset{N}{\leqq} \bigoplus_i h_i \quad \text{for all } j\in J$$

imply the corresponding inequality for $g := \sup_{j\in J} g^j$.

2. Next, each g^j is a supremum of functions $\alpha 1_{G_1\times\ldots\times G_n}$ with $0\leqq\alpha\in\mathbf{Q}$ and $G_i\in\mathfrak{G}(X_i)$. As there are only countably many values α, in view of (1.13) it is sufficient to prove that for arbitrary sets $G_i^j\in\mathfrak{G}(X_i)$ the equations

(b) $$(G_1^j\times\ldots\times G_n^j)\cap\{\bigoplus_i h_i<\alpha\} \underset{N}{=} \emptyset \quad \text{for all } j\in J$$

yield the corresponding equation for $G := \bigcup_{j\in J}(G_1^j\times\ldots\times G_n^j)$.

Finally, the set $\{\bigoplus_i h_i<\alpha\}$ is a countable union of products $B_1\times\ldots\times B_n$ with $B_i=\{h_i<\alpha_i\}$. Therefore, again according to (1.13), it remains to verify that for arbitrary sets $B_i\in\mathfrak{B}(X_i)$ the equations

(c) $$(G_1^j\times\ldots\times G_n^j)\cap(B_1\times\ldots\times B_n) \underset{N}{=} \emptyset \quad \text{for all } j\in J$$

lead to the corresponding equation for G as above.

3. To this end define

$$J_i := \{j\in J : \mu_i(G_i^j\cap B_i)=0\} \quad \text{and} \quad N_i := \bigcup_{j\in J_i}(G_i^j\cap B_i).$$

Then the set N_i is contained in $\mathfrak{B}(X_i)$ and satisfies $\mu_i(N_i)=0$, as follows from $G_i^j\cap B_i\in\mathfrak{G}(B_i)$ and the fact that the restriction of μ_i to B_i is again τ-continuous. Moreover, due to the equation

$$(G_1^j\cap B_1)\times\ldots\times(G_n^j\cap B_n) \underset{N}{=} \emptyset,$$

for each $j\in J$ there is at least one index i satisfying $\mu_i(G_i^j\cap B_i)=0$, and this amounts to the equation

$$(G_1^j\times\ldots\times G_n^j)\cap(B_1\times\ldots\times B_n)\subset\bigcup_i \pi_i^{-1}[N_i] \quad \text{for all } j\in J,$$

thus completing the proof. □

Again (1.28) cannot be carried over to the decreasing case as is seen by (1.24), where $I(G^k) \geqq S(G^k) \geqq 1$ for all $k \in \mathbf{N}$ in contrast to $I(G) = I(\emptyset) = 0$. However, (1.29) has the following counterpart:

(1.30) **Proposition.** *The functional I is τ-continuous downwards on $\mathscr{F}^b(X)$.*

Proof. Let $(h^j)_{j \in J}$ be a decreasing net in $\mathscr{F}^b(X)$ with limit h and choose an arbitrary $\delta > I(h)$; moreover, formally applying (1.6a), assume $h^j \leqq 0$ for all $j \in J$. Then, in view of (1.8b), there exist bounded functions $h_i \in \mathscr{L}(\mu_i)$ such that

$$h \leqq \bigoplus_i h_i \quad \text{and} \quad \sum_i \mu_i(h_i) < \delta.$$

Denoting by $\gamma \leqq 0$ a common finite lower bound for the functions h_i, according to Lusin's theorem there are compact subsets K_i of X_i satisfying

$$\sum_i \mu_i(h_i) - n\gamma \sum_i \mu_i(\complement K_i) < \delta$$

such that the restrictions of h_i to K_i are continuous. Hence, for arbitrary $\varepsilon > 0$, Dini's theorem applies to the restrictions of h^j and $(\bigoplus_i h_i) + \varepsilon$ to the compact set $K := K_1 \times \ldots \times K_n$ and provides an index $j_0 \in J$ with

$$h^{j_0}(x) < (\bigoplus_i h_i)(x) + \varepsilon \qquad \text{for all } x \in K.$$

Now, replacing the functions h_i by

$$h_i' := h_i + \frac{\varepsilon}{n} - n\gamma 1_{\complement K_i}$$

and using the bounds $h^{j_0} \leqq 0$ and $h_i \geqq \gamma$, it is easily verified that

$$h^{j_0} \leqq \bigoplus_i h_i' \quad \text{and} \quad \sum_i \mu_i(h_i') < \delta + \varepsilon.$$

For $\delta \downarrow I(h)$ and $\varepsilon \downarrow 0$, combined with (1.5), this implies

$$\inf_{j \in J} I(h^j) \leqq I(h) \leqq \inf_{j \in J} I(h^j),$$

as was to be shown. □

1.4. Topological Versions of I

As already mentioned in the introduction, the duality theorems dealt with in the second part are closely related to the Hahn-Banach theorem, provided the function h is bounded and continuous. Therefore it is of interest - and will be essential for some applications treated in the third part - to settle the question, to what extent continuity properties of h can be reflected by corresponding properties of the functions h_i appearing in the definition of I.

Here, lower semicontinuity raises no problem, as any function $h_i \in \mathscr{L}(\mu_i)$ is majorized by functions $h_i' \in \mathscr{L}(\mu_i) \cap \mathscr{G}_b(X_i)$ with arbitrarily small deviations

$\mu_i(h_i') - \mu_i(h_i)$, due to the regularity of μ_i. Concerning upper semicontinuity the main result reads:

(1.31) **Proposition.** *For $h \in \mathscr{F}(X)$ let there exist functions $h_i^0 \in \mathscr{L}_f(\mu_i) \cap \mathscr{F}(X)$ with $h \leqq \bigoplus_i h_i^0$; then*

$$I(h) = \inf\{\sum_i \mu_i(h_i): h_i \in \mathscr{L}_f(\mu_i) \cap \mathscr{F}(X) \text{ and } h \leqq \bigoplus_i h_i\}.$$

Proof. 1. Consider first the case $h \in \mathscr{F}^b(X)$, in which the additional condition is trivially satisfied, and assume without loss of generality $h \leqq 0$. Then h is an (infinite) infimum of functions of the type $\alpha 1_{G_1 \times \ldots \times G_n}$ with $\alpha < 0$ and $G_i \in \mathfrak{G}(X_i)$, which in view of (1.30) for this proof may be replaced by a finite infimum; hence finally

$$h \in \mathscr{F}(X) \cap (\mathscr{E}(X_1) \otimes \ldots \otimes \mathscr{E}(X_n)).$$

Now, in view of (1.8b), in determining $I(h)$ it suffices to consider bounded functions $h_i \in \mathscr{L}(\mu_i)$; thus

$$h_i \in \mathscr{E}(X_i) \quad \text{and} \quad h \leqq \bigoplus_i h_i$$

may be assumed as well. But apparently h_1 may be decreased to

$$h_1'(x_1) := \sup\{h(x_1, \ldots, x_n) - \sum_{i \neq 1} h_i(x_i): x_i \in X_i \text{ for } i \neq 1\}$$

without violating these statements and in addition, due to $h \in \mathscr{F}(X)$, achieving $h_1' \in \mathscr{F}(X_1)$; here - as for for $h_1' \in \mathscr{E}(X_1)$ - use is made of the fact that the (infinite) supremum is actually a finite one. Continuing this way, after n steps all functions are smoothed and the special case is settled.

2. In the general case the functions h_i^0 have to be used. For an arbitrary $\varepsilon > 0$ choose $k \in \mathbf{N}$ large enough to fulfil

$$\sum_i \mu_i(h_i^1) < \varepsilon \qquad \text{for } h_i^1 := \left(h_i^0 - \frac{k}{2n}\right)^+ \in \mathscr{F}^b(X_i)$$

and apply part 1 of the proof to the function $h \wedge k \in \mathscr{F}^b(X)$. For arbitrary $\delta > I(h)$ this provides functions $h_i^2 \in \mathscr{L}_f(X_i) \cap \mathscr{F}(X_i)$ satisfying

$$h \wedge k \leqq \bigoplus_i h_i^2 \quad \text{and} \quad \sum_i \mu_i(h_i^2) < \delta.$$

Then the functions $h_i := 2h_i^1 + h_i^2$ lie again in $\mathscr{L}_f(\mu_i) \cap \mathscr{F}(X_i)$ and fulfil $h \leqq \bigoplus_i h_i$, as is obvious in the case $h(x) \leqq k$ and follows otherwise by means of the inequality

$$h(x) \leqq 2h(x) - k \leqq 2 \bigoplus_i \left(h_i^0 - \frac{k}{2n}\right)(x).$$

Since in addition $\sum_i \mu_i(h_i) < 2\varepsilon + \delta$, the assertion is proved. □

The existence of functions h_i^0 as required in this proposition is an essential condition and not a consequence of $I(h)<\infty$:

(1.32) *Example.* Let X be the product of $X_1=\mathbf{N}$ and $X_2=[0,1]$ with the discrete and euclidean topology, respectively, and μ_i be given by

$$\mu_1(\{x_1\}):=2^{-x_1} \quad \text{and} \quad \mu_2(\{0\}):=1.$$

Then $h:(x_1,x_2)\mapsto 2^{x_1}x_2$ defines a continuous function with $h\underset{N}{=}0$ and therefore $I(h)=0$. Now assume

$$h\leqq h_1\oplus h_2 \quad \text{with} \quad h_i\in\mathscr{L}_f(\mu_i)\cap\mathscr{F}(X_i).$$

Then $h_2(0)<\infty$ and there is a constant $\gamma<\infty$ such that $h_2(x_2)\leqq\gamma$ in some neighborhood of $x_2=0$; but this implies

$$h_1(k)\geqq h\left(k,\frac{1}{k}\right)-h_2\left(\frac{1}{k}\right)\geqq 2^k\frac{1}{k}-\gamma \qquad \text{for almost all } k\in\mathbf{N}$$

and contradicts the integrability of h_1.

The extension of (1.31) to the continuous case is straightforward under the natural topological condition:

(1.33) **Proposition.** *For $h\in\mathscr{C}(X)$ let there exist functions $h_i^0\in\mathscr{L}_f(\mu_i)\cap\mathscr{C}(X_i)$ with $h\leqq\bigoplus_i h_i^0$; then, provided all spaces X_i are completely regular,*

$$I(h)=\inf\{\sum_i \mu_i(h_i)\colon h_i\in\mathscr{L}_f(\mu_i)\cap\mathscr{C}(X_i) \text{ and } h\leqq\bigoplus_i h_i\}.$$

Proof. 1. Assume first $h\in\mathscr{C}^b(X)$ and apply (1.31), which yields the analogous equation with $\mathscr{F}(X_i)$ instead of $\mathscr{C}(X_i)$. But as in (1.8b) the functions h_i may be replaced by bounded ones, which then allows to apply the complete regularity of the spaces X_i in conjunction with the τ-continuity of the measures μ_i.

2. The extension to unbounded functions h proceeds as in part 2 of the proof of (1.31). □

The fact that the condition of complete regularity in (1.33) cannot be dispensed with is related to a recently solved topological problem. Suppose the proposition to hold in arbitrary Hausdorff spaces and apply it to a fixed bounded function $h\in\mathscr{C}(X)$ and to all point measures μ_i on X_i. This leads to the equation

$$h=\inf\{\bigoplus_i h_i\colon h_i\in\mathscr{C}_f(X_i) \text{ and } h\leqq\bigoplus_i h_i\}.$$

Hence h is upper semicontinuous and, replacing h by $-h$, even continuous with respect to the product of the (usually weaker) topologies generated on X_i by $\mathscr{C}(X_i)$. This, however, fails to be true in general as is shown in [18] (for which reference the author is indebted to Z. Frolik).

2. Duality Theorems

2.1. Special Duality Theorems

Even without the additional tools collected in Sect. 2.2, the continuity properties of the functionals S and I established in the first part lead to fairly comprehensive situations implying for a function $h \in \mathscr{P}(X)$ the "duality theorem"

$$(D) \qquad S(h) = I(h).$$

The starting point is, of course, the case of finite spaces X_i, where the validity of (D) for finite functions h follows from the well-known duality theorem of linear programming, which in turn is an easy consequence of the Hahn-Banach theorem applied to a finite dimensional space.

A natural extension yields:

(2.1) **Proposition.** *(D) holds on $\mathscr{E}(X_1) \otimes \ldots \otimes \mathscr{E}(X_n)$.*

Proof. 1. Given a function $h \in \mathscr{E}(X_1) \otimes \ldots \otimes \mathscr{E}(X_n)$, there are finite spaces X_i' and Borel measurable maps $\varphi_i: X_i \to X_i'$ such that, abbreviating $X' := X_1' \times \ldots \times X_n'$ and $\varphi := (\varphi_1, \ldots, \varphi_n)$, there is a factorization

$$h = h' \circ \varphi \quad \text{with} \quad h': X' \to \mathbf{R}.$$

The above-mentioned duality theorem of linear programming, applied to the measures $\mu_i' := \varphi_i(\mu_i)$ on X_i' and the function h', yields a measure μ' on X' and functions $h_i': X_i' \to \mathbf{R}$ satisfying

$$(a') \qquad \pi_i'(\mu') = \mu_i' \quad \text{for } 1 \leqq i \leqq n,$$

$$(b') \qquad h' \leqq \sum_i h_i' \circ \pi_i',$$

$$(c') \qquad \mu'(h') = \sum_i \mu_i'(h_i'),$$

where π_i' denotes the corresponding projection from X' to X_i'.

2. Since the spaces X_i' are finite, there is a density f' of μ' with respect to $\mu_1' \otimes \ldots \otimes \mu_n'$. Defining

$$d\mu := f\, d(\mu_1 \otimes \ldots \otimes \mu_n) \quad \text{with} \quad f := f' \circ \varphi$$

and $h_i := h_i' \circ \varphi_i$ yields a measure $\mu \in \mathsf{M}(X)$ and functions $h_i \in \mathscr{L}(\mu_i)$ which in view of $\varphi_i \circ \pi_i = \pi_i' \circ \varphi$ are easily seen to satisfy

$$(a) \qquad \pi_i(\mu) = \mu_i \quad \text{for } 1 \leqq i \leqq n,$$

$$(b) \qquad h \leqq \sum_i h_i \circ \pi_i,$$

$$(c) \qquad \mu(h) = \sum_i \mu_i(h_i).$$

Thus $S(h) \geqq I(h)$ and combined with (1.4) the assertion follows. □

Now a first major result can be stated:

(2.2) **Theorem.** *(D) holds on $\overline{\mathscr{G}}_m(X)$.*

Proof. 1. Consider first the case $h \in \mathscr{G}_b(X)$ and assume without loss of generality $h \geqq 0$. Then h is a supremum of functions $\alpha 1_{G_1 \times \ldots \times G_n}$ with $\alpha \geqq 0$ and $G_i \in \mathfrak{G}(X_i)$, hence the limit of the increasing net of finite suprema of these functions, which belong to $\mathscr{E}(X_1) \otimes \ldots \otimes \mathscr{E}(X_n)$ as well. Thus (D) results from (2.1) by applying (1.23) and (1.29).

2. Now, taking into account (1.16), (D) extends to the closure of $\mathscr{G}_b(X)$ with respect to d, which in view of (1.20) is $\overline{\mathscr{G}}_m(X)$. □

The preceding result has a natural counterpart:

(2.3) **Proposition.** *(D) holds on* $\overline{\mathscr{F}}^m(X)$.

Proof. The proof of (2.2) carries over, (1.26) and (1.30) replacing now (1.23) and (1.29). □

The following result, which will be needed in Sect. 2.3, implies that both (2.2) and (2.3) really extend (2.1):

(2.4) **Proposition.** $\mathscr{E}(X_1) \otimes \ldots \otimes \mathscr{E}(X_n) \subset \overline{\mathscr{F}}(X) \cap \overline{\mathscr{G}}(X)$.

Proof. Since each function $h \in \mathscr{E}(X_1) \otimes \ldots \otimes \mathscr{E}(X_n)$ is a finite supremum of functions $\alpha 1_{B_1 \times \ldots \times B_n}$ with $\alpha \in \mathbf{R}$ and $B_i \in \mathfrak{B}(X_i)$ and, moreover, $\overline{\mathscr{F}}(X)$ and $\overline{\mathscr{G}}(X)$ by (1.18) are again lattices, it suffices to consider the special case $h = \alpha 1_{B_1 \times \ldots \times B_n}$. Now, given an arbitrary $\varepsilon > 0$, there are sets $F_i \in \mathfrak{F}(X_i)$ and $G_i \in \mathfrak{G}(X_i)$ satisfying

$$F_i \subset B_i \subset G_i \quad \text{and} \quad \mu_i(G_i \setminus F_i) < \varepsilon.$$

If then h' denotes any of the functions $\alpha 1_{F_1 \times \ldots \times F_n}$ and $\alpha 1_{G_1 \times \ldots \times G_n}$, the difference $|h - h'|$ can be estimated by $|\alpha| (\bigoplus_i 1_{G_i \setminus F_i})$. This provides the inequality $d(h, h') \leqq |\alpha| n \varepsilon$, and by proper choice of h' – depending on the sign of α – the assertion $h \in \overline{\mathscr{F}}(X) \cap \overline{\mathscr{G}}(X)$ follows. □

As a consequence of the last result, combined with (1.18), the statements of (2.3) and (2.2) apply in particular to functions of the form

$$h = \inf_{k \in \mathbf{N}} (f^k \vee h^k) \quad \text{and} \quad h = \sup_{k \in \mathbf{N}} (g^k \wedge h^k),$$

where $f^k \in \mathscr{F}^m(X)$, $g^k \in \mathscr{G}_m(X)$ and $h^k \in \mathscr{E}(X_1) \otimes \ldots \otimes \mathscr{E}(X_n)$.

A question still to be settled concerns the necessity of a minorizing or majorizing function of the type $\bigoplus_i h_i$ with $h_i \in \mathscr{L}(\mu_i)$ in the statements of (2.2) and (2.3). A first answer is in the affirmative:

(2.5) *Example.* Choose $n = 2$ and X_i, μ_i as in (1.24). Then the function

$$h := -\infty \cdot 1_{\{x_1 \geqq x_2\}} \in \mathscr{G}(X)$$

satisfies the equations

$$S(h) = -\infty \quad \text{and} \quad I(h) = 0.$$

To verify the first one, suppose $\mu \in \mathrm{M}$ to fulfil $\mu^*(h) > -\infty$; this implies $\mu(\{x_1 \geqq x_2\}) = 0$, or $x_1 < x_2$ μ-almost everywhere, and leads to $\int_X x_1 \, d\mu < \int_X x_2 \, d\mu$, contradicting the fact $\pi_1(\mu)=\pi_2(\mu)$. To verify the second equation, it suffices to show

$$\mu_1(h_1)+\mu_2(h_2)\geqq 0 \quad \text{whenever } h_1(x_1)+h_2(x_2)\geqq 0 \quad \text{on } \{x_1<x_2\};$$

but this follows from the equation

$$\int_0^{1-\varepsilon} h_1(s)\,ds+\int_\varepsilon^1 h_2(s)\,ds=\int_\varepsilon^1 (h_1(s-\varepsilon)+h_2(s))\,ds\geqq 0,$$

letting $\varepsilon>0$ tend to 0.

Thus the result of (2.2) cannot be extended from $\overline{\mathscr{G}}_m(X)$ to $\overline{\mathscr{G}}(X)$. The analogous extension of (2.3), however, is possible:

(2.6) **Theorem.** *(D) holds on* $\overline{\mathscr{F}}(X)$.

Proof. Since in the case $S(h)=\infty$ the assertion is a consequence of (1.4), consider $h\in\overline{\mathscr{F}}(X)$ with $S(h)<\infty$. This means in particular $\mu^*(h)<\infty$ for all $\mu\in\mathsf{M}$, hence as well $\mu^*(h^+)<\infty$ for all $\mu\in\mathsf{M}$ and thus $S(h^+)<\infty$ by (1.7). On the other hand

$$I(h^+)=\lim_{k\to\infty} I(h^+\wedge k)=\lim_{k\to\infty} S(h^+\wedge k)=S(h^+),$$

where (1.28), (2.3), (1.21) are applied in this order. By (1.5) this implies $I(h)<\infty$ and shows that h is in fact contained in $\overline{\mathscr{F}}^m(X)$. □

The preceding result solves the case of discrete spaces without any boundedness - or even finiteness - condition:

(2.7) **Corollary.** *(D) holds on $\mathscr{P}(X)$, whenever all spaces X_i are discrete.*

Proof. The product space X being again discrete, $\mathscr{P}(X)$ coincides with $\mathscr{F}(X)$ and (2.6) applies. □

As a consequence (2.7) yields the analogous result, whenever all spaces X_i are countable, because in this case the topology in X_i may be refined to the discrete topology without affecting M and $\mathscr{L}(\mu_i)$.

2.2. Suslin Functions

To extend the duality theorems from semicontinuous to measurable functions requires some facts connected with the passage from Suslin sets to Suslin functions. The most up-to-date source for these concepts is the recently published part C of "Probabilités et potentiel" [2]. There, however, the functions are restricted to non-negative values, which makes it more natural to work with projections. To treat functions $f\in\mathscr{P}(Y)$ it is simpler - and somewhat more general - to use the classical Suslin operation, which yields:

(2.8) *Definition.* (a) For $\mathfrak{A}\subset\mathfrak{P}(Y)$ a set B is called an "$\mathfrak{A}$-Suslin set" if it has a representation

$$B=\bigcup\bigcap_{l\in\mathbf{N}} A_{k_1\ldots k_l} \quad \text{with} \quad A_{k_1\ldots k_l}\in\mathfrak{A},$$

where the union is taken over all sequences $(k_l)_{l\in\mathbf{N}}$ in $\mathbf{N}$;

(b) for $\mathscr{A} \subset \mathscr{P}(Y)$ a function g is called an "$\mathscr{A}$-Suslin function", if it has a representation

$$g = \sup \inf_{l \in \mathbf{N}} f_{k_1 \dots k_l} \quad \text{with} \quad f_{k_1 \dots k_l} \in \mathscr{A},$$

where the supremum is taken over all sequences $(k_l)_{l \in \mathbf{N}}$ in $\mathbf{N}$.

The main fact concerning these notions that will be needed in the sequel is the following one:

(2.9) **Proposition.** *The class of all $\mathfrak{A}$-Suslin sets resp. $\mathscr{A}$-Suslin functions is a σ-lattice containing $\mathfrak{A}$ resp. $\mathscr{A}$.*

Proof. The classical proof, for instance in [7, p. 106], showing the extension to Suslin sets to be idempotent, carries over to Suslin functions without any change. Utilizing this fact, the assertion is easily established. □

The most important concept connected with the Suslin operation extends immediately from sets to functions:

(2.10) *Definition.* Given $\mathscr{A} \subset \mathscr{P}(Y)$ a functional $C\colon \mathscr{P}(Y) \to \tilde{\mathbf{R}}$ is called an "$\mathscr{A}$-capacity", if
(0) $\mathscr{A}$ is a lattice,
(1) C is isotone,
(2) C is σ-continuous upwards on $\mathscr{P}(Y)$,
(3) C is σ-continuous downwards on $\mathscr{A}$.

As in the corresponding situation for sets Choquet's theorem holds true:

(2.11) **Proposition.** *Let $C\colon \mathscr{P}(Y) \to \tilde{\mathbf{R}}$ be an $\mathscr{A}$-capacity, where in addition*
(4) $\mathscr{A}$ is stable with respect to countable infima.
Then the approximation

$$C(g) = \sup\{C(f)\colon \mathscr{A} \ni f \leqq g\}$$

applies to all $\mathscr{A}$-Suslin functions g.

Proof. As already mentioned at the end of [1], Choquet's original proof carries over to the present situation (since the complete lattice $\tilde{\mathbf{R}}$ is totally ordered and thus completely distributive). □

Finally, the underlying topology enters:

(2.12) *Definition.* If Y is a topological space, $\mathfrak{S}(Y)$ resp. $\mathscr{S}(Y)$ denotes the class of all $\mathfrak{F}(Y)$-Suslin sets resp. $\mathscr{F}(Y)$-Suslin functions.

The relationship between $\mathfrak{F}(Y)$ and $\mathscr{F}(Y)$ is maintained:

(2.13) **Proposition.** *A function $g \in \mathscr{P}(Y)$ belongs to $\mathscr{S}(Y)$ if and only if*

$$\{g \geqq \alpha\} \in \mathfrak{S}(Y) \quad \textit{for all } \alpha \in \tilde{\mathbf{R}}.$$

Proof. 1. Simplifying the index sets, first consider a function

$$g = \sup_j \inf_k f_{jk} \quad \text{with} \quad f_{jk} \in \mathscr{F}(Y).$$

For arbitrary $\alpha\in\tilde{\mathbf{R}}$ this yields

$$\{g\geqq\alpha\}=\bigcap_{l\in\mathbf{N}}\bigcup_{j}\bigcap_{k}\left\{f_{jk}\geqq\alpha-\frac{1}{l}\right\}.$$

Since all sets on the right-hand side belong to $\mathfrak{F}(Y)$, the set $\{g\geqq\alpha\}$ belongs to $\mathfrak{S}(Y)$ according to (2.9).

2. To prove the converse, for arbitrary $A\subset Y$ and $\alpha\in\mathbf{R}$ denote the function $f\colon Y\to\{-\infty,\alpha\}$ with $\{f=\alpha\}=A$ by $[A;\alpha]$. If now g is a function satisfying the condition in question, in view of (2.9) and the equation

$$g=\sup\{[\{g\geqq\alpha\};\alpha]:\alpha\in\mathbf{Q}\}$$

without loss of generality $g=[B;\alpha]$ may be supposed. However,

$$B=\bigcup_{j}\bigcap_{k}A_{jk}\quad\text{with}\quad A_{jk}\in\mathfrak{F}(Y)$$

according to the assumption, and this implies the representation

$$g=\sup_{j}\inf_{k}f_{jk}\quad\text{with}\quad f_{jk}:=[A_{jk};\alpha]\in\mathscr{F}(Y).\quad\square$$

2.3. General Duality Theorems

With the aid of the last section the special duality theorems of Sect. 2.1 can now be generalized. The central result reads:

(2.14) **Theorem.** *(D) holds on* $\overline{\mathscr{S}}_m(X)$.

Proof. Fix some $k\in\mathbf{N}$ and define

$$S'(h):=S((-k)\vee(h\wedge k))\qquad\text{for } h\in\mathscr{P}(X),$$
$$I'(h):=I((-k)\vee(h\wedge k))\qquad\text{for } h\in\mathscr{P}(X).$$

By (1.5) combined with (1.21), (1.26) and (1.28), (1.30) the functionals S' and I' are $\mathscr{F}(X)$-capacities, which agree on $\mathscr{F}(X)$ according to (2.6). As condition (4) in (2.11) is satisfied, the coincidence of S' and I' therefore extends to $\mathscr{S}(X)$, i.e.

$$S(h)=I(h)\qquad\text{for all bounded } h\in\mathscr{S}(X).$$

Applying (1.21) and (1.28) once more, this result is extended to all functions $h\in\mathscr{S}_b(X)$; finally, taking into account (1.16) and (1.20), it is generalized to $\overline{\mathscr{S}}_m(X)$. $\square$

A first application to measurable functions is immediate:

(2.15) **Corollary.** *(D) holds for all functions* $h\in\mathscr{P}_m(X)$ *that are measurable with respect to* $\mathfrak{C}(X)$.

Proof. The functions enjoying the stated measurability constitute the σ-lattice generated by $\mathscr{C}(X)$, hence the assertion results from (2.14) and

$$\sigma(\mathscr{C}(X))\subset\sigma(\mathscr{F}(X))\subset\mathscr{S}(X).\quad\square$$

The next application to measurable functions is more profound:

(2.16) **Corollary.** *(D) holds for all functions* $h \in \mathscr{P}_m(X)$ *that are measurable with respect to* $\mathfrak{B}(X_1) \otimes \ldots \otimes \mathfrak{B}(X_n)$.

Proof. The functions enjoying the present measurability constitute the σ-lattice generated by $\mathscr{E}(X_1) \otimes \ldots \otimes \mathscr{E}(X_n)$, where in view of (2.4)

$$\sigma(\mathscr{E}(X_1) \otimes \ldots \otimes \mathscr{E}(X_n)) \subset \sigma(\overline{\mathscr{F}}(X)) \subset \sigma(\overline{\mathscr{S}}(X)).$$

According to (2.9) and (1.18) $\overline{\mathscr{S}}(X)$ is again a σ-lattice and the assertion follows from (2.14). □

Even more interesting are, of course, duality theorems for functions that are measurable only with respect to the generally larger σ-algebra $\mathfrak{B}(X)$. A first answer is provided by:

(2.17) **Corollary.** *(D) holds on* $\mathscr{B}_m(X)$, *whenever the space X is strongly Lindelöf or perfect.*

Proof. 1. Since for a strongly Lindelöf space X the σ-algebra $\mathfrak{B}(X)$ coincides with the product $\mathfrak{B}(X_1) \otimes \ldots \otimes \mathfrak{B}(X_n)$, the assertion is in this case a consequence of (2.16).

2. If X is perfect, each open set is of type F_σ and thus $\mathfrak{B}(X)$ is the σ-lattice generated by $\mathfrak{F}(X)$ alone. Therefore any function $h \in \mathfrak{B}(X)$ satisfies

$$\{h \geqq \alpha\} \in \sigma(\mathfrak{F}(X)) \subset \mathfrak{S}(X) \quad \text{for all } \alpha \in \tilde{\mathbf{R}},$$

hence by (2.13) is contained in $\mathscr{S}(X)$, and (2.14) applies. □

As none of the topological properties required in (2.17) is finitely multiplicative, explicit conditions on the different factors X_i are more practicable. The main result in this direction reads:

(2.18) **Corollary.** *(D) holds on* $\mathscr{B}_m(X)$, *whenever each space* X_i *has at least one of the following properties*

(a) X_i *is second countable,*
(b) X_i *is metrizable,*
(c) X_i *is a Suslin space.*

Proof. Each space X_i can be exhausted by a σ-compact set up to a μ_i-null set, which is irrelevant for the functionals S and I. Therefore in any of the three cases X_i may be assumed to be the countable union of Suslin spaces, thus being a Suslin space itself. Then X is again a Suslin space, hence strongly Lindelöf, and (2.17) applies. □

To conclude this section, it has to be pointed out, that an open problem evidently is left: are the toplogical conditions in (2.17) and (2.18) really essential? To state only the simplest case, so far the following question is undecided: assume all spaces X_i to be compact and $B \subset X$ to be a set of type G_δ satisfying $\mu(B)=0$ for all $\mu \in \mathrm{M}$; does this imply $B \in \mathfrak{N}$? Here the first condition amounts to $S(B)=0$, while the second one is equivalent to $I(B)=0$.

2.4. Extremal Solutions

This section treats a problem disregarded so far, but important in applications; it is the question to what extent the supremum $S(h)$ and the infimum $I(h)$ are really attained for suitable $\mu \in \mathsf{M}$ and $h_i \in \mathscr{L}(\mu_i)$, respectively. On the basis of the lemmata (1.25) and (1.27) the answer is not too difficult. Concerning first the functional S it reads:

(2.19) **Theorem.** *For any function $h \in \overline{\mathscr{F}}(X)$ there exists a "maximal" measure $\mu \in \mathsf{M}$ such that*

$$S(h) = \mu^*(h).$$

Proof. Since the case $S(h) = \infty$ – for arbitrary $h \in \mathscr{P}(X)$ – is settled by (1.7), in the sequel $S(h) < \infty$ will be assumed. According to (2.6) this implies $I(h) < \infty$ as well, hence $h \in \overline{\mathscr{F}}^m(X)$. Thus by (1.20) there are functions $h^k \in \mathscr{F}^b(X)$ such that

$$\delta_k := d(h, h^k) < \frac{1}{k} \quad \text{for} \quad k \in \mathbf{N}.$$

Taking up the notation of (1.25), this leads to the equation

$$\mathsf{M}(h; \delta) = \bigcap_{k \in \mathbf{N}} \mathsf{M}\left(h^k; \delta - \frac{1}{k}\right) \qquad \text{for all } \delta < S(h),$$

which is easily established taking into account

$$\mu^*(h^k) - \delta_k \leqq \mu^*(h) \leqq \mu^*(h^k) + \delta_k \qquad \text{for all } \mu \in \mathsf{M}.$$

Thus the non-void sets $\mathsf{M}(h; \delta)$ are compact by (1.25) and obviously decrease for $\delta \uparrow S(h)$, i.e.

$$\bigcap \{\mathsf{M}(h; \delta) \colon \delta < S(h)\} \neq \emptyset.$$

Now any μ from this intersection is maximal. □

According to the remark following (2.4) this result applies not only to $h \in \mathscr{F}(X)$ but also to countable infima of functions in $\mathscr{E}(X_1) \otimes \ldots \otimes \mathscr{E}(X_n)$, hence in particular to uniform limits of such functions – a fact that is proved in [19, p. 297] for polish spaces X_i.

That (2.19) does not extend to bounded – or even indicator – functions $h \in \mathscr{G}(X)$, is easily seen:

(2.20) *Example.* Choose $n = 2$ and X_i, μ_i as in (1.24). Then, considering the measures μ^k there, the equation $S(G) = 1$ can be shown for $G := \{x_1 < x_2\} \in \mathfrak{G}(X)$. The existence of a measure $\mu \in \mathsf{M}$ satisfying $\mu(G) = 1$, however, would imply $\int_X x_1 \, d\mu < \int_X x_2 \, d\mu$, contrary to the fact $\pi_1(\mu) = \pi_2(\mu)$.

Turning now to the functional I, the assumption $I(h) < \infty$ is inevitable; combined with the analogous boundedness below it also suffices:

(2.21) **Theorem.** *For any function* $h \in \mathscr{P}_m(X)$ *with* $I(h) < \infty$ *there exist "minimal" functions* $h_i \in \mathscr{L}_f(\mu_i)$ *such that*

$$h \leqq \bigoplus_i h_i \quad \textit{and} \quad I(h) = \sum_i \mu_i(h_i).$$

Proof. After redefining h on a set $A \in \mathfrak{N}$ all summands in the minorizing function $\bigoplus_i h_i$ may be chosen finite, hence (1.6a) is applicable and $h \geqq 0$ is seen to be no real restriction. Then, taking up the notation of (1.27),

$$\bigcap \{\mathscr{L}(h; \delta): \delta > I(h)\} \neq \emptyset,$$

because the sets $\mathscr{L}(h; \delta)$ are compact by (1.27) and obviously decrease for $\delta \downarrow I(h)$. Choosing any sequence $(h_1^k, \dots, h_n^k)_{k \in \mathbf{N}}$ from this intersection and defining $h_i := \limsup_{k \to \infty} h_i^k$ as in the proof of (1.28) results in functions $h_i \in \mathscr{L}(\mu_i)$ fulfilling $\sum_i \mu_i(h_i) \leqq I(h)$. Moreover these functions may be redefined suitably to be non-negative and to satisfy $h \leqq \bigoplus_i h_i$ everywhere, thus having all properties required for being minimal. □

To recognize the condition $h \in \mathscr{P}_m(X)$ as essential for this result requires a more involved construction:

(2.22) *Example.* For $n = 2$ choose $X_i = \mathbf{N}$ with the discrete topology and define a measure μ on the product X – deleting needless brackets – by

$$\mu(k, k) = \frac{1}{2k(k+1)} = \mu(k+1, k) \qquad \text{for } k \in \mathbf{N}$$

(μ vanishing otherwise) and a function $h \in \mathscr{C}(X)$ by

$$h(k_1, k_2) := \begin{cases} k_1 - k_2 & \text{for } k_1 - k_2 \leqq 1, \\ 1 \ \ - k_2 & \text{otherwise.} \end{cases}$$

Moreover, choose $\mu_i := \pi_i(\mu)$ and define functions $h_i^l \in \mathscr{L}(\mu_i)$ by

$$h_1^l(k) := +k \text{ for } k \leqq l \text{ and } l \text{ otherwise,}$$
$$h_2^l(k) := -k \text{ for } k < l \text{ and } 0 \text{ otherwise.}$$

By simple computation this yields

$$h \leqq h_1^l \oplus h_2^l \quad \text{and} \quad \mu_1(h_1^l) + \mu_2(h_2^l) = \frac{1}{2} + \frac{1}{l+1} \qquad \text{for all } l \in \mathbf{N},$$

hence the inequalities

$$\tfrac{1}{2} = \mu(h) \leqq S(h) \leqq I(h) \leqq \tfrac{1}{2},$$

i.e. the measure μ is maximal for μ_1, μ_2 and h.

If now $h_i \in \mathscr{L}(\mu_i)$ are assumed to be minimal, an inspection of the proof of (1.4) leads to

$$h(k_1, k_2) = h_1(k_1) + h_2(k_2) \quad \mu\text{-almost everywhere.}$$

After the normalization $h_1(1)=1$ a simple recursion along the points (k, k) and $(k+1, k)$ results in

$$h_1(k) = +k \quad \text{and} \quad h_2(k) = -k \qquad \text{for all } k \in \mathbf{N},$$

in contradiction to $h_i \in \mathscr{L}(\mu_i)$.

To conclude this section, it should be mentioned that the result of (2.21) fails to carry over to the topological versions of I in (1.31) and (1.33), as is seen by:

(2.23) *Example.* Choose $n=2$ and X_i, μ_i as in (1.32); in addition, with the function h introduced there, define the bounded function $h' := h \wedge 1 \in \mathscr{C}(X)$, satisfying again $I(h')=0$. Then functions $h_i \in \mathscr{L}_f(\mu_i) \cap \mathscr{F}(X_i)$ with $h \leqq h_1 \oplus h_2$, which may be assumed to be non-negative as in (1.8a), always yield a strictly positive sum $\mu_1(h_1) + \mu_2(h_2)$. This is obvious for $h_1 \neq 0$, while otherwise $h_2\left(\frac{1}{k}\right) \geqq 1$ has to hold for all $k \in \mathbf{N}$, implying $h_2(0) \geqq 1$ due to $h_2 \in \mathscr{F}(X_2)$.

3. Applications

3.1. Equi-integrable Functions

As a first application those functions in $\mathscr{P}(X)$ are investigated that are "equi-integrable", either in the weaker sense of being integrable for all $\mu \in \mathrm{M}$ or in the stronger one of having the same integral for all $\mu \in \mathrm{M}$. Again, in each of the corresponding statements one half is trivial.

The first result reads:

(3.1) **Proposition.** *Let the topological conditions of* (2.17) *or* (2.18) *be satisfied. Then*

$$\bigcap \{\mathscr{L}(\mu) \colon \mu \in \mathrm{M}\} = \mathscr{B}_m(X) \cap \mathscr{B}^m(X).$$

Proof. If $h \in \mathscr{B}(X)$ is integrable with respect to all $\mu \in \mathrm{M}$, the same holds for $|h|$, hence $S(|h|) < \infty$ by (1.7) and thus $I(|h|) < \infty$ by (2.17) or (2.18) as was to be shown. □

Less immediate is the second result:

(3.2) **Proposition.** *Let the topological conditions of* (2.17) *or* (2.18) *be satisfied and consider a function* $h \in \mathscr{B}_m(X) \cap \mathscr{B}^m(X)$. *Then the integral* $\mu(h)$ *is independent of* $\mu \in \mathrm{M}$ *if and only if* h *has a representation*

$$h = \bigoplus_{N\; i} h_i \qquad \textit{with finite } h_i \in \mathscr{L}(\mu_i).$$

Proof. The "if" assertion being trivial, assume the integral $\mu(h)$ to be constant on M, i.e.

$$\sup_{\mu\in M}\mu(h)=\inf_{\mu\in M}\mu(h)=-\sup_{\mu\in M}\mu(-h).$$

The resulting equation $S(-h)=-S(h)$ implies the corresponding equation $I(-h)=-I(h)$ by (2.17) or (2.18), and by application of (2.21) to $\pm h$ finite functions $\underline{h}_i, \bar{h}_i\in\mathscr{L}(\mu_i)$ are obtained such that

(a) $$\underline{h}:=\bigoplus_i \underline{h}_i \underset{N}{\leqq} h \underset{N}{\leqq} \bigoplus_i \bar{h}_i=:\bar{h},$$

(b) $$\sum_i \mu_i(\underline{h}_i)=\sum_i \mu_i(\bar{h}_i).$$

These relations together lead to

$$d(\underline{h},\bar{h})=I(\bar{h}-\underline{h})=\sum_i \mu_i(\bar{h}_i-\underline{h}_i)=0$$

and thus yield by (1.15) the desired equation $\underline{h}\underset{N}{=}h\underset{N}{=}\bar{h}$. □

From (3.2) it follows easily that - under the same topological assumptions - the measure $\mu(B)$ of a set $B\in\mathfrak{B}(X)$ is independent of $\mu\in M$ if and only if for some i

$$B\underset{N}{=}\pi_i^{-1}[B_i] \quad \text{with} \quad B_i\in\mathfrak{B}(X_i).$$

3.2. *Measures with Given Support*

In the investigations up to now it made no difference whether the product space X consisted of two or more factors X_i. This changes considerably as soon as h is specialized to indicator functions. Under restriction to the case $n=2$, it turns out that the special structure of h may be transferred to the functions h_i.

For the functional I this means:

(3.3) **Proposition.** *Let $n=2$ and $B\in\mathfrak{P}(X)$ be arbitrary; then*

$$I(B)=\inf\{\sum_i \mu_i(B_i)\colon B_i\in\mathfrak{B}(X_i) \text{ and } B\subset\bigcup_i \pi_i^{-1}[B_i]\}.$$

Proof. For $h=1_B$ the functions h_i appearing in the definition of I can be confined to $0\leqq h_i\leqq 1$ according to (1.8a). But then the inequality $1_B\leqq\bigoplus_i h_i$ leads to the relations

(a) $$B\subset(\{h_1\geqq s\}\times X_2)\cup(X_1\times\{h_2\geqq 1-s\}) \quad \text{for } 0\leqq s\leqq 1,$$

(b) $$\sum_i \mu_i(h_i)=\int_0^1 \mu_1(\{h_1\geqq s\})\,ds+\int_0^1 \mu_2(\{h_2\geqq 1-s\})\,ds \geqq \inf_{0\leqq s\leqq 1}(\mu_1(\{h_1\geqq s\})+\mu_2(\{h_2\geqq 1-s\})).$$

Choosing $B_1=\{h_1\geqq s\}$ and $B_2=\{h_2\geqq 1-s\}$ yields the assertion. □

That the statement of (3.3) indeed fails to extend to the case $n>2$, can be demonstrated even in the simplest non-trivial case:

(3.4) *Example.* For $n=3$ let X_i be the discrete space $\{0,1\}$ and the measure μ_i assign $\frac{1}{2}$ to each point; in addition define

$$B:=\{x\in X:\ \sum_i \pi_i(x)=1\}.$$

Then $1_B\leqq\bigoplus_i h_i$ for $h_i:=\frac{1}{2}1_{\{0\}}$ and thus $I(B)\leqq\frac{3}{4}$, while sets $B_i\subset X_i$ with $\sum_i \mu_i(B_i)<1$ have a total cardinality of at most one, so that B cannot be covered by $\bigcup_i \pi_i^{-1}[B_i]$.

Returning to the case $n=2$, the result of (2.21) can be carried over:

(3.5) **Proposition.** *Let $n=2$ and $B\in\mathfrak{P}(X)$ be arbitrary; then there exist sets $B_i\in\mathfrak{B}(X_i)$ such that*

$$B\subset\bigcup_i \pi_i^{-1}[B_i] \quad \textit{and} \quad \sum_i \mu_i(B_i)=I(B).$$

Proof. Minimal functions h_i corresponding to $h=1_B$ according to (2.21) may be confined to $0\leqq h_i\leqq 1$ as in (1.8a), hence relations (a) and (b) in the proof of (3.3) can be used. But, due to the minimality of h_i, the inequality in (b) is in fact an equation and thus the sets $B_1=\{h_1\geqq s\}$ and $B_2=\{h_2\geqq 1-s\}$ are minimal for almost all $s\in[0,1]$. □

After having adapted the relevant result of Sect. 2.4 to sets instead of functions the same can be done concerning Sect. 1.4. Again it is no problem to restrict the sets B_i in (3.3) to $\mathfrak{G}(X_i)$; but also the deeper result of (1.31) can be carried over:

(3.6) **Proposition.** *Let $n=2$ and $B\in\mathfrak{F}(X)$; then*

$$I(B)=\inf\{\sum_i \mu_i(B_i):\ B_i\in\mathfrak{F}(X_i) \textit{ and } B\subset\bigcup_i \pi_i^{-1}[B_i]\}.$$

Proof. Functions $h_i\in\mathscr{F}(X_i)$ corresponding to $h=1_B$ according to (1.31) may again be confined to $0\leqq h_i\leqq 1$. Then proceeding as in the proof of (3.3) yields sets $B_i\in\mathfrak{F}(X_i)$ straight away. □

The natural combination of the statements in (3.5) and (3.6), however, is not possible, as is seen by:

(3.7) *Example.* Choose $n=2$ and X_i, μ_i as in (1.32). Then the set

$$B:=\left\{\left(k,\frac{1}{k}\right):\ k\in\mathbf{N}\right\}\in\mathfrak{F}(X)$$

satisfies $B\underset{N}{=}\emptyset$ and thus $I(B)=0$, while for sets $B_i\in\mathfrak{F}(X_i)$ the condition $B\subset\bigcup_i \pi_i^{-1}[B_i]$ always implies the inequality $\sum_i \mu_i(B_i)>0$; this is obtained similarly as in (2.23).

The concluding result of the present section concerns the problem mentioned in its headline; its purely topological version reads:

(3.8) **Proposition.** *Let* $n=2$ *and* $B\in\mathfrak{F}(X)$; *then there exists a measure* $\mu\in\mathsf{M}$ *with* supp $\mu\subset B$ *if and only if*

$$\sum_i \mu_i(B_i)\geqq 1 \quad \text{for all } B_i\in\mathfrak{F}(X_i) \text{ with } B\subset\bigcup_i \pi_i^{-1}[B_i].$$

Proof. Since the supremum $S(B)$ is attained in view of (2.19) and the duality theorem (2.6) applies to $h=1_B$, the condition on μ is equivalent to the inequality $I(B)\geqq 1$, which stating explicity the representation of (3.6) may be used. □

Finally some remarks concerning the existing literature are in order. First the work of Strassen [23, p. 436] has to be mentioned, who treats the problem of (3.8) for the case of polish spaces. Extensions to completely regular spaces are mainly due to Hoffmann-Jørgensen [8, p. 36], where functions $h_i\in\mathscr{C}(X_i)$ are used instead of sets $B_i\in\mathfrak{F}(X_i)$, and Edwards [4, p. 68], who restricts the essential inequality on μ_i and B_i to the case of a closed set B_1 and an open set B_2.

Of course, there is also an extension of (3.8) to $\overline{\mathfrak{F}}(X)$, i.e. to the existence of measures $\mu\in\mathsf{M}$ supported by some set B with $1_B\in\overline{\mathscr{F}}(X)$. In accordance with the remark following (2.19) this applies in its simplest version to countable intersections of sets belonging to the algebra generated on X by $\mathfrak{B}(X_1),\ldots,\mathfrak{B}(X_n)$. In this non-topological form the statement covers the main result in the abstract set-up of Sudakov [24, p. 822] as well as the recent results of Shortt [22, p. 316] for a certain subclass of second countable metrizable spaces X_i.

3.3. Stochastic Order

Another application concerns the case of topological spaces Y with a partial order "$\leqq$", where the order and topological structure are supposed to be compatible in the usual sense, i.e.

$$R(Y):=\{(x,y)\colon Y\ni x\leqq y\in Y\}\in\mathfrak{F}(Y\times Y),$$

in which case Y is called an ordered topological space.

Here, the main additional difficulty consists in measurability problems (in this context compare [13]), which can be circumvented by the tightness of the occurring measures via the following statement:

(3.9) **Lemma.** *Let* Y *be an ordered topological space,* $\nu\in\mathsf{M}(Y)$ *be arbitrary and* $g\in\mathscr{P}(Y)$ *be isotone. Then*

$$\nu_*(g)=\sup\{\nu(f)\colon g\geqq f\in\mathscr{L}^b(\nu) \text{ and } f \text{ isotone}\},$$
$$\nu^*(g)=\inf\{\nu(f)\colon g\leqq f\in\mathscr{L}_b(\nu) \text{ and } f \text{ isotone}\}.$$

Proof. 1. It suffices to prove the first equation. As any function $f\in\mathscr{L}(\nu)$ is minorized by functions $f'\in\mathscr{L}(\nu)\cap\mathscr{F}^b(Y)$ with arbitrarily small deviations

$\nu(f)-\nu(f')$, due to the regularity of ν, the assertion to be shown amounts to the following

$$(*)\quad \begin{cases} \text{given } f\in\mathscr{L}(\nu)\cap\mathscr{F}^b(Y) \text{ with } f\leqq g \text{ and } \varepsilon>0 \text{ there exists an isotone function} \\ f'\in\mathscr{L}(\nu) \text{ with } f'\leqq g \text{ and } \nu(f')>\nu(f)-\varepsilon. \end{cases}$$

2. Consider first the case $g\in\mathscr{P}_b(Y)$ and assume without loss of generality $g\geqq 0$ as well as $f\geqq 0$. Choosing a compact subset K of Y with $\nu(1_{\complement K}f)<\varepsilon$ condition (*) is satisfied by the function

$$f'(y):=\sup_{x\in K} 1_{R(Y)}(x,y)f(x):$$

(1) $$f' \text{ is isotone},$$

since $y\mapsto 1_{R(Y)}(x,y)\,f(x)$, due to $f\geqq 0$, is isotone for all $x\in K$;

(2) $$f'\leqq g,$$

in view of $f\leqq g$, the isotony of g and $g\geqq 0$;

(3) $$f'\in\mathscr{F}^b(Y),$$

since $(x,y)\mapsto 1_{R(Y)}(x,y)\,f(x)$ is upper semicontinuous and K is compact;

(4) $$\nu(f')>\nu(f)-\varepsilon,$$

in view of $(f-f')^+\leqq 1_{\complement K}f$ and the choice of K.

3. In the general case define $g_k:=g\vee(-k)$, $f_k:=f\vee(-k)$ and choose $\varepsilon_k>0$ with $\sum_{k\in\mathbf{N}}\varepsilon_k\leqq\varepsilon$. Applying part 2 of the proof to g_k, f_k and ε_k provides functions f_k' with the corresponding properties (1)–(4). For $f':=\inf_{k\in\mathbf{N}} f_k'$ this means

(1′) f' is isotone (by (1)),

(2′) $f'\leqq g$ (by (2) and $\inf_{k\in\mathbf{N}} g_k=g$),

(3′) $f'\in\mathscr{F}^b(Y)$ (by (3)),

(4′) $\nu(f')>\nu(f)-\varepsilon$ (by (4) and $(f-f')^+\leqq\sum_{k\in\mathbf{N}}(f_k-f_k')^+$),

i.e. the function f' meets all requirements. □

By means of this lemma it is possible to transfer isotony properties of h to the functions h_i in the definition of I. For the case of an arbitrary n this means:

(3.10) **Proposition.** *Let X_i be ordered topological spaces and the function $h\in\mathscr{P}(X)$ be isotone; then*

$$I(h)=\inf\{\sum_i \mu_i(h_i):\ h_i\in\mathscr{L}_f(\mu_i) \text{ isotone and } h\leqq\bigoplus_i h_i\}.$$

Proof. Consider functions $h_i \in \mathscr{L}_f(\mu_i)$ satisfying $h \leqq \bigoplus_i h_i$ and an arbitrary $\varepsilon > 0$. With the notation $A_i := \{h_i < \infty\}$ define

$$h'_1(x_1) := \sup\{h(x_1, \ldots, x_n) - \sum_{i \neq 1} h_i(x_i) \colon x_i \in A_i \text{ for } i \neq 1\}.$$

Then h_1 may be decreased to h'_1, obtaining an isotone function without violating the inequality $h \leqq \bigoplus_i h_i$. Applying next (3.9) to $\nu = \mu_1$ and $g = h'_1$ yields an isotone function $h''_1 \in \mathscr{L}_f(\mu_1)$ such that

$$h'_1 \leqq h''_1 \quad \text{and} \quad \mu_1(h''_1) < \mu_1^*(h'_1) + \varepsilon \leqq \mu_1(h_1) + \varepsilon.$$

Continuing this way, all functions h_i may be replaced by functions h''_i having all desired properties and satisfying

$$\sum_i \mu_i(h''_i) < \sum_i \mu_i(h_i) + n\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the proof is completed. □

Specialized now to the case $n = 2$, (3.10) yields in particular:

(3.11) **Proposition.** *Let $n = 2$ and $X_i = Y$ with an ordered topological space Y; then*

$$I(R(Y)) = 1 - \sup\{\mu_1(A) - \mu_2(A) \colon A \in \mathfrak{B}(Y) \text{ isotone}\}.$$

Proof. The function $h = 1_{R(Y)}$ is antitone in the first and isotone in the second argument, hence by (3.10) (reversing the order in X_1)

$$I(R(Y)) = \inf\{\sum_i \mu_i(h_i) \colon h_i \in \mathscr{L}_f(\mu_i) \text{ and } h \leqq \bigoplus_i h_i,$$
$$h_1 \text{ antitone and } h_2 \text{ isotone}\},$$

where h_i may be confined to $0 \leqq h_i \leqq 1$ as in (1.8a). But then the inequality $h \leqq \bigoplus_i h_i$ amounts to

$$1 \leqq h_1(x_1) + h_2(x_2) \quad \text{for} \quad x_1 \leqq x_2;$$

due to the isotony of $f := h_2$ therefore h_1 may be decreased to $1 - f$. This provides

$$I(R(Y)) = \inf\{1 - \mu_1(f) + \mu_2(f) \colon f \in \mathscr{B}(Y) \text{ isotone with } 0 \leqq f \leqq 1\}$$
$$= 1 - \sup\{\mu_1(f) - \mu_2(f) \colon \ldots\},$$

where – similarly as in the proof of (3.3) – the isotone functions f may be replaced by the isotone sets $A = \{f \geqq s\}$. □

The preceding result can even be strengthened by restricting A to $\mathfrak{F}(Y)$ or $\mathfrak{G}(Y)$ as it is done in the main result of this section:

(3.12) **Proposition.** *Let $n = 2$ and $X_i = Y$ with an ordered topological space Y; then the following conditions are equivalent:*

(0) *there exists* $\mu \in \mathsf{M}$ *with* supp $\mu \subset R(Y)$,

(1) $\mu_1(A) \leqq \mu_2(A)$ *for all isotone* $A \in \mathfrak{F}(Y)$,

(2) $\mu_1(A) \leqq \mu_2(A)$ *for all isotone* $A \in \mathfrak{G}(Y)$.

Proof. 1. Since the supremum $S(R(Y))$ is attained according to (2.19) and the duality theorem (2.6) applies to $h = 1_{R(Y)}$, condition (0) is equivalent to $I(R(Y)) \geqq 1$, hence in view of (3.11) to

$$\mu_1(A) \leqq \mu_2(A) \qquad \text{for all isotone } A \in \mathfrak{B}(Y).$$

Thus (0) implies both (1) and (2).

2. Let now condition (1) be fulfilled and for isotone $A \in \mathfrak{B}(Y)$ and arbitrary $\varepsilon > 0$ choose $K \subset A$ compact with $\mu_1(A \setminus K) < \varepsilon$. Then

$$F := \pi_2[R(Y) \cap (K \times Y)]$$

as the projection of a closed set along a compact space is again closed and in addition isotone; moreover $K \subset F$ and $F \subset A$, due to $K \subset A$ and the isotony of A. This leads to

$$\mu_1(A) - \varepsilon < \mu_1(K) \leqq \mu_1(F) \leqq \mu_2(F) \leqq \mu_2(A)$$

and for $\varepsilon \downarrow 0$ to the desired inequality $\mu_1(A) \leqq \mu_2(A)$.

3. To derive (0) from (2) instead of (1), it is only necessary to reverse the order in Y, to interchange the role of μ_1 and μ_2 and to replace $\mu_i(A)$ by $1 - \mu_i(\complement A)$. □

For ordered polish spaces the corresponding result has been stated first by Kamae-Krengel-O'Brien [9, p. 900], while extensions to completely regular spaces are due to Hoffmann-Jørgensen [8, p. 46] and Edwards [4, p. 75] – both under superfluous semicontinuity conditions on the map $y \mapsto \{x \in Y : x \leqq y\}$.

3.4. General Marginal Problems

A last application of duality theorems will treat a generalization of the marginal problem behind the first definition (1.1) and proposition (1.2). If instead of the one-dimensional measures $\mu_1, \ldots, \mu_n$ some multi-dimensional marginals are prescribed, the corresponding analogue of M may well be empty. In the most important case $X_i = \mathbf{R}$ a necessary and sufficient condition for the existence of a measure $\mu \in \mathsf{M}(X)$ with the given marginals was first derived in 1964 in the paper [11] by the author. Now this criterion can be extended to topological spaces without any additional assumptions; moreover, the corresponding product space may be composed of an infinite number of factors.

The following notation will be used in the sequel: let Y_t, $t \in T$, be (non-void) topological spaces with product space Y and denote for $\emptyset \neq U \subset V \subset T$ the product $\prod_{t \in U} Y_t$ by Y_U and the canonical projection from Y_V onto Y_U simply by π_U. Then the main result reads:

(3.13) **Proposition.** *For each of the non-void subsets $T_1, \dots, T_n$ of T let be given a measure $\nu_{T_i} \in \mathsf{M}(Y_{T_i})$. Then the existence of a measure $\nu \in \mathsf{M}(Y)$ satisfying the equations*

(1) $$\pi_{T_i}(\nu) = \nu_{T_i} \quad \textit{for all } i$$

is equivalent to the following condition

(2) $$\sum_i \nu_{T_i}(f_{T_i}) \geqq 0 \quad \textit{for all bounded } f_{T_i} \in \mathscr{F}(Y_{T_i}) \quad \textit{with} \quad \sum_i f_{T_i} \circ \pi_{T_i} \geqq 0.$$

Proof. Putting $U := \bigcup_i T_i$, for any measure $\nu_U \in \mathsf{M}(Y_U)$ there is obviously a measure $\nu \in \mathsf{M}(Y)$ with $\pi_U(\nu) = \nu_U$; therefore the assumption $U = T$ means no real restriction. Then define $X_i := Y_{T_i}$ (with X again standing for the corresponding product), abbreviate $\mu_i := \nu_{T_i}$ and consider the set

$$B := \{x \in X : \pi_{T_i \cap T_k}(x_i) = \pi_{T_i \cap T_k}(x_k) \text{ whenever } T_i \cap T_k \neq \emptyset\},$$

which is closed, since all Y_t are tacitly assumed to be Hausdorff spaces. Due to the assumption $U = T$, the map

$$\varphi : Y \ni y \mapsto (\pi_{T_1}(y), \dots, \pi_{T_n}(y)) \in B$$

defines a homeomorphism and induces, moreover, a bijection between $\mathsf{M}(Y)$ and the subset $\mathsf{M}_B(X)$ of $\mathsf{M}(X)$ consisting of all μ with supp $\mu \subset B$, where the equation $\pi_{T_i}(\nu) = \nu_{T_i}$ corresponds to the equation $\pi_i(\mu) = \mu_i$. Therefore, choosing $h := 1_B - \infty \cdot 1_{\complement B} \in \mathscr{F}(X)$, the existence of a measure ν satisfying (1) is equivalent to $I(h) \geqq 1$ by (2.6) and (2.19). Employing in addition (1.8 b) and (1.31), this amounts to the condition

$$\sum_i \mu_i(h_i) \geqq 1 \qquad \text{for all bounded } h_i \in \mathscr{F}(X_i) \text{ with } \bigoplus_i h_i \geqq h.$$

In view of $h(x) = -\infty$ outside B this is equivalent to

$$\sum_i \nu_{T_i}(h_i) \geqq 1 \text{ for all bounded } h_i \in \mathscr{F}(Y_{T_i}) \text{ with } \sum_i h_i \circ \pi_{T_i} \geqq 1.$$

By introducing $f_{T_i} := h_i - \dfrac{1}{n}$ condition (2) emerges. □

If the spaces Y_t are completely regular, the functions f_{T_i} occurring in this criterion, as usual, can be limited to $\mathscr{C}(Y_{T_i})$.

By means of the last result the restriction to tight measures throughout the foregoing investigations can be shown to be essential. To this end a counterexample from [11] is slightly modified:

(3.14) *Example.* Let $Y_0 = [0, 1]$ be partitioned into two subspaces Y_1 and Y_2 with inner Lebesgue measure $\lambda_*(Y_i) = 0$ and choose

$$T = \{0, 1, 2\} \quad \text{and} \quad T_i = \{0, i\} \quad \text{for} \quad i = 1, 2.$$

Consider the normed measures λ_i defined by

$$\lambda_i(B) := \lambda^*(B) \quad \text{for } B \in \mathfrak{B}(Y_i)$$

and let ν_{T_i} be their images by the Borel measurable maps

$$\psi_i\colon\; Y_i \ni y \mapsto (y, y) \in Y_{T_i},$$

which, due to $\lambda^*(Y_i) = 1$, fulfil

$$\nu_{T_i}(B \times Y_i) = \lambda(B) \quad \text{for } B \in \mathfrak{B}(Y_0).$$

To bounded functions $f_{T_i} \in \mathscr{C}(Y_{T_i})$ assign bounded functions g_i by

$$g_i(y_0) := \inf\{f_{T_i}(y_0, y_i)\colon\; y_i \in Y_i\} \quad \text{for } y_0 \in Y_0,$$

which are measurable, since the infimum may be restricted to a countable dense subset of Y_i. Then $\sum_i f_{T_i} \circ \pi_{T_i} \geqq 0$ implies

$$\sum_i g_i(y_0) = \inf\{\sum_i f_{T_i}(y_0, y_i)\colon\; y_i \in Y_i \text{ for } i \neq 0\} \geqq 0 \qquad \text{for } y_0 \in Y_0,$$

which in turn yields

$$\sum_i \nu_{T_i}(f_{T_i}) \geqq \sum_i \nu_{T_i}(g_i \circ \pi_0) = \sum_i \lambda(g_i) \geqq 0,$$

i.e. condition (2) in (3.13) is satisfied.

Assume now the existence of a measure $\nu \in \mathrm{M}(Y)$ satisfying condition (1) in (3.13). Then $Y_1 \cap Y_2 = \emptyset$ leads to

$$\begin{aligned}\nu(Y) &\leqq \nu(\bigcup_i \{y \in Y\colon\; y_i \neq y_0\}) \\ &\leqq \sum_i \nu_{T_i}(\{(y_0, y_i)\colon\; y_i \neq y_0\}) = 0\end{aligned}$$

and thus to a contradiction. Therefore the tightness of the measures ν_{T_i} cannot be weakened to their regularity and τ-continuity (which are given here, since the spaces Y_{T_i} are metrizable and second countable).

In addition to the reference [11] given at the beginning of this section the following two related results have to be referred to. First the preceding proposition can be deduced from the paper [16, p. 145] by Maharam, except that it provides a measure ν defined on a subalgebra of $\mathfrak{B}(Y)$ only. Besides that it can be derived from the paper [14, p. 101] by Lembcke, at least if condition (2) is formally strengthened by enlarging $\mathscr{F}(Y_{T_i})$ to $\mathscr{L}(\nu_{T_i})$.

Finally, it should be mentioned that (3.13) may be extended to the case of a countable number of given marginals, and above all – the general marginal problem supposed to have a solution – there arises again the question of corresponding duality theorems. But these and related generalizations will be taken up in a subsequent paper.

References

1. Choquet, G.: Forme abstraite du théorème de capacitabilité. Ann. Inst. Fourier **9**, 83–89 (1959)
2. Dellacherie, C., Meyer, P.A.: Probabilités et potentiel, Chapitres IX à XI. Paris: Herrmann 1983
3. Dudley, R.M.: Probabilities and metrics. Lecture Notes Series No. 45. Aarhus: Matematisk Institut 1976
4. Edwards, D.A.: On the existence of probability measures with given marginals. Ann. Inst. Fourier **28/4**, 53–78 (1978)
5. Fernique, X.: Sur le théorème de Kantorovitch-Rubinštein dans les espaces polonais. Séminaire de Probabilités XV. Lecture Notes in Mathematics No. **850**, pp. 6–10. Berlin-Göttingen-Heidelberg: Springer 1981
6. Gaffke, N., Rüschendorf, L.: On a class of extremal problems in statistics. Math. Operationsforschung Stat., Ser. Optimization **12**, 123–135 (1981)
7. Hausdorff, F.: Set Theory. New York: Chelsea Publishing Company 1957
8. Hoffmann-Jørgensen, J.: Probability in Banach spaces. Ecole d'Eté de Probabilités de Saint-Flour VI. Lecture Notes in Mathematics No. **598**, pp. 1–186. Berlin-Heidelberg-New York: Springer 1977
9. Kamae, T., Krengel, U., O'Brien, G.L.: Stochastic inequalities on partially ordered spaces. Ann. Probability **5**, 899–912 (1977)
10. Kantorovitch, L.V., Rubinštein, G.S.: On a space of completely additive functions (in Russian). Vestnik Leningrad Univ. **13/7**, 52–59 (1958)
11. Kellerer, H.G.: Verteilungsfunktionen mit gegebenen Marginalverteilungen. Z. Wahrscheinlichkeitstheorie verw. Gebiete **3**, 247–270 (1964)
12. Kellerer, H.G.: Duality theorems and probability metrics. Proc. 7th Brasov Conf. 1982 (to appear)
13. Kemperman, J.: On the FKG-inequality for measures on a partially ordered space. Proc. Nederl. Akad. Wet. **80**, 313–331 (1977)
14. Lembcke, J.: Reguläre Maße mit einer gegebenen Familie von Bildmaßen. Sitzungsber. Bayer. Akad. Wiss. 1976, 61–115 (1977)
15. Levin, V.L., Milyutin, A.A.: The problem of mass transfer with a discontinuous cost function and a mass statement of the duality problem for convex extremal problems. Russian Math. Surveys **34/3**, 1–78 (1978)
16. Maharam, D.: Consistent extensions of linear functionals and of probability measures. Proc. 6th Berkeley Sympos. Math. Statist. Probab. **2**, 127–147, Univ. Calif. (1971)
17. Monge, G.: Mémoire sur la théorie des déblais et ramblais. Mem. Math. Phys. Acad. Roy. Sci. Paris 666–704 (1781)
18. Oka, S.: Tychonoff functor and product spaces. Proc. Japan Acad. **54**, 97–100 (1978)
19. Rüschendorf, L.: Sharpness of Fréchet-Bounds. Z. Wahrscheinlichkeitstheorie verw. Gebiete **57**, 293–302 (1981)
20. Schaefer, H.H.: Topological vector spaces. New York-Heidelberg-Berlin: Springer 1966
21. Schwartz, L.: Radon measures on arbitrary topological spaces and cylindrical measures. London: Oxford University Press 1973
22. Shortt, R.M.: Strassen's marginal problem in two or more dimensions. Z. Wahrscheinlichkeitstheorie verw. Gebiete **64**, 313–325 (1983)
23. Strassen, V.: The existence of probability measures with given marginals. Ann. Math. Statist. **36**, 423–439 (1965)
24. Sudakov, V.N.: Measures on subsets of direct products. J. Soviet Math. **3**, 825–839 (1975)
25. Wasserštein, L.N.: Markov processes with countable state space describing large systems of automata. Problems of Information Transmission **5/3**, 47–52 (1969)

Received March 30, 1984; in final form June 24, 1984