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Intrinsically Homogeneous Sets, Splitting Times, and the Big Shift

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Summary. Let X be a strong Markov process. Let M be an optional set with the property that $1_{M\circ\theta_T}(s)=1_M(s+T)$ whenever $s>0$ and T is an optional time with $[T]\subset M$. If $L=\sup\{t>0: t\in M\}\in M$, we show that L is a splitting time of X: the pre-L events and the post-L events are conditionally independent given X_L. To prove this, we extend work of Sharpe's to show that the big shift operators Θ_T and $\hat{\Theta}_T$ commute with optional projection and dual optional projection, respectively, whenever T is an optional time. Examples are given which are not contained within previous work of Millar and Getoor.

0. Introduction

Let $X=(\Omega,\mathbf{F},\mathbf{F}_t,X_t,\theta_t,P^x)$ be a right process [3] canonically defined on the space Ω of all right continuous paths with lifetime ζ and taking values in a Lusin measurable space $(E,\mathbf{E})$. If T is an optional time, the strong Markov property states that $\mathbf{F}_T$ (the events occurring up to time T) and the events occurring after time T are conditionally independent given X_T. Recently there has been some interest in exploring random times which have this conditional independence property, but which are not necessarily stopping times. These times are called *splitting times* of the process. A good introduction to this topic is provided in the survey article of Millar [9]. Much of the work has been motivated by the following example. Let f be a continuous function on $E\times E$, and let R be the time at which the process $f(X_{t-},X_t)$ achieves its last minimum (here, of course, we assume X_{t-} exists!). Then the process $(X_{R+t})_{t>0}$ is conditionally independent of $\mathbf{F}_R$ given the vector $(\inf f(X_{s-},X_s),X_R)$, and, given this vector, $(X_{R+t},\mathbf{F}_{R+t})_{t>0}$ is a homogeneous strong Markov process. The most recent proof of this fact given by Millar [8] has relied on constructing an auxiliary Markov process and appealing to a well-known decomposition of a Markov process at a coterminal time [6, 7]. Of course, this approach yields much more information than the fact that R is a splitting time. Getoor [4]

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recently recast Millar's approach into a general setting so that other splitting times fit into the framework.

The purpose of this paper is to introduce a different auxiliary procedure which exhibits new classes of splitting times. These times are the ends of certain *intrinsically homogeneous* sets. We say a set M is intrinsically homogeneous if it is optional and if $1_{M\circ\theta_T}(s)=1_M(T+s)$ whenever T is an optional time with $[T]\subset M$ and $s>0$ (see (2.1) for a precise definition). These are generalizations of optional homogeneous sets. Getoor and Sharpe [5] gave a proof (relying on Motoo's theorem) that the ends of optional homogeneous sets are splitting times: we modify Motoo's theorem and their proof to fit this situation. A major hurdle is in showing that if L is the end of an optional intrinsically homogeneous set, then the dual optional projection A_t of the increasing process $1_{\{0<L\leqq t\}}$ is an intrinsically additive functional: i.e. $A_{t+T}=A_T+A_t\circ\theta_T$ whenever T is optional and $[T]\subset M$. In order to show this, in Sect. 1 we extend some work of Sharpe [10] on the big shift operator Θ_t. He introduced this operator on measurable processes and a dual operator $\hat{\Theta}_t$ taking random measures into random measures and showed that Θ_t commutes with optional projection and $\hat{\Theta}_t$ commutes with dual optional projection. We show that this still holds when we replace Θ_t and $\hat{\Theta}_t$ with Θ_T and $\hat{\Theta}_T$ whenever T is an optional time. We develop what we need in Sect. 1. Much of the material discussed in Sect. 1 is also discussed in Sect. 3 of a recent comprehensive paper on Semimartingales and Markov Processes by Çinlar, Jacod, Protter and Sharpe [11].

In Sect. 2, we discuss a few simple facts about intrinsically homogeneous sets and functionals. In Sect. 3, we show that the end L of an optional intrinsically homogeneous set M with $L\in M$ is a splitting time. (All intrinsically homogeneous sets in this paper will be optional, so we shall just refer to an "intrinsically homogeneous set.") That is, the pre-L and post-L events are conditionally independent given X_L, alone.

We then discuss several examples which are not contained in the approach of Getoor and Millar. In the discussion of Example (3.6), we show that this approach cannot subsume their work, either, so both approaches have their uses in different circumstances.

Now we turn to the usual task of laying out the habitual notations and hypotheses. The process has been introduced in the first sentence of the paper. We shall need a few more filtrations. Let $\mathbf{F}_t^0=\sigma\{X_s: s\leqq t\}$; $\mathbf{F}_t^e=\sigma\{f(X_s): s\leqq t, f$ is 1-excessive$\}$; $\mathbf{F}_t^\mu$ is the customary augmentation of $\mathbf{F}_t$ with all of the P^μ-null sets in the completion of $\mathbf{F}$. We shall use the notation $\mathbf{T}$ to denote the collection of all $(\mathbf{F}_t)$-optional times. A random time R is a nonnegative $\mathbf{F}$-measurable random variable. If $(\mathbf{G}_t)$ is any filtration on Ω, let $\mathbf{O}(\mathbf{G}_t)$ denote the smallest σ-field on $R^+\times\Omega$ containing the right continuous processes adapted to $\mathbf{G}_t$: these are the $(\mathbf{G}_t)$-optional processes. If R is a random time, we define the stopped fields as follows: $\mathbf{G}_R=\sigma\{Z_R: Z\in\mathbf{O}(\mathbf{G}_{t+})\}$. It is important to use this definition in discussing $\mathbf{F}_R^0$ and $\mathbf{F}_R^e$ when $T\in\mathbf{T}$. We let $\mathbf{F}^*$ denote the universal completion of $\mathbf{F}^0$.

A map κ: $\mathbf{B}(R^+) \to \mathbf{F}$ is said to be a random measure if (i) $\kappa(\cdot, A) \in \mathbf{F}$ for all $A \in \mathbf{B}(R^+)$; (ii) $\kappa(\omega, \cdot)$ is a measure for each ω. The class of random measures will be denoted by $\mathbf{R}$, and we let $\mathbf{AR} = \{\kappa \in \mathbf{R}$: $\kappa(\omega, (0,t]) \in \mathbf{F}_t$ for all $t\}$.

If $\mathbf{G}$ is a sigma-algebra on either Ω or $R^+ \times \Omega$, $b\mathbf{G}$ denotes the collection of bounded $\mathbf{G}$-measurable functions, and $\mathbf{G}^+$ denotes the collection of positive $\mathbf{G}$-measurable functions.

If $\kappa \in \mathbf{R}$ and $Z \in \mathbf{B}(R^+) \times \mathbf{F}$, we define $Z^* \kappa \in \mathbf{R}$ by setting $Z^* \kappa(\omega, B) = \int_B Z_s \kappa(\omega, ds)$ for $B \in \mathbf{B}(R^+)$. Finally, we let $\mathbf{E}^e = \sigma\{f$: f is 1-excessive$\}$ and $\mathbf{E}^* = \sigma\{f$: f is universally measurable on $E\}$.

1. The Big Shift

Let $T \in \mathbf{T}$, and define a map Θ_T: $\mathbf{B}(R^+) \times \mathbf{F} \to \mathbf{B}(R^+) \times \mathbf{F}$ by setting

$$(\Theta_T Z)(s, \omega) = Z_{s - T(\omega)}(\theta_T \omega)\, 1_{[T, \infty)}(s)$$

for all $Z \in \mathbf{B}(R^+) \times \mathbf{F}$. Sharpe introduced this operator in [10] with $T = t$, a fixed time. All of the ideas in this section are basically due to Sharpe [10] and Benveniste and Jacod [1]. Our contribution is to introduce and develop the big shift operator with random times T (see also [11]).

(1.1) **Theorem.** *There is a mapping* $Z \to {}^1Z$ *of* $b\mathbf{B}(R^+) \times \mathbf{F}$ *into* $b\mathbf{O}(\mathbf{F}_{t+})$ *so that*

(i) ${}^1Z$ *is a version of the optional projection of* Z *with respect to* $(\Omega, \mathbf{F}_t^\mu, P^\mu)$ *for all* μ.

(ii) ${}^1(\Theta_T Z)$ *and* $\Theta_T\, {}^1Z$ *are indistinguishable for each* T *in* $\mathbf{T}$.

Proof. Let $Z \in b\mathbf{B}(R^+) \times \mathbf{F}$. There is a process ${}^1Z$ satisfying (i) [10, 6, 11]. We now show that ${}^1Z$ satisfies (ii). Fix $T \in \mathbf{T}$ and an initial law μ. There is a process $Z^0 \in b\mathbf{B}(R^+) \times \mathbf{F}^0$ so that Z and Z^0 are P^μ-indistinguishable. It follows that ${}^1(\Theta_T Z)$ and ${}^1(\Theta_T Z^0)$ are P^μ-indistinguishable, as are $\Theta_T\, {}^1Z$ and $\Theta_T\, {}^1Z^0$. Therefore, it suffices to show that ${}^1(\Theta_T Z^0)$ and $\Theta_T\, {}^1Z^0$ are P^μ-indistinguishable for all $Z^0 \in b\mathbf{B}(R^+) \times \mathbf{F}^0$. So let $Z_s^0 =$

$$g(s) \prod_{i=1}^{n} \int_0^\infty e^{-a(i)u} F_i \circ \theta_u\, du, \tag{1.2}$$

where $(a(i))$ is a sequence of positive numbers, g is a bounded continuous function on R^+, and $F_i = f_i(X_0)$, with each f_i being a bounded continuous function on E. Then

$$Z_s^0 = g(s) \prod_{i=1}^{n} \left(\int_0^s e^{-a(i)u} F_i \circ \theta_u\, du + e^{-a(i)s} \int_0^\infty e^{-a(i)v} F_i \circ \theta_v \circ \theta_s\, dv \right). \tag{1.3}$$

Upon multiplying out, we find that Z^0 is a sum of products, a typical one of which is of the form

$$(1.4)\qquad \begin{aligned} W_s = g(s) \prod_{i=1}^{k} \int_0^s e^{-a(i)u} F_i \circ \theta_u \, du \\ \cdot \prod_{i=k+1}^{n} e^{-a(i)s} \int_0^\infty e^{-a(i)v} F_i \circ \theta_v \circ \theta_s \, dv. \end{aligned}$$

To compute ${}^1Z^0$ it suffices to compute ${}^1W$. Now

$$(1.5)\qquad {}^1W_s = g(s) \prod_{i=1}^{k} \int_0^s e^{-a(i)u} F_i \circ \theta_u \, du \, E^{X_s} \left[\prod_{i=k+1}^{n} e^{-a(i)s} \int_0^\infty e^{-a(i)v} F_i \circ \theta_v \, dv \right].$$

Notice that $E^x \left[\prod_{i=k+1}^{n} \int_0^\infty e^{-a(i)v} F_i \circ \theta_v \, dv \right]$ is $\left(\sum_{i=k+1}^{n} a(i) \right)$-excessive, so (1.5) is a.s. right continuous. It follows that (1.5) is a right continuous version of the optional projection of W_s. Then

$$(1.6)\qquad \begin{aligned} (\Theta_T {}^1W)_s = g(s-T) \prod_{i=1}^{k} e^{a(i)T} \int_T^s e^{-a(i)u} F_i \circ \theta_u \, du \\ \cdot \prod_{i=k+1}^{n} e^{-a(i)(s-T)} E^{X_s} \left[\prod_{i=k+1}^{n} \int_0^\infty e^{-a(i)v} F_i \circ \theta_v \, dv \right] 1_{[T,\infty)}(s). \end{aligned}$$

But

$$(1.7)\qquad \begin{aligned} (\Theta_T W)_s = g(s-T) \prod_{i=1}^{k} \int_0^{s-T} e^{-a(i)u} F_i \circ \theta_{u+T} \, du \\ \cdot \prod_{i=k+1}^{n} e^{-a(i)(s-T)} \int_0^\infty e^{-a(i)v} F_i \circ \theta_{v+s} \, dv \Bigg] 1_{[T,\infty)}(s), \end{aligned}$$

and a simple computation shows that a right continuous version of ${}^1(\Theta_T W)_s$ is given by (1.6).

Let $\mathbf{M} = \{Z \in b\mathbf{B}(R^+) \times \mathbf{F}^0\colon\ {}^1(\Theta_T Z^0)$ and $\Theta_T {}^1Z^0$ are P^μ-indistinguishable$\}$. Then $\mathbf{M}$ is a vector space containing constants, and $\mathbf{M}$ is closed under uniform and monotone convergence. Since $\mathbf{M}$ contains the multiplicative class consisting of functions of the form (1.2), which generates $\mathbf{B}(R^+) \times \mathbf{F}^0$, $\mathbf{M} = b\mathbf{B}(R^+) \times \mathbf{F}^0$ by the monotone class theorem, and this completes the proof. Q.E.D.

In order to discuss analogous results for a dual shift operator $\hat{\Theta}_T$, we shall need a few auxiliary facts. Once again, the approach was inspired by Sharpe's treatment of $\hat{\Theta}_T$, with $T=t$ fixed. The following result is a consequence of an easy monotone class argument.

(1.8) **Lemma.** *Let T be a $(\mathbf{F}^0_{t+})$-optional time. Then*

$$\mathbf{F}^0 = \sigma\{G \cdot H \circ \theta_T\colon\ G \in b\mathbf{F}^0_T,\ H \in b\mathbf{F}^0\}.$$

(1.9) **Proposition.** *Let T be an $(\mathbf{F}^0_{t+})$-optional time. The trace of $\mathbf{B}(R^+) \times \mathbf{F}^0$ on $[T, \infty)$ is $\sigma\{G \cdot \Theta_T Z\colon\ G \in b\mathbf{F}^0_T,\ Z \in b\mathbf{B}(R^+) \times \mathbf{F}^0\}$.*

Proof. The trace of $\mathbf{B}(R^+)\times\mathbf{F}^0$ on $[T,\infty)$ is generated by processes of the form $Z_s=1_{[T\vee c,\infty)}(s)F(\omega)$, where $c\geqq 0$, $F\in b\,\mathbf{F}^0$, and hence is equal to the sigma-algebra generated by processes of the form $Z_s=1_{[T\vee c,\infty)}(s)\,G\cdot F\circ\theta_T$, where $c\geqq 0$, $F\in b\,\mathbf{F}^0$, $G\in b\,\mathbf{F}_T^0$. Set $D(n)=\sup\{t: t=k2^{-n}$ for some k; $t+T\leqq c\}$ $(\sup\emptyset=0)$. It follows that $D(n)+T$ increases to $T\vee c$. But if $F\in b\,\mathbf{F}^0$, $G\in b\,\mathbf{F}_T^0$, and $Z^{k,n}=1_{[k/2^n,\infty)}\cdot F$, then

$$G\cdot\sum_k 1_{\{D(n)=k/2^n\}}(\Theta_T Z^{k,n})_s=1_{[D(n)+T,\infty)}(s)\,G\cdot F\circ\theta_T. \tag{1.10}$$

As n increases, this converges to $1_{[T\vee c,\infty)}G\cdot F\circ\theta_T$. Since $\{D(n)=k/2^n\}\in\mathbf{F}_T^0$, we have shown that $1_{[T\vee c,\infty)}G\cdot F\circ\theta_T\in\sigma\{G\cdot\Theta_T Z\colon G\in b\,\mathbf{F}_T^0,\ Z\in b\,\mathbf{B}(R^+)\times\mathbf{F}^0\}$, and the conclusion of the proposition follows. Q.E.D.

We now introduce Sharpe's dual shift operator $\hat{\Theta}_T\colon \mathbf{R}\to\mathbf{R}$ defined by

$$\hat{\Theta}_T\,\kappa(\omega,B)=\kappa(\theta_T\,\omega,B-T) \tag{1.11}$$

for all random measures κ, for all $B\in\mathbf{B}(R^+)$. Here we are assuming that κ is considered as a measure on all of R^1 so that (1.11) is well-defined. Sharpe proved that $\hat{\Theta}_T$ commutes with the dual optional projection for $T=t$ fixed. Using the preceding results, we need modify Sharpe's proof in only minor ways to show that $\hat{\Theta}_T$ commutes with dual optional projections whenever $T\in\mathbf{T}$.

(1.12) **Theorem.** *Let $\kappa\in\mathbf{R}$ with $E^x(\kappa(\omega,R^+))<\infty$ for all x. There is a map $\kappa\to\kappa^1$ of $\mathbf{R}$ into $\mathbf{AR}$ so that*

(i) *κ^1 is a version of the dual optional projection of κ with respect to $(\Omega,\mathbf{F}_t^\mu,P^\mu)$ for all μ.*

(ii) *$(\Theta_T\kappa)^1=\Theta_T(\kappa^1)$ for all $T\in\mathbf{T}$.*

Proof. A standard construction in Markov processes guarantees existence of κ^1 satisfying (i). To prove (ii), since $(\Theta_T\kappa)^1$ and $\Theta_T\kappa^1$ are measures carried by $[T,\infty)$, it suffices to show that

$$E^\mu\int 1_{[T,\infty)}(s)Z_s(\Theta_T\kappa)^1(ds)=E^\mu\int 1_{[T,\infty)}(s)Z_s(\Theta_T\kappa^1)(ds) \tag{1.13}$$

for all $Z\in(\mathbf{B}(R^+)\times\mathbf{F})^+$. But for fixed μ, it suffices to prove (1.13) for all $Z\in(\mathbf{B}(R^+)\times\mathbf{F}^0)^+$ (since a process in $\mathbf{B}(R^+)\times\mathbf{F}$ is P^μ-indistinguishable from a process in $\mathbf{B}(R^+)\times\mathbf{F}^0$) and for all $(\mathbf{F}_{t+}^0)$-optional times T (since for each $T\in\mathbf{T}$, there is an $(\mathbf{F}_{t+}^0)$-optional time T^0 with $P^\mu(T^0\neq T)=0$). Recalling (1.9), we find that it suffices to verify that

$$E^\mu\int G\cdot\Theta_T Z(\hat{\Theta}_T\kappa)^1(ds)=E^\mu\int G\cdot\Theta_T Z(\hat{\Theta}_T\kappa^1)(ds) \tag{1.14}$$

for all $G\in b\,\mathbf{F}_T^0$, $Z\in(\mathbf{B}(R^+)\times\mathbf{F}^0)^+$. The left hand side of (1.14) may be written as

$$\begin{aligned}
&E^\mu[G\textstyle\int {}^1(\Theta_T Z)_s(\hat{\Theta}_T\kappa)(ds)] &&=E^\mu[G\textstyle\int \Theta_T\,{}^1Z_s(\hat{\Theta}_T\kappa)(ds)]\\
&=E^\mu[G\cdot({}^1Z^*\kappa)(\theta_T,R^+)] &&=E^\mu[G\cdot E^{X_T}[({}^1Z^*\kappa)(\cdot,R^+)]]\\
&=E^\mu[G\cdot E^{X_T}[Z^*\kappa^1(\cdot,R^+)]]&&=E^\mu[G\textstyle\int \Theta_T Z_s(\hat{\Theta}_T\kappa^1)(ds)]
\end{aligned}$$

Q.E.D.

We now introduce some definitions which will be of use in Section 3. Let $\kappa \in \mathbf{R}$.

(1.15) *Definition.* Let Γ be a set in $\mathbf{O}(\mathbf{F}_t)$. Then κ is said to be Γ-homogeneous if the following holds: whenever $T \in \mathbf{T}$ with $[T] \subset \Gamma$, $1_{(T,\infty)} * \hat{\Theta}_T \kappa = 1_{(T,\infty)} * \kappa$.

The following statement is an immediate consequence of (1.15) and Theorem (1.12).

(1.16) **Corollary.** *Let $\kappa \in \mathbf{R}$ with $E^x(\kappa(\omega, R^+)) < \infty$. Let $\Gamma \in \mathbf{O}(\mathbf{F}_t)$. If κ is Γ-homogeneous, then κ^1 is Γ-homogeneous.*

If we let $A_t(\omega) = \kappa(\omega, (0, t])$, and κ and T are as in (1.15), then

$$A_s(\theta_T \omega) = \kappa(\theta_T \omega, (0, s]) = \hat{\Theta}_T \kappa(T, T+s] = A_{T+s} - A_T.$$

Therefore, the Γ-homogeneous measures $\kappa \in \mathbf{AR}$ correspond to (what we may call) the Γ-additive functionals.

This concludes our development of the big shift operator with random times. It is worth noting that there are analogues of all of the above theorems concerning commutation with the predictable and dual predictable projections, and the reader should have no difficulty filling in the details (the predictable case is discussed in [11]).

2. Intrinsically Homogeneous Random Sets

We define the class of intrinsically homogeneous random sets as follows.

(2.1) *Definition.* Let $\mathbf{H} = \{\Gamma \in \mathbf{O}(\mathbf{F}_t)$: if $T \in \mathbf{T}$ and $[T] \subset \Gamma$, then $1_{(T,\infty)} \Theta_T 1_\Gamma = 1_{(T,\infty)} 1_\Gamma\}$.

In this section we make some simple observations about sets in $\mathbf{H}$ and related objects, and we give some examples which are further discussed in Sect. 3.

We should clarify a point about $\mathbf{H}$. Sets in $\mathbf{H}$ should properly be called "intrinsically homogeneous on $(0, \infty)$." We can define $\mathbf{H}^* = \{\Gamma \in \mathbf{H}$: if $T \in \mathbf{T}$ and $[T] \subset \Gamma$, then $1_{[T,\infty)} \Theta_T 1_\Gamma = 1_{[T,\infty)} 1_\Gamma\}$. These sets should be called "intrinsically homogeneous on $[0, \infty)$." The analogy with sets which are homogeneous on $(0, \infty)$ and on $[0, \infty)$ is obvious. We shall stick to the terminology "intrinsically homogeneous" for sets in $\mathbf{H}$.

Let A_t be an increasing right continuous adapted process with $A_0 = 0$, and let Γ denote the set of points of right increase of A. We say that A_t is an intrinsically additive functional if $A_{S+T} = A_T + A_S \circ \theta_T$ whenever S, $T \in \mathbf{T}$, $S > 0$, and $[T] \subset \Gamma$. We leave it to the reader to check the following result (the proof is virtually the same as the one given when time-changing a Markov process by the inverse of a continuous additive functional).

(2.2) **Theorem.** Let A_t be an intrinsically additive functional as described above which is also continuous. Let τ_t denote the right continuous inverse of A. Then $X_\tau = (\Omega, \mathbf{F}, \mathbf{F}_{\tau_t}, X_{\tau_t}, \theta_{\tau_t}, P^x)$ is a homogeneous strong Markov process.

We now give two main examples of intrinsically homogeneous sets.
(2.3) Let A_t be an optional process satisfying $A_{t+s}(\omega)=A_t(\omega)+A_s(\theta_t\omega)$. This process need not be increasing or right continuous. Then $\Gamma=\{t: A_t=0\}$ belongs to **H**.

(2.4) Again, let A_t be a left continuous optional process satisfying $A_{t+s}=A_t+A_s\circ\theta_t$. Set $B_t=\sup\{A_s: s\leqq t\}$. Let $\Gamma=\{t: B_t>B_s \text{ for all } s<t\}$. Then $\Gamma\in\mathbf{H}$ and B has the following property. Let $T\in\mathbf{T}$, $[T]\subset\Gamma$. Then $B_{t+T}=\sup\{A_s: s\leqq t+T\}$. Since $[T]\subset\Gamma$, $A_T\geqq A_s$ for all $s<T$, so

$$B_{t+T}=\sup\{A_{s+T}: s\leqq t\}=\sup\{A_s\circ\theta_T: s\leqq t\}+A_T=B_t\circ\theta_T+B_T.$$

In the next section we shall examine some specific choices for A_t to derive information about splitting times.

3. Splitting Times and Intrinsically Homogeneous Sets

We fix $M\in\mathbf{H}$, and we let $L=\sup\{t>0: t\in M\}$ ($\sup\emptyset=0$). Assume that $L\in M$ a.s. on $\{L>0\}$. If $T\in\mathbf{T}$ and $[T]\subset M$, then $L\circ\theta_T=(L-T)^+$, so L behaves much as a coterminal time does. Let $\kappa(\omega,ds)$ be the random measure $\varepsilon_{L(\omega)}(ds)1_{\{0<L(\omega)<\infty\}}$. Now κ is M-homogeneous. Moreover, κ^1 is supported by M, for

$$E^\mu\int 1_{M^c}(s)\,\kappa^1(ds)=E^\mu\int 1_{M^c}(s)\,\kappa(ds)=P^\mu(0<L\in M^c)=0$$

for all μ.

The main object of this section is to prove the following theorem by adapting a proof of Getoor and Sharpe [5] to this setting.

(3.1) **Theorem.** *Let $M\in\mathbf{H}$ be such that $L\in M$* a.s. *on $\{L>0\}$. Then for every $F\in b\mathbf{F}^*$, there is a bounded $\mathbf{E}^*$-measurable function f such that for every $Z\in b\mathbf{O}(\mathbf{F}_t)$ and for every μ,*

$$E^\mu[Z_L F\circ\theta_L; L<\infty]=E^\mu[Z_L f(X_L); L<\infty].$$

Before embarking on the proof of (3.1), we prove the following generalization of Motoo's theorem.

(3.2) **Proposition.** *Let B and A be continuous, increasing, adapted processes with $B_0=A_0=0$, and let $M\in\mathbf{O}(\mathbf{F}_t)$. Assume*

(i) If S, $T\in\mathbf{T}$ *with* $[T]\subset M$, *then* $A_{S+T}=A_T+A_S\circ\theta_T$ *and* $B_{S+T}=B_T+B_S\circ\theta_T$;
(ii) $A\ll B$;
(iii) $E^\mu\int 1_{M^c}(s)\,dB_s=0$ *for all* μ.

Then there is a function $g\in\mathbf{E}^*$ so that $A_t=\int_0^t g(X_s)\,dB_s$.

Proof. Let $\tau_t=\tau(t)$ denote the right continuous inverse of B_t. Unfortunately, we cannot apply Theorem (2.2) (since B may not be intrinsically additive), but we can state that $(X_{\tau(t)}, \theta_{\tau(t)}, F_{\tau(t)}, P^x)$ is a time-homogeneous strong Markov process on the random set $\Phi=\{t: \tau(t)\in M\}$. (Notice that $\Phi\in\mathbf{O}(\mathbf{F}_{\tau(t)})$). In other words, if T is an optional time for the filtration $(\mathbf{F}_{\tau(t)})$ and $[T]\subset\Phi$, then $\tau_{T+t} =\tau_T+\tau_t\circ\theta_{\tau(T)}$. Set $\tilde{A}_t=A_{\tau(t)}$ and $\tilde{B}_t=B_{\tau(t)}=t$ on $\{\tau_t<\zeta\}$, so that $\tilde{A}_t\ll t$. Set $Z_t =\liminf_{n\to\infty} n(\tilde{A}_{t+1/n}-\tilde{A}_t)$. By Lebesgue's differentiation theorem, we have $\tilde{A}_t =\int_0^t Z_s\,ds$.

Set $g(x)=E^x[Z_0]\in\mathbf{E}^{*+}$. For the remainder of the proof, fix a finite measure μ on $(E,\mathbf{E})$, and define a measure ν on $\mathbf{E}$ by setting $\nu(C)=\int_0^\infty e^{-t}P^\mu(X_{\tau_t}\in C)\,dt$. Choose g' and g'' in $\mathbf{E}^+$ with $g'\leqq g\leqq g''$ and $\nu(g''-g')=0$. Then $\int e^{-t}(g''(X_{\tau_t}) -g'(X_{\tau_t}))\,dt=0$ a.s. (P^μ), which implies that the process $g(X_{\tau_t})$ is in the completion of $\mathbf{B}(R^+)\times\mathbf{F}$ with respect to the measure $P^\mu\times dt$. It remains to show that $A_t=\int_0^t g(X_s)\,dB_s$. If T is optional for $(\mathbf{F}_{\tau_t})$ with $[T]\subset\Phi$, we have $g(X_{\tau_T}) =E^\mu[Z_T|\mathbf{F}_{\tau_T}]=Z_T$ a.s. P^μ. In particular, if we let $T=t$ on $\{t\in\Phi\}$ and $T=\infty$ on $\{t\in\Phi\}^c$, we get $Z_t 1_\Phi(t)=g(X_{\tau_t})1_\Phi(t)$ a.s. P^μ for each $t\geqq 0$. But $Z_t 1_\Phi(t)\in\mathbf{B}(R^+) \times\mathbf{F}$, so $\{t\colon Z_t\cdot 1_\Phi(t)\neq g(X_{\tau_t})\cdot 1_\Phi(t)\}$ has Lebesgue measure 0 for P^μ-almost all ω. By (iii), $\{t\colon t\notin\Phi\}$ has zero Lebesgue measure for P^μ-almost all ω. Therefore, $\tilde{A}_t =\int_0^t g(X_{\tau_s})\,ds$, whence $A_v=\int_0^{B_v} g(X_{\tau_s})\,ds$ whenever $v\in$Range $(\tau_t)\cap M$. Thus $A_v =\int_0^v g(X_s)\,dB_s$ whenever $v\in$Range $(\tau_t)\cap M$. Since A_v and B_v increase only on the Range $(\tau_t)\cap M$, $A_v=\int_0^v g(X_s)\,dB_s$ for all $v\geqq 0$. Q.E.D.

(3.3) **Lemma.** *Let A_t be an increasing right continuous process which is $(\mathbf{F}_t)$-optional with $A_0=0$. If $g\in\mathbf{E}^{*+}$, then $B_t=\int_0^t g(X_s)\,dA_s$ is an $(\mathbf{F}_t)$-optional process.*

Proof. Fix a finite measure μ on $(E,\mathbf{E})$, and define a measure ν on $(E,\mathbf{E})$ by setting $\nu(h)=E^\mu\int_0^t h(X_s)\,dA_s$ for all $h\in\mathbf{E}^+$. If $g\in\mathbf{E}^{*+}$, we may find g_1 and g_2 in $\mathbf{E}^+$ with $g_1\leqq g\leqq g_2$ and $\nu(g_2-g_1)=0$. Therefore, $B_t=\int_0^t g_1(X_s)\,dA_s$ a.s. P^μ, which implies that $B_t\in\mathbf{F}_t^\mu$. Since this is true for all μ, $B_t\in\mathbf{F}_t$, and the right continuity of the process B_t implies that B_t is an $(\mathbf{F}_t)$-optional process. Q.E.D.

(3.4) *Comment.* Assume $E^\mu[B_\infty]<\infty$ for all finite μ in the lemma above. Letting $t=\infty$, we obtain $B_\infty=\int_0^\infty g_1(X_s)\,dA_s$ a.s. P^μ. Since $g_1\leqq g$, it follows that $B_t=\int_0^t g_1(X_s)\,dA_s$ a.s. P^μ for each t, and these two processes are therefore P^μ-indistinguishable.

Proof of Theorem (3.1)

Let $R=\inf\{t>0: t\in M\}$, and let $p(x)=E^x[e^{-R}]\in\mathbf{E}^*$. For $1\leqq n<\infty$, set $N_n=\{x: (n-1)/n\leqq p(x)<n/(n+1)\}$; $N_0=\{x: p(x)=1\}$. Let $F\in b\mathbf{F}^*$, $0\leqq F\leqq 1$, and set $\gamma(ds)=F\circ\theta_s\kappa(ds)$. Notice that γ^1 is M-homogeneous and $\gamma^1\ll\kappa^1$. Let $A_t^0=\int_0^t 1_{N_0}(X_s)\gamma^1(ds)$ and let $B_t^0=\int_0^t 1_{N_0}(X_s)\kappa^1(ds)$. By Lemma (3.3), A_t^0 and B_t^0 are in $\mathbf{O}(\mathbf{F}_t)$. Fix a finite measure μ on $(E,\mathbf{E})$. As in Lemma (3.3) and (3.4), we may choose $0\leqq g_1\leqq 1_{N_0}$ so that $g_1\in\mathbf{E}$ and $\int_0^t g_1(X_s)\gamma^1(ds)$ and A_t^0 are P^μ-indistinguishable. Therefore, to show that A_t^0 is continuous, we need only take $T\in\mathbf{T}$ with $[T]\subset M$ and examine

(3.5) $$E^\mu[\gamma^1(\{T\})g_1(X_T): 0<T<\infty]=E^\mu[F\circ\theta_T g_1(X_T); L=T, 0<T<\infty].$$

But on $\{g_1(X_T)>0,\ 0<T<\infty\}\subset\{X_T\in N_0, 0<T<\infty\}$, $R\circ\theta_T=0$, so $L\circ\theta_T>0$. Since $[T]\subset M$, $L\circ\theta_T>0$ is equivalent to $L>T$. The right hand side of (3.5) is therefore zero, so A_t^0 is continuous. Similarly, B_t^0 is continuous. It is simple to check that the hypotheses of Proposition (3.2) are satisfied, so there is a function $g\in\mathbf{E}^{*+}$ so that $A_t^0=\int_0^t g(X_s)\,dB_s^0$.

Now let $A_t^n=\int_0^t 1_{N_n}(X_s)\gamma^1(ds)$ and let $B_t^n=\int_0^t 1_{N_n}(X_s)\kappa^1(ds)$. Fix $n\geqq 1$ and a finite measure μ on $(E,\mathbf{E})$. Choose $h_n\in\mathbf{E}$ as in Lemma (3.3) and (3.4) with $0\leqq h_n\leqq 1_{N_n}$ so that B_t^n and $\int_0^t h_n(X_s)\kappa^1(ds)$ are P^μ-indistinguishable. Since $\gamma^1\ll\kappa^1$, there is a process $Z\in\mathbf{O}(\mathbf{F}_t)$ so that $\gamma^1(0,t]=\int_0^t Z_s\kappa^1(ds)$. It follows that A_t^n and $\int_0^t h_n(X_s)\gamma^1(ds)$ are P^μ-indistinguishable. Now we show that the $(\mathbf{F}_t)$-optional process $Z_t^n=h_n(X_t)\cdot 1_M(t)$ is discrete a.s. (P^μ). Let $R_n=\inf\{t>0: Z_t^n>0\}$, and let $p_n(x)=E^x(e^{-R_n})\leqq p(x)$. On $\{h_n>0\}\subset N_n$, $p<n/(n+1)<1$. An application of the argument given in Theorem 4 of [2] shows that $\{t: Z_t^n>0\}$ is an optional discrete set (and dA_t^n is therefore a discrete measure a.s. P^μ). If $T\in\mathbf{T}$, $[T]\subset\{t>0: Z_t^n>0\}$, and $W_t\in\mathbf{O}(\mathbf{F}_t)^+$, then

(3.6) $$\begin{aligned}E^\mu\int_0^\infty W_t 1_{[T]}(t)\,dA_t^n&=E^\mu\int_0^\infty W_t 1_{[T]}(t)\,h_n(X_t)\,\gamma^1(dt)\\&=E^\mu[W_T h_n(X_T)F\circ\theta_T; L=T, 0<T<\infty].\end{aligned}$$

On $\{0<T<\infty\}$, $L\geqq T$; so on $\{L\geqq T\}$, we have $L=T$ if and only if $L\circ\theta_T=0$ (since $[T]\subset M$). Thus we may rewrite (3.6) as

(3.7) $$\begin{aligned}&E^\mu[W_T h_n(X_T)E^{X(T)}[F; L=0]; 0<T<\infty]\\&=E^\mu\int W_t 1_{[T]}(t)\,h_n(X_t)\frac{k^F(X_t)}{k(X_t)}\kappa^1(dt)\end{aligned}$$

where $k^F(x)=E^x[F;L=0]$, $k(x)=P^x(L=0)$, and $0/0=0$. It follows that $\Delta A_T^n 1_{\{0<T<\infty\}}=(k^F(X_T)/k(X_T))\,\Delta B_T^n 1_{\{0<T<\infty\}}$ a.s. P^μ. If we set $f=\frac{k^F}{k}1_{\{k>0\}}+g\,1_{\{k=0\}}\in\mathbf{E}^*$, and sum over all $n\geqq 0$, we have that

$$A_t=\int_0^t f(X_s)\,dB_s \quad \text{a.s. } P^\mu,$$

where $A_t=\gamma^1(0,t]$ and $B_t=\kappa^1(0,t]$. But μ is arbitrary and f is independent of μ. Therefore, if $Z\in\mathbf{O}(\mathbf{F}_t)^+$,

$$E^\mu[Z_L F\circ\theta_L;0<L<\infty]=E^\mu\int Z_t\,dA_t=E^\mu\int Z_t f(X_t)\,dB_t. \tag{3.8}$$

But if we define a measure ν on $(E,\mathbf{E})$, by setting

$$\nu(j)=E^\mu\int Z_t j(X_t)\,dB_t=E^\mu[Z_L j(X_L),0<L<\infty], \tag{3.9}$$

for all $j\in\mathbf{E}^+$, it follows that (3.9) holds for all $j\in\mathbf{E}^{*+}$. Thus we may rewrite (3.8) as $E^\mu[Z_L f(X_L);0<L<\infty]$.

To complete the proof, we follow Getoor and Sharpe and apply the Markov property at zero:

$$\begin{aligned}E^\mu[Z_L f(X_L);L=0]&=E^\mu[Z_0 f(X_0);L=0]=E^\mu[Z_0 f(X_0)\,k(X_0)]\\&=E^\mu[Z_0 k^F(X_0)]=E^\mu[Z_0 E^{X(0)}[F;L=0]]\\&=E^\mu[Z_L F\circ\theta_L;L=0].\quad \text{Q.E.D.}\end{aligned}$$

We now consider some examples.

(3.5) *Example.* Let A_t be a $\mathbf{B}(R^+)\times\mathbf{F}^e$-measurable process satisfying:

(i) $A_{t+s}(w)=A_t(w)+A_s(\theta_t w)$,

(ii) A_t is adapted,

(iii) $t\to A_t$ is a.s. left continuous.

Then $M=\{t:A_t=0\}$ is intrinsically homogeneous and $L\in M$ a.s. Thus, for example, let L_t^0 and L_t^1 be the local times of a Brownian motion (killed exponentially) at 0 and 1, respectively. Then the last time $L_t^0=L_t^1$ is a splitting time for Brownian motion. Another way to look at this example is the following: if m_t is a continuous multiplicative functional, then $L=\sup\{t:m_t=1\}$ is a splitting time.

(3.6) *Example.* Let A_t be as in (3.5), and let $B_t=\inf\{A_s:s\leqq t\}$. Set $M=\{t:B_t<B_{t-e}$ for all $e\}$. Then M is intrinsically homogeneous and $L\in M$. If we take $A_s=f(X_s)-f(X_0)$ and assume that f is continuous and X has continuous paths (up to the lifetime), then L is the time of the *first* minimum of $f(X_t)$. This example seems to have escaped mention in [4] and [8], although an appropriate shift functional in [4] yields this example.

We now discuss a very special case of Millar's motivating example. Let f be a continuous function on E, and assume X has continuous paths. Let $B_t=\inf\{f(X_s):s\leqq t\}$, and assume $B_t>-\infty$. If we set

$$M=\{t>0:f(X_t)=B_t\}, \tag{3.7}$$

then M is intrinsically homogeneous (this relies on the continuity of f and X) and $L \in M$ a.s. Therefore, the conclusion of Theorem (3.1) applies (and yields the same conclusion as does Millar's approach). It does not seem likely that this example can be extended to the generality of Millar's motivating example within the present framework. One would need to deal with processes of the form $F(X_{s-}, X_s)$, and the presence of the left limit seems to be incompatible with the notion of intrinsically homogeneous set as we have formulated it. (The reader is encouraged to examine why M is no longer intrinsically homogeneous in (3.7) if we replace $f(X_s)$ with $F(X_{s-}, X_s)$ and B_t with $\inf\{F(X_{s-}, X_s): s \leqq t\}$. It may be possible to broaden the definition and to prove a "predictable" version of (3.1) which would permit a statement such as "pre-minimum events and post-minimum events are condtitionally independent given a *germ field* around the time of the minimum." Such a result is less precise than the ones obtained by Getoor and Millar, and we have therefore not pursued this.

(3.8) *Example.* We assume X is a discrete-time process for simplicity. The reader is invited to formulate a continuous analogue of this example (which will involve left limits of the process). Let $M = \{k$: there exists $m < k$ so that $X_m = X_k$ and, for every i with $m < i < k$, $X_j \neq X_i$ for all $j < i\}$. Intuitively, we are looking at the loops of the process which do not contain other loops, and we let M consist of the times these loops terminate. It is easy to check that if $T \in \mathbf{T}$ with $[T] \subset M$, then $M \circ \theta_T \cap [1, \infty) = (M - T)^+ \cap [1, \infty)$ (i.e. M is intrinsically homogeneous on $(0, \infty)$). It is also easy to see that M is *not* homogeneous. Since M is discrete, $L \in M$, and L is a splitting time by Theorem (3.1).

A Final Comment. The reader may find the assumption in (3.1) that "$L \in M$" somewhat annoying. If M is optional and homogeneous, the closure of M, $\bar{M}$, is also optional and homogeneous, and $L \in \bar{M}$. However, this is not true for intrinsically homogeneous sets. For example, let X be uniform motion to the right on the line with speed 1, and let A_t be the additive functional of X which increases linearly with slope 1. Let B_t be the additive functional of x which jumps up by 1 when the process passes through 0. Note that P^0 ($B_t = 0$ for all $t) = 1$. Let $C_t = \sup\{A_s - B_s: s \leqq t\}$, and let $M = \{t > 0$: $C_t > C_s$ for all $s < t\}$. Then M is intrinsically homogeneous, but $\bar{M}$ is not. For if $\omega(0) = x < 0$, $M(\omega) = (0, -x) \cup [1-x, \infty)$, and $\bar{M}(\omega) = [0, -x] \cup [1-x, \infty)$. Let T be the time X_t hits 0. Then $T \in \bar{M}$ and $\bar{M}(\theta_T \omega) = [0, \infty)$ while $(\bar{M} - T)^+ = [1 - x - T, \infty) \cap (0, \infty)$.

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