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Power Series Whose Coefficients Form Homogeneous Random Processes

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Summary. Power series $f(z)=\sum a_i z^i$ are considered, where the sequence $\{a_i\}$ forms a homogeneous random process. If the sequence is exchangeable and the variance of the marginal distributions exists, it is proved that r, the random radius of convergence of $f(z)$, takes the values 0 and 1. If the sequence is a second order stationary time series then $r=1$ with probability 1. If $\{a_i\}$ is a regular denumerable Markov chain, it can be proved that $r=c\leqq 1$ with probability 1, but both $c=0$ and $c=1$ can arise. A number of criteria are given for deciding the value of c in this situation.

1. Introduction

Let $r(f)$ denote the radius of convergence of the power series

$$f(z)=\sum_{i=0}^{\infty} a_i z^i,$$

where z is a complex variable and the $\{a_i\}$ are complex valued random variables. Then $r(f)$ is also a random variable. Its distribution has been studied by Arnold [1, 2]. If the $|a_i|$ are independently and identically distributed, the situation is completely described by the following theorem.

Zero-one law ([1], Satz 6). If the $\{|a_i|\}$ are independently and identically distributed, then $r(f)=c$ with probability one. If $E\log^+|a_i|=\infty$, then $c=0$; if $E\log^+|a_i|<\infty$, then $c=1$.

Arnold also studied the situation where the $\{|a_i|\}$ have identical marginal distributions but the dependence between them is arbitrary. He showed that a random power series with this property can be constructed for which the distribution of $r(f)$ is any preassigned discrete distribution. However, there is the following limitation.

Theorem on identically distributed coefficients ([1], Satz 5). If the $\{|a_i|\}$ are identically distributed, and $E\log^+|a_i|<\infty$, then $r(f)\geqq 1$ with probability one.

The object of this paper is to investigate some cases where the $\{|a_i|\}$ have identical marginal distributions, but where the dependence between them is of one of the regular forms which have proved of interest in other branches of probability theory. They are in this way intermediate between those leading to the two theorems quoted above.

2. Exchangeable Processes

The sequence $\{|a_i|\}$ is said to be exchangeable if the joint distribution function of any finite subset containing k members, say $|a_{i_1}|,\ldots,|a_{i_k}|$ depends only on k and not on the particular indices $i_1,\ldots,i_k$. The fundamental properties of such sequences were discovered by De Finetti, but it is convenient here to quote a result in the terminology of Blum *et al.* [4]. Let F be the class of one dimensional distribution functions, $F(x, y)$ be the subset satisfying $F(x) \leqq y$, and A the Borel field of subsets of F generated by the $\{F(x, y)\}$. Then there is a probability measure μ on A such that if $Pr(B)$ is the probability of any event B defined on the sample space of the sequence $\{|a_i|\}$, and $Pr^*(B)$ is its probability calculated on the assumption that the $|a_i|$ are independent, then

$$Pr(B) = \int_F Pr^*(B)\, d\mu(F).$$

The question of the convergence of the series $\sum a_i z^i$ can thus be referred to the case of independent, identically distributed coefficients.

Proposition 1. *If* $\{|a_i|\}$ *is an exchangeable sequence,* $r(f)$ *can take the values* 0 *and* 1. $Pr\{r(f)=0\} = \mu(F_1)$ *where* F_1 *is the subset of* F *containing those distribution functions for which* $\int \log^+ |x|\, dF(x) = \infty$.

Proof. This is immediate from the above representation and the zero-one law.

3. Stationary Time Series

Let $\{|a_j|\}$ be a second order stationary time series, so that $E|a_j|$ is a constant (which can be taken as 0 without loss of generality), and $E|a_j a_{j+t}|$ depends only on t. The sequence can be represented in the form

$$|a_j| = \int_\Omega e^{ijw}\, d\zeta(w)$$

where $\zeta(w)$ is an orthogonal set process and the integral is defined in the sense that the approximating sums converge in quadratic mean ([5], p. 483).

Proposition 2. *If* $\{|a_j|\}$ *is a second order stationary time series,* $r(f)=1$ *with probability one.*

Proof. Let $\sum e^{ijw_s} \zeta(w_s)$ be a typical approximating sum for the above integral. Then if $|z|<1$,

$$\sum |a_j|\, z^j = \sum_j \lim \sum e^{ijw_s} \zeta(w_s)\, z^j$$

$$= \lim \sum \frac{\zeta(\omega_s)}{1 - z\, e^{iw_s}}$$

$$= \int_\Omega \frac{d\zeta(w)}{1 - z\, e^{iw_s}}.$$

The change in order is permissible since for fixed $|z|<1$, the convergence of the series is uniform in w_s, and the integral on the last line exists in quadratic mean. For $|z|>1$, the series diverges for almost every w. Thus $r(f)=1$ with probability one.

4. Denumerable Markov Chains in Equilibrium

Let the sequence $\{|a_i|\}$ be a time homogeneous Markov chain on a denumerable state space $\{y\}$, consisting of a single positive recurrent ergodic class without cyclic sub-classes. Such a Markov chain is sometimes said to be regular. Let the one-step and n-step transition probabilities be denoted by

$$p_{xy} = Pr\{|a_i| = y \,|\, |a_{i-1}| = x\} \qquad i = 1, 2, \ldots,$$
$$p_{xy}^{(n)} = Pr\{|a_i| = y \,|\, |a_{i-n}| = x\} \qquad i = n, n+1, \ldots.$$

The ergodic property of regular Markov chains asserts the existence of a discrete probability distribution p_y^* such that

$$\lim_{n \to \infty} p_{xy}^{(n)} = p_y^*.$$

The following partial extensions of results for independent $|a_i|$ hold.

Proposition 3. *If the sequence $\{|a_i|\}$ is a regular Markov chain, then $r(f) = c$ with probability one.*

Proof. No matter which state is occupied at $i = i_0$, every other state will be entered for some $i > i_0$. Hence the event that the series converges is independent of the state occupied at i_0, and consequently of the sequence of states occupied at $i = 1, \ldots, i_0$. Thus the result follows by a simple extension of the zero-one law ([5], p. 398).

Proposition 4. *If the sequence $\{|a_i|\}$ forms a regular Markov chain, then $r(f) \leqq 1$ with probability one.*

Proof. Let η be a fixed state, and let $i_1, i_2, \ldots,$ be the random sequence of values for which $|a_{i_j}| = \eta$. Then by the recurrence property of regular Markov chains, $\{i_j\}$ is an infinite set. Now if $|z| > 1$,

$$\sum_{i=0}^{N} |a_i|\,|z|^i \geqq \sum_{i_j \leqq N} |a_{i_j}|.$$

Since the right hand side diverges as $N \to \infty$, the proposition is established.

The 'one' half of the zero-one law can be extended to regular Markov chains.

Proposition 5. *If the sequence $\{|a_i|\}$ forms a regular Markov chain with equilibrium distribution p_y^*, and $\sum p_y^* \log^+ |y| < \infty$, then $r(f) = 1$ with probability one.*

Proof. Suppose that each $|a_i|$ has the marginal distribution p_y^*. This is convenient and consistent with the object of the paper, but since it can be satisfied by assigning an appropriate distribution to $|a_0|$, it is not an essential restriction. It then follows from the theorem on identically distributed coefficients that $r(f) \geqq 1$. But by Proposition 4, $r(f) \leqq 1$. Hence $r(f) = 1$ with probability one as asserted.

On the other hand the condition $E \log^+ |a_i| = \infty$ is no longer sufficient to ensure that $r(f) = 0$. It is in fact possible to find distributions satisfying this condition for which there exist a pair of Markov chains, each having the specified distribution as equilibrium distribution, but such that $r(f) = 1$ for one of them and $r(f) = 0$ for the other.

Example 1. Consider a Markov chain whose state space is the positive integers, and for which for each j, the only transitions possible are to the adjacent states

$j-1, j+1$. Let α_j^* be the stationary measure for the chain, that is the solution of

$$\alpha_j^* = \alpha_{j-1}^* p_{j-1,j} + \alpha_{j+1}^* (1 - p_{j,j+1}),$$

where in view of the definition $p_{j,j-1} = 1 - p_{j,j+1}$ and $p_{12} = 1$.

On introducing the convenient substitution $p_{j,j+1} = \frac{1}{2} - b_j$, this difference equation can be written

$$\alpha_{j+1}^* = \{\alpha_j^* - \alpha_{j-1}^* (\tfrac{1}{2} - b_{j-1})\} / (\tfrac{1}{2} + b_j).$$

It can be seen from this that by suitable choice of the sequence b_j, for instance by setting $b_j = 0$ for suitable segments of the integers, it can be arranged that

$$\sum_{j=1}^{\infty} \alpha_j^* = c < \infty, \qquad \sum_{j=1}^{\infty} \alpha_j^* \log j = \infty.$$

Then $p_j^* = c^{-1} \alpha_j^*$ is the stationary distribution for the chain, and satisfies $\sum p_j^* \log j = \infty$. Clearly $Pr\{|a_j - a_{j+t}| > t\} = 0$, and hence conditional on $|a_0| = \eta$, and for $|z| < 1$,

$$\sum_{i=0}^{N} |a_i| \, |z|^i \leqq \sum_{t=0}^{N} |\eta + t| \, |z|^t.$$

From this result and Proposition 4 it follows that $r(f) = 1$ with probability one.

Example 2. Let $\{p_j^*\}$ be the stationary distribution of Example 1 and let the infinite transition matrix (p_{ij}) be defined by $p_{jk} = p_k^*$. Then the Markov chain is a sequence of independent random variables and by the zero-one law, $r(f) = 0$.

Thus the distribution of $r(f)$ is not completely determined by the equilibrium distribution of the chain. The next results concern the rows of the one-step transition matrix.

Proposition 6. *Let the sequence $\{|a_i|\}$ form a regular Markov chain with transition matrix (p_{xy}). If for every x, $\sum p_{xy} \log^+ |y| < \infty$ and the convergence is uniform in x, then $r(f) = 1$ with probability one. If for every x, $\sum p_{xy} \log^+ |y| = \infty$, and the divergence is uniform in x, then $r(f) = 0$ with probability one.*

Proof. The arguments are identical with those used by Arnold in [1], Satz 6, depending on Lemma 3, Lemma 4 and Korollar 2 of that paper. Indeed, although the condition of identical marginal distribution of the $|a_i|$ is the natural one to study, it seems that in general the actual requirements are that the convergence or divergence of the integral for $E \log^+ |a_i|$ should be uniform in i and with respect to conditioning on the $\{|a_j|\}$ for $j < i$.

The second part of the above result is superseded by the following one.

Proposition 7. *Let the sequence $\{|a_i|\}$ form a regular Markov chain with transition matrix (p_{xy}). If for any value of x, $\sum p_{xy} \log^+ |y| = \infty$, then $r(f) = 0$ with probability one.*

Proof. Let η be a fixed state of the chain such that $\sum_y p_{\eta y} \log^+ |y| = \infty$. Let $i_1, i_2, \ldots$, be the random sequence of indices for which $|a_{i_j}| = \eta$, which as noted above is infinite with probability one. As a consequence of the strong Markov property, the random variables $\{|a_{i_j+1}|\}$ form an independent, identically distributed sequence, for which the common distribution is $p_{\eta y}$. From the properties of positive recurrent Markov chains the sequence $i_1, i_2 - i_1, \ldots$, is an independent sequence, all except the first member being identically distributed and $E(i_{j+1} - i_j) = p_\eta^{*-1} (= m$, say$) < \infty$. Hence by the strong law of large numbers $Pr\left(\lim_{n\to\infty} i_n/n = m\right) = 1$. In particular for every sample sequence $\{|a_i|\}$ other than a set of probability zero, there exists an integer n_0 such that $i_n < 2mn$ for all $n > n_0$. Now if $|z| < 1$

$$\begin{aligned}\sum_{i=0}^{N} |a_i|\,|z|^i &\geqq \sum_{i_j \leqq N} |a_{i_j+1}|\,|z|^{i_j+1} \\ &\geqq \sum_{\cap (j \leqq n_0)(i \leqq N)} |a_{ij+1}|\,|z|^{i_j+1} + \sum_{\cap (j \geqq n_0+1)(i \leqq N)} |a_{i_j+1}|\,|z|^{2mj}.\end{aligned}$$

For finite N the second part of the right hand side may not exist, but as $N \to \infty$ the probability of this tends to zero. But in view of the zero-one law and the above remarks, the series

$$|a_{i_j+1}|\,|z^{2m}|^j$$

diverges with probability one. Hence, independently of the random value of n_0 involved in the above inequality, the series $\sum |a_i|\,|z|^i$ also diverges with probability one.

The case where $\sum p_{xy} \log^+ |y| < \infty$ for every x, but the convergence is not uniform remains outstanding. Example 1 shows that $r(f) = 1$ is possible under these conditions. The following example shows that $r(f) = 0$ is also possible.

Example 3. As in Example 2 let the state space be the positive integers and consider the transition matrix (p_{jk}) given by $p_{jk} = p_k^*$. Let $b_0, b_1, \ldots$, be a sequence of constants with the property that

$$Pr\{|a_i| > b_i \text{ infinitely often}\} = 0.$$

By the Borel-Cantelli lemma this is true when

$$\sum_{i=0}^{\infty} \sum_{j=b_i+1}^{\infty} p_j^* < \infty$$

and such a sequence of constants is said to belong to the upper class ([5], p. 260). By increasing the early members if necessary it can be arranged that

$$Pr\{a_i \leqq b_i, \text{ all } i\} > 0.$$

Now let a new transition matrix (p'_{jk}) be defined by

$$\begin{aligned} p'_{jk} &= p_{jk} && \text{if} \quad k < b_j, \\ &= \sum_{l=b_j}^{\infty} p_{jl} && \text{if} \quad k = b_j, \\ &= 0 && \text{if} \quad k > b_j. \end{aligned}$$

The right hand end of each row of the original matrix is thus "telescoped" onto the b_jth element. Now let the sample sequences $a_0, a_1, \dots,$ of the original chain be mapped into the sample sequences $a'_0, a'_1, \dots,$ of the new chain by the following transformation.

$$\begin{aligned} a'_0 &= a_0, \\ a'_j &= a_j \quad \text{if} \quad a'_{j-1} < b_{j-1}, \\ &= b_j \quad \text{if} \quad a'_{j-1} \geqq b_{j-1}. \end{aligned}$$

Reference to the definition of (p'_{jk}) shows that the mapping of sample sequences is measure preserving. Now the sample sequences for which $a_i \leqq b_i$ for all i, are left unaltered by the transformation. But because the original process is an independent sequence for which $r(f)=0$, all these sample sequences except for a subset of probability zero also have the property that $a_i > c^i$ infinitely often, for every c ([1], Satz 3h). Thus in the new process defined by (p'_{jk}) there is a set of sample sequences of positive probability, for which $a_i > c^i$ infinitely often for every c. Thus there is a positive probability that $r(f)=0$, and by Proposition 3 this must be one. However, each row of the transition matrix contains only a finite number of non-zero entries.

The fact that many of the properties of a Markov chain can be deduced from the recurrence properties of a fixed state, which in the present problem leads to the consideration of subsequences of the original sequence as in the proofs of Propositions 4 and 7, suggests a connection between this problem and that of random power series with gaps, treated by Arnold in [3], which also contains unsolved questions.

If η is a state of the chain, let $\sigma(\eta) = \max\{|y-\eta|; p_{\eta y} > 0\}$ be called the span of η. In Example 1, $\sigma(\eta)=1$ for all η, while in Example 3 $\sigma(\eta)$ is always finite but increases rapidly as η increases. By bounding its rate of growth a criterion is obtained.

Proposition 8. *Let the sequence $\{|a_i|\}$ form a regular Markov chain on the positive integers for which $\sigma(j) = O(j^k)$ for some k, then $r(f)=1$ with probability one.*

Proof. Conditional on $a_0 = \eta$, and for $|z|<1$,

$$\begin{aligned} \sum_{i=1}^{N} a_i |z|^i &\leqq \beta \sum_{i=0}^{\infty} \sum_{j=1}^{i} |\eta + j^k|\,|z|^i \\ &\leqq \beta' \sum_{i=0}^{\infty} i^{k+1} |z|^i < \infty. \end{aligned}$$

Finally, in a different direction it can be shown that a regularity property of the ergodic behaviour of the chain ensures that the "zero" half of the zero-one law also carries over from the case of an independent sequence.

Proposition 9. *If the sequence $\{|a_i|\}$ forms a regular Markov chain for which the convergence $p_{xy}^{(n)} \to p_y^*$ is uniform in x, and for which $\sum_y p_y^* \log^+ |y| = \infty$, then $r(f)=0$.*

Proof. Let m be chosen sufficiently large, that $\sum_{y \geqq \eta} |p_{xy}^{(m)} - p_y^*| < \varepsilon$ for all x, η. Now if $b_0, b_1, \dots,$ is a sequence of independent, identically distributed random variables with common distribution p_y^*, then since the series $\sum b_i z^i$ has radius of convergence zero, it follows that $\lim_{i\to\infty} Pr\{b_j > c^j \text{ for some } j > i\} = 1$, for every c. Hence $\lim_{i\to\infty} Pr\{a_{mj} > c^{(i/m)mj} \text{ for some } j > i\} > 1 - \varepsilon$, for every c. This establishes the result.

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