Z. Wahrscheinlichkeitstheorie verw. Geb. 10, 335—353 (1968)

Generalized Morse Sequences

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Received February 6, 1968

Summary. A method for construction of almost periodic points in the shift space on two symbols is developed, and a necessary and sufficient condition is given for the orbit closure of such a point to be strictly ergodic. Points satisfying this condition are called generalized Morse sequences. The spectral properties of the shift operator in strictly ergodic systems arising from generalized Morse sequences are investigated. It is shown that under certain broad regularity conditions both the continuous and discrete parts of the spectrum are non-trivial. The eigenfunctions and eigenvalues are calculated. Using the results, given any subgroup of the group of roots of unity, a generalized Morse sequence can be constructed whose continuous spectrum is non-trivial and whose eigenvalue group is precisely the given group. New examples are given for almost periodic points whose orbit closure is not strictly ergodic.

Introduction

In order to explain our results we shall need some notation. A sequence $b = (b_0, \ldots, b_m)$ of zeroes and ones is called a block. The block obtained from b by changing zeroes into ones and vice-versa is called the mirror image of b and denoted by $\tilde{b}$. A fixed block $c = (c_0, \ldots, c_n)$ may be used as a rule to construct new blocks from old ones: if b is a block, then we form the block $b \times c$ by putting $n + 1$ copies of either b or $\tilde{b}$ next to each other, choosing the i^{th} copy as b if $c_i = 0$ and as $\tilde{b}$ if $c_i = 1$. Now if $c_0 = 0$, then the block $b \times c$ is simply an extension of the block b.

Using the notation, we may define the well-known Morse sequence x (see e.g. [4], [7], [8]) as an infinite "product" of blocks: set $b = (01)$ and $x = b \times b \times b \times \cdots$. In words, this rule says: first write down 01, and then at each succeeding step write the mirror image of the complete previous production to the right of the same. The first 32 members of x are

$$01 \,|\, 10 \,|\, 1001 \,|\, 10010110 \,|\, 1001011001101001 \,|\, \ldots .$$

Let us denote by Ω the space of two-sided sequences of zeroes and ones, and by T the shift transformation on Ω. The following results were announced by S. Kakutani in [4]. If the Morse sequence x is continued to the left in a suitable manner to produce a point of Ω, then the orbit closure $\mathcal{O}_x$ of this point under T is a strictly ergodic subsystem of (Ω, T). Furthermore, T possesses partly continuous and partly discrete spectrum on $\mathcal{O}_x$ with respect to the uniquely determined probability measure on $\mathcal{O}_x$, and the group $\mathcal{G}_x$ of eigenvalues of T on $\mathcal{O}_x$ coincides with the group of all 2^k-th roots of unity.

In this paper we consider the infinite sequences which can be produced by the above-mentioned method of generating new sequences from old ones. For instance, if we set $b = (001)$, then $x = b \times b \times b \times \ldots$ defines a "ternary" sequence

$$x = (001\ 001\ 110\ 001\ 001\ 110\ 110\ 110\ 001\ \ldots),$$

and the statements above are shown to be valid for this x, the group of eigenvalues this time being the group of all 3^k-th roots of unity. In general, if $b^0, b^1, \ldots$ are blocks all beginning with a zero and having length greater than two, then $x = b^0 \times b^1 \times b^2 \times \ldots$ defines an infinite sequence. We determine a necessary and sufficient condition for x to be periodic. Then, restricting our attention to non-periodic x, we show that a necessary and sufficient condition for the corresponding (canonically defined) subset $\mathcal{O}_x$ of Ω to be strictly ergodic is that a sufficient portion of both zeroes and ones occur in the blocks $b^0, b^1, \ldots$ which define x. If $r_0(b), r_1(b)$ denote the relative frequencies of occurrence of zeroes and ones respectively in the block b, then this condition is simply that the sum

$$\sum_{t=0}^{\infty} \min(r_0(b^t), r_1(b^t))$$

diverge. Non-periodic sequences x for which this condition is satisfied are called (generalized) Morse sequences.

If x is a Morse sequence, then we show that the spectrum of T on the subspace of functions invariant under mirroring is discrete; the eigenvalue group is the group of all $n(b^0)\, n(b^1) \ldots n(b^t)$-th roots of unity, where $n(b)$ denotes the length of a block b.

We derive a necessary and sufficient condition on x for T to have continuous spectrum on the orthogonal complement of the above subspace[1]. It is somewhat surprising that T always has continuous spectrum on this subspace if $n(b^t)$ is even for an infinite number of t; if this is not the case, then the condition is, roughly speaking, that a sufficient portion of odd (or even) zeroes and ones exist in the blocks $b^0, b^1, \ldots$ and is expressed in a sum as above. In the course of investigation we prove that if (Ω_0, T_0) is any strictly ergodic system and if n is such that T_0^n is ergodic, then (Ω_0, T_0^n) is strictly ergodic.

We show that the entropy of T on $\mathcal{O}_x$ is zero if x is a Morse sequence.

In answer to a question raised by K. JACOBS as to which groups of roots of unity can possibly occur as eigenvalue groups of strictly ergodic subsystems of (Ω, T), we construct, using the above results, for any infinite group of roots of unity a continuous Morse sequence having the given group as its group of eigenvalues.

Finally, the necessary and sufficient condition for strict ergodicity of $\mathcal{O}_x$ and the fact that $\mathcal{O}_x$ is always a minimal set provide numerous examples of minimal sets which are not strictly ergodic. We give these and other examples and discuss further problems in the last paragraph.

§ 1. Preliminaries

We shall be dealing with finite or infinite sequences $b = (b_0, b_1, \ldots)$ of zeroes and ones, called finite or infinite blocks. Let B be the set of finite blocks, X the set of infinite blocks. The length of $b \in B$ is denoted by $n(b)$. Denote by Ω the set of two-sided sequences of zeroes and ones. We have

$$X = \prod_{N} \{0, 1\} = \{x = (x_0, x_1, \ldots) \mid x_i \in \{0, 1\}\ (i \in \mathbf{N})\}$$

$$\Omega = \prod_{Z} \{0, 1\} = \{\omega = (\ldots, \omega_{-1}, \omega_0, \omega_1, \ldots) \mid \omega_i \in \{0, 1\}\ (i \in \mathbf{Z})\},$$

[1] Such sequences are called continuous Morse sequences.

where $\mathbf{Z}$ denotes the integers, $\mathbf{N}$ the non-negative integers. Ω' will stand for either Ω or X and $\mathbf{Z}'$ for $\mathbf{Z}$ or $\mathbf{N}$ respectively in parallel statements about Ω and X.

Provided with the product topology, Ω' is a compact, metrisable, totally disconnected Hausdorff space, a clopen base for the topology being given by the set of all (finite-dimensional) cylinders

$$_t[b] := \{\omega' \in \Omega' \mid \omega'_t = b_0, \ldots, \omega'_{t+n} = b_n\},$$

$n \in \mathbf{N}$, $t \in \mathbf{Z}'$, $b = (b_0, b_1, \ldots, b_n) \in B$.

We define $T : \Omega' \to \Omega'$ by

$$(T\omega')_i := \omega'_{i+1} \quad (i \in \mathbf{Z}'),$$

the shift. T^{-1} carries cylinders into cylinders and thus T is continuous. On Ω, T is a homeomorphism.

The orbit $\mathcal{O}(\omega')$ of a point $\omega' \in \Omega'$ is given by

$$\mathcal{O}(\omega') := \{T^t\omega' \mid t \in \mathbf{Z}'\},$$

its orbit closure is $\overline{\mathcal{O}(\omega')}$.

We use the symbol "$\sim$" to designate mirroring, i.e. interchange of zeroes and ones, in all situations. Thus

$$\tilde{0} = 1, \quad \tilde{1} = 0,$$
$$(\tilde{\omega}')_i = 1 - \omega'_i \quad (\omega' \in \Omega',\ i \in \mathbf{Z}'),$$
$$A \subseteqq \Omega' \to \tilde{A} = \{\tilde{\omega}' \mid \omega' \in A\},$$
$$\tilde{f}(\omega') = f(\tilde{\omega}')\ (\omega' \in \Omega',\ f \text{ a function on } \Omega'), \quad \text{etc.}$$

We shall need the following definitions and theorem only for Ω' and T; for a general treatment and proofs see GOTTSCHALK-HEDLUND [2]. A subset Ω_0 of Ω' is invariant if $T\Omega_0 \subseteqq \Omega_0$.

Definition 1. A subset M of $\mathbf{Z}'$ is *dense* if there exists a $D \in \mathbf{N}$ such that $t \in \mathbf{Z}'$ implies $\{t, t+1, \ldots, t+D\} \cap M \neq \emptyset$.

Definition 2. A point ω' of Ω' is *almost periodic* if U open, $\omega' \in U$ implies that $\{t \in \mathbf{Z}' \mid T^t\omega' \in U\}$ is dense.

Definition 3. A subset Ω_0 of Ω' is *minimal* if it is non-empty, closed, invariant, and contains no proper subset with these properties.

Theorem 1 (GOTTSCHALK [1]). *$\overline{\mathcal{O}(\omega')}$ is minimal if and only if ω' is almost periodic.*

We fix now the notation used for calculating relative frequencies of occurrence of blocks in points of B, X, and Ω. If $\omega' \in B$, X, or Ω and $t \in \mathbf{Z}$, $n \in \mathbf{N}$, then we set

$$\omega'(t, n) = (\omega'_t, \omega'_{t+1}, \ldots, \omega'_{t+n})$$

whenever it is possible, and anything not occurring in this exposition otherwise. If $b \in B$ and $t \in \mathbf{Z}$, let

$$1_{t[b]}(\omega') := \begin{cases} 1 & \text{if } \quad b = \omega'(t, n(b) - 1) \\ 0 & \text{otherwise.} \end{cases}$$

If $t = 0$, then it is omitted along with the square brackets. We say that b occurs in ω' at t iff

$$1_{t[b]}(\omega') = 1 .$$

If $c \in B$, then it is obvious that the quantity

$$r_b(c) := \frac{1}{n(c)} \sum_{t=0}^{n(c)-1} 1_{t[b]}(c)$$

is the relative frequency of occurrence of b in c. In this vein, $r_0(c)$ and $r_1(c)$ denote the relative frequencies of zeroes and ones in c respectively; moreover,

$$r_0(c) + r_1(c) = 1 .$$

For a point $\omega' \in \Omega'$ we set

$$r_b(\omega') := \lim_{t\to\infty} \frac{1}{t} \sum_{j=0}^{t-1} 1_b(T^j \omega')$$

if this limit exists, and say then that $r_b(\omega')$ exists. The existence of $r_b(\omega')$ is obviously equivalent with the existence of

$$\lim_{n\to\infty} r_b(\omega'(t, n)) \tag{1}$$

for any one $t \in \mathbf{Z}'$; in this case all the limits (1) are equal. $r_b(\omega')$ is said to exist uniformly if the limits (1) are uniform in $t \in \mathbf{Z}'$. In keeping with the usual terminology we make the following definitions (for details see OXTOBY [10]).

Definition 4. $\omega' \in \Omega'$ is quasi-regular if $r_b(\omega')$ exists for each $b \in B$.

Definition 5. $\omega' \in \Omega'$ is strictly transitive if $r_b(\omega')$ exists uniformly for each $b \in B$.

Turning now to measure-theoretical properties of Ω we denote by $\mathscr{B}$ the set of Borel subsets of Ω. All measures considered will be defined on $\mathscr{B}$, normalized, and T-invariant (i.e. if m is a measure, then it is to be understood that $m(\Omega) = 1$ and $m(A) = m(TA)$ $(A \in \mathscr{B})$); for short we say simply invariant measure. For a development of the following in a more general setting and proofs see OXTOBY [10], KRYLOFF-BOGOLIOUBOFF [6].

An invariant measure is ergodic if $A \in \mathscr{B}$, $TA = A$ implies $m(A)\, m(\Omega - A) = 0$.

Definition 6. A compact invariant non-empty subset Ω_0 of Ω is

a) *uniquely ergodic* if there exists exactly one invariant measure carried by Ω_0,

b) *strictly ergodic* if Ω_0 is uniquely ergodic and minimal.

Theorem 2 (OXTOBY [10], see also KAKUTANI [4]).
Let ω be a point of Ω. Then

a) *$\overline{\mathcal{O}(\omega)}$ is uniquely ergodic if and only if ω is strictly transitive,*

b) *$\overline{\mathcal{O}(\omega)}$ is strictly ergodic if and only if ω is strictly transitive and almost periodic.*

We remark here that if ω is strictly transitive, then

$$L(1_{t[b]}) := r_b(\omega)$$

defines a positive T-invariant linear form L on the set of all (continuous) indicator functions of cylinders with $L(1) = 1$. Since the linear hull of this set is dense in the space of continuous functions on Ω, L can be extended to a positive invariant normalized functional on the continuous functions, which is nothing other than an invariant measure on $\mathscr{B}$. We denote this measure by m_ω. Thus the unique measure of a cylinder ${}_t[b]$ is given in this case by

$$m_\omega({}_t[b]) = r_b(\omega);$$

more generally, if f is a continuous function on $\overline{\mathcal{O}(\omega)}$, then

$$\int f\, dm_\omega = \lim_{t\to\infty} \frac{1}{t} \sum_{j=0}^{t-1} f(T^j\omega).$$

m_ω is obviously carried by $\overline{\mathcal{O}(\omega)}$.

It is clear that if $\overline{\mathcal{O}(\omega)}$ is uniquely ergodic, then the unique invariant measure m_ω is ergodic. Denote by $\mathscr{L}^2_\omega$ the space of complex-valued square integrable functions with respect to m_ω on $\overline{\mathcal{O}(\omega)}$, and suppose that $\overline{\mathcal{O}(\omega)}$ and m_ω are invariant under mirroring, i.e. $\widetilde{\overline{\mathcal{O}(\omega)}} = \overline{\mathcal{O}(\omega)}$, $\tilde{m}_\omega = m_\omega$. Then T and "$\sim$" induce isometries in $\mathscr{L}^2_\omega$. Set

$$\mathscr{D}_\omega := \{f \in \mathscr{L}^2_\omega \mid f = \tilde{f}\}$$
$$\mathscr{C}_\omega := \{f \in \mathscr{L}^2_\omega \mid f = -\tilde{f}\}.$$

Since T and "$\sim$" commute, $\mathscr{D}_\omega$ and $\mathscr{C}_\omega$ are T-invariant. It is easily shown that $\mathscr{D}_\omega$ and $\mathscr{C}_\omega$ are closed linear subspaces of $\mathscr{L}^2_\omega$ and that

$$\mathscr{L}^2_\omega = \mathscr{D}_\omega \oplus \mathscr{C}_\omega.$$

§ 2. Block Arithmetic and Recurrent Sequences

For $b \in B$, $c \in X \cup B$ we define

$$b + c := (b_0, b_1, \ldots, b_{n(b)-1}, c_0, c_1, \ldots)$$
$$b \times (0) := b$$
$$b \times (1) := \tilde{b}$$
$$b \times c := (b \times c_0) + (b \times c_1) + \cdots.$$

If $c \in B$, then $b + c$, $b \times c \in B$ and

$$n(b + c) = n(b) + n(c)$$
$$n(b \times c) = n(b)\, n(c).$$

Suppose now that $b^0, b^1, \ldots \in B$ and that the first member of each b^i is a zero. Since it is easily shown that the operation "$\times$" is associative,

$$x := b^0 \times b^1 \times b^2 \times \cdots$$

defines a sequence x of zeroes and ones in an obvious manner. If $n(b^i) \geqq 2$ for all i, then $x \in X$.

Definition 7. Every sequence of the form $x = b^0 \times b^1 \times b^2 \times \cdots \in X$ is called a (one-sided) *recurrent sequence*.

We now list some elementary formulas for calculating relative frequencies which can easily be checked by the reader. Suppose that $b, c, d \in B$. Then:

$$r_0(c) + r_1(c) = 1$$
$$r_d(c) = r_{\tilde{d}}(\tilde{c})$$
$$r_i(b+c) = \frac{n(b)}{n(b)+n(c)} r_i(b) + \frac{n(c)}{n(b)+n(c)} r_i(c) \quad (i = 0, 1)$$
$$r_0(b \times c) = r_0(b)\, r_0(c) + r_1(b)\, r_1(c)$$
$$r_1(b \times c) = r_0(b)\, r_1(c) + r_1(b)\, r_0(c)\,.$$

For technical purposes we define

$$s_d(b) := |r_d(b) - r_{\tilde{d}}(b)|\,.$$

Obviously $s_d(b) = s_{\tilde{d}}(b) = s_d(\tilde{b}) = s_{\tilde{d}}(\tilde{b})$, and if $d = (0)$, then $s_0(b)$ measures in a certain sense the "balance" of zeroes and ones in the block. We have

$$s_0(b+c) \leqq \max(s_0(b), s_0(c))$$
$$s_0(b \times c) = s_0(b)\, s_0(c)\,.$$

If we assume that b is long compared with d, then we obtain approximations of the above rules for arbitrary d. More precisely, suppose that $\frac{n(d)}{n(b)} < \varepsilon$. Then

$$|r_d(b \times c) - \{r_d(b)\, r_0(c) + r_{\tilde{d}}(b)\, r_1(c)\}| < \varepsilon$$
$$|r_{\tilde{d}}(b \times c) - \{r_d(b)\, r_1(c) + r_{\tilde{d}}(b)\, r_0(c)\}| < \varepsilon\,,$$

and consequently

(2) $$|s_d(b \times c) - s_d(b)\, s_0(c)| < 2\varepsilon\,.$$

We now prove some lemmas about relative frequencies in recurrent sequences and almost periodicity. In the following suppose that

$$x = b^0 \times b^1 \times b^2 \times \cdots$$

is a recurrent sequence. We set $c^t = b^0 \times b^1 \times \cdots \times b^t$ $(t \in \mathbf{N})$.

Lemma 1. *x is periodic if and only if there exists a $k \in \mathbf{N}$ such that either*

a) $b^k \times b^{k+1} \times b^{k+2} \times \cdots = (0, 0, 0, 0, \ldots)$, *or*

b) $b^k \times b^{k+1} \times b^{k+2} \times \cdots = (0, 1, 0, 1, \ldots)$.

Proof Either a) or b) is obviously sufficient for periodicity of x. Thus we assume that x is periodic with minimal period p and we choose t such that $n(c^t) \geqq p$. Select now q with $0 \leqq q < p$ and $q = n(c^t) \bmod p$. If $q = 0$, then it is clear that $b^{t+1} \times b^{t+2} \times \cdots = (0, 0, 0, 0, \ldots)$, so assume $q > 0$. Then we assert that $2q = p$. One way of seeing this is the following:

On the one hand,

$$x(n(c^t), p-1) = x(0, p-1) \quad \text{or} \quad \widetilde{x(0, p-1)}$$

because $x = c^t \times b^{t+1} \times b^{t+2} \times \cdots$;
on the other hand,

$$x(n(c^t), p-1) = (x_q, x_{q+1}, \ldots, x_{p-1}, x_0, \ldots, x_{q-1})$$

because of the p-periodicity of x. It follows that the function $n \to x_n$ defined on $\{0, 1, \dots, p-1\}$ is invariant under translation by $2q \bmod p$. Therefore, the greatest common divisor of $2q$ and p is a period of this function, and since p is the smallest period, we have $2q = p$.

It follows that $n(c^t) = \frac{p}{2} \bmod p$;

$$\widetilde{(x_0, \dots, x_{p/2-1})} = (x_{p/2}, \dots, x_{p-1}),$$

and $b^{t+1} \times b^{t+2} \times \cdots = (0, 1, 0, 1, \dots)$. ⌋

Lemma 2. *Every recurrent sequence is almost periodic.*

Proof. Let $x = b^0 \times b^1 \times b^2 \dots$ be a recurrent sequence. Since the set of cylinders ${}_0[c^t]$ $(t \in \mathbf{N})$ forms a base of neighborhoods for $x \in X$, it suffices to show that for each $t \in \mathbf{N}$, the set

$$\{k \mid T^k x \in {}_0[c^t]\}$$

is dense. Fix t and look at $b^{t+1} \times b^{t+2} \times \cdots$. If this sequence is $(0, 0, 0, 0, \dots)$, then c^t occurs quite regularly in x and the given set is dense. If not, then choose $s > t$ such that $b^{t+1} \times \cdots \times b^s$ contains both a zero and a one. Then the blocks c^t and $\widetilde{c^t}$ both occur in c^s, hence also in $\widetilde{c^s}$. But

$$x = c^s \times (b^{s+1} \times b^{s+2} \times \cdots)$$

may be considered as an infinite sum of the blocks c^s and $\widetilde{c^s}$, and each of these blocks contains the block c^t. It follows that the given set is dense. ⌋

Lemma 3. *Suppose that x is non-periodic. Then x is strictly transitive if and only if*

$$\sum_{k=0}^{\infty} \min(r_0(b^k), r_1(b^k))$$

diverges.

Proof. I. Suppose that the given sum diverges.

1. We show that $\lim_t r_0(c^t) = \frac{1}{2}$.

Obviously $\lim_t r_0(c^t) = \frac{1}{2}$ iff $\lim s_0(c^t) = 0$. But

$$\begin{aligned} s_0(c^t) &= \prod_{k=0}^{t} s_0(b^k) = \prod_{k=0}^{t} |r_0(b^k) - r_1(b^k)| = \\ &= \prod_{k=0}^{t} \{1 - 2\min(r_0(b^k), r_1(b^k))\}, \end{aligned}$$

and this product "converges" to zero if the given sum diverges, according to the well-known product convergence criterion (see e.g. Knopp [5]).

2. $\lim_t s_d(c^t) = 0$ for every $d \in B$.

To show this, let $\varepsilon > 0$ and select T with $\frac{n(d)}{n(c^T)} < \varepsilon$. We set

$$e^t := b^{T+1} \times b^{T+2} \times \cdots \times b^t \quad \text{for} \quad t > T.$$

Then

$$c^t = c^T \times e^t \qquad (t > T)$$

because of associativity and therefore

$$|s_d(c^t) - s_d(c^T)\, s_0(e^t)| < 2\,\varepsilon$$

by way of (2). Now if

$$\sum_{k=0}^{\infty} \min(r_0(b^k), r_1(b^k)) = \infty\,,$$

then

$$\sum_{k=T+1}^{\infty} \min(r_0(b^k), r_1(b^k)) = \infty\,,$$

and we may apply 1. to the sequence $b^{T+1}, b^{T+2}, \ldots$ to obtain

$$\lim_t s_0(e^t) = 0\,.$$

Therefore

$$s_d(c^t) < 3\,\varepsilon$$

for sufficiently large t, which implies

$$\lim_t s_d(c^t) = 0\,.$$

3. $\lim\limits_t r_d(c^t)$ exists for each $d \in B$.

Let $\varepsilon > 0$ and choose according to 2. T such that

$$s_d(c^T) = |r_d(c^T) - r_{\tilde d}(c^T)| < \varepsilon \tag{3}$$

and

$$\frac{n(d)}{n(c^T)} < \varepsilon\,.$$

Then

$$|r_d(c^t) - \{r_d(c^T)\, r_0(e^t) + r_{\tilde d}(c^T)\, r_1(e^t)\}| < \varepsilon$$

for $t > T$, where $e^t = b^{T+1} \times b^{T+2} \times \cdots \times b^t$. Because of (3) and $r_0(e^t) + r_1(e^t) = 1$, we conclude that

$$|r_d(c^t) - r_d(c^T)| < 2\,\varepsilon \qquad (t > T)\,,$$

and thus for t and t' larger than T, we have

$$|r_d(c^t) - r_d(c^{t'})| < 4\,\varepsilon\,.$$

Therefore $\lim\limits_t r_d(c^t)$ exists.

4. x is strictly transitive.

To show this, let d be an arbitrary finite block and $\varepsilon > 0$. According to 2. and 3., $\lim\limits_t s_d(c^t) = 0$ and $\lim\limits_t r_d(c^t) = \alpha$ for a suitable α. We may therefore choose T such that

$$|\alpha - r_d(c^T)| < \varepsilon$$
$$|\alpha - r_{\tilde d}(c^T)| < \varepsilon\,,$$

and

$$\frac{n(d)}{n(c^T)} < \varepsilon\,.$$

Next choose M such that $\frac{1}{M-2} < \varepsilon$ and set $N_0 = MT$. For any $n > N_0$ and any k, the block $x(k, n)$ can be written as a sum of blocks as follows:

$$a' + a^1 + a^2 + \cdots + a^m + a'',$$

where $m \geqq M - 2$, $n(a')$, $n(a'') < n(c^T)$, and where each a^i $(1 \leqq i \leqq m)$ is either c^T or $\widetilde{c^T}$. This follows because we may write x in the form

$$c^T \times (b^{T+1} \times b^{T+2} \times \cdots).$$

Using now the formulas at the beginning of this paragraph along with a bit of elbow grease, one calculates without difficulty that

$$|\alpha - r_d(x(k, n))| < 6\varepsilon.$$

This implies, since ε was arbitrary, that $r_d(x)$ exists uniformly for each $d \in B$, i.e. x is strictly transitive. An application of Lemma 2 produces the desired result.

II. Suppose that x is strictly transitive and set $r_0(x) = \alpha$. For $\varepsilon > 0$ there exists an M such that

$$|\alpha - r_0(x(k, n))| < \varepsilon$$

for all $n \geqq M$ and all $k \in \mathbf{N}$. Select t such that $n(c^t) \geqq M$. Then

$$|\alpha - r_0(c^t)| < \varepsilon$$

and, since x is not periodic, $b^{t+1} \times b^{t+2} \times \cdots$ contains at least one 1. It follows that $\widetilde{c^t}$ occurs in x at a certain place, thus

$$|\alpha - r_0(\widetilde{c^t})| < \varepsilon.$$

We conclude that

$$|1 - \alpha - r_0(c^t)| < \varepsilon$$

and

$$|1 - 2\alpha| < 2\varepsilon.$$

This implies that $r_0(x) = \alpha = \frac{1}{2}$ or

$$\lim_t s_0(c^t) = 0.$$

Using the reverse implication of the product convergence criterion, either the given sum diverges and we are finished, or there exists a k with $r_0(b^k) = \frac{1}{2}$. In the latter case, we simply apply the above argument to the (strictly transitive) sequence $b^{k+1} \times b^{k+2} \times \cdots$ to conclude that there exists a $k' > k$ with $r_0(b^{k'}) = \frac{1}{2}$. By induction it follows that there exist infinitely many k's with this property, i.e. the sum in question diverges. ⌋

§ 3. Morse Sequences and Strictly Ergodic Subsets of Ω

Let ω' be a point of Ω, X, or B. Then we define the set $B_{\omega'} \subseteqq B$ as the set of all finite blocks occurring in ω':

$$B_{\omega'} := \{b \in B \mid \text{there exists a } k \in \mathbf{Z} \text{ such that } \omega'(k, n(b) - 1) = b\}.$$

For each $x \in X$ define now a subset $\mathcal{O}_x$ of Ω by setting

$$\mathcal{O}_x := \{\omega \in \Omega \mid B_\omega \subseteqq B_x\}.$$

Definition 8. A one-sided non-periodic recurrent sequence $x = b^0 \times b^1 \times b^2 \times \cdots$ is called a (one-sided) *Morse sequence* if

$$\sum_{k=0}^{\infty} \min(r_0(b^k), r_1(b^k)) = \infty .$$

The points of $\mathcal{O}_x$ are then called *two-sided Morse sequences.*

In the following $x = b^0 \times b^1 \times b^2 \times \cdots$ is a recurrent sequence;

$$c^t = b^0 \times \cdots \times b^t; \; n_t = n(c^t) \qquad (t \in \mathbf{N}) .$$

Lemma 4. *There exists an* $\omega \in \mathcal{O}_x$ *with* $x = (\omega_0, \omega_1, \omega_2, \ldots)$.

Proof. Obviously we may assume that x is non-periodic. For any

$$b = (b_0, \ldots, b_n) \in B \text{ set } \hat{b} = (b_n, \ldots, b_0).$$

Define for $i \in \mathbf{N}$

$$d^i := \begin{cases} b^i & \text{if } b^i_{n(b^i)-1} = 0 \\ \tilde{b}^i & \text{if } b^i_{n(b^i)-1} = 1 . \end{cases}$$

Then $\hat{x} := \hat{d}^0 \times \hat{d}^1 \times \hat{d}^2 \times \cdots$ is well defined and belongs to X. Define now ω by:

$$(\omega_0, \omega_1, \ldots) := x$$
$$(\omega_{-1}, \omega_{-2}, \ldots) := \hat{x} .$$

To show that $\omega \in \mathcal{O}_x$ it suffices to prove that for any $t \in \mathbf{N}$ there exists a $k \in \mathbf{N}$ such that

$$x(k, 2n_t - 1) = \omega(-n_t, 2n_t - 1) .$$

But if b, $c \in B$, then $\hat{\hat{b}} = b$, $\hat{b} \times \hat{c} = \widehat{b \times c}$, and $b \times c = \widehat{\hat{b} \times \hat{c}}$, as is easy to see, and by definition of ω we have

$$\begin{aligned} \omega(-n_t, 2n_t - 1) &= \overline{(\hat{d}^0 \times \hat{d}^1 \times \cdots \times \hat{d}^t)} + c^t \\ &= (d^0 \times d^1 \times \cdots \times d^t) + c^t . \end{aligned}$$

Also if b, $c \in B$, then $b \times c = \tilde{b} \times \tilde{c} = \widetilde{\widetilde{b \times \tilde{c}}} = \widetilde{\tilde{b} \times c}$, so that

$$d^0 \times d^1 \times \cdots \times d^t = \widetilde{c^t} \quad \text{or} \quad c^t$$

since each d^i is either b^i or $\widetilde{b^i}$. It follows that

$$\omega(-n_t, 2n_t - 1) = c^t + c^t \quad \text{or} \quad \widetilde{c^t} + c^t .$$

But since x is not periodic, both the blocks (00) and (10) appear in $b^{t+1} \times b^{t+2} \times \cdots$; namely, since every sequence $b^{t'} \times b^{t'+1} \times \cdots$ contains a one, the statement "(00) appears" is equivalent with the statement "(11) appears" and likewise for (01) and (10). If only (00) and (11) appear, then each $b^{t'}$ consists only of zeros; if only (01) and (10) appear, then $b^{t+1} \times b^{t+2} \times \cdots = (0, 1, 0, 1, \ldots)$.

Therefore there exists a $k \in \mathbf{N}$ such that

$$x(k, 2n_t - 1) = \omega(-n_t, 2n_t - 1) . \qquad \lrcorner$$

Theorem 3. *Suppose that* x *is a non-periodic recurrent sequence. Then* $\mathcal{O}_x$ *is strictly ergodic if and only if* x *is a Morse sequence.*

Proof. By Lemma 4 there exists an $\omega \in \mathcal{O}_x$ with $x = (\omega_0, \omega_1, \ldots)$, which implies $B_\omega = B_x$. But by definition of the topology on Ω,

$$\mathcal{O}_\omega = \overline{\mathcal{O}(\omega)} = \{\eta \in \Omega \mid B_\eta \subseteqq B_\omega\} = \mathcal{O}_x\,,$$

so that $\mathcal{O}_x$ is the orbit closure of ω in Ω. Since x is a recurrent sequence, x is almost periodic, and the fact that x is a Morse sequence implies that x is strictly transitive (Lemmas 2 and 3). From the definitions follows easily that ω is almost periodic and strictly transitive, and Theorem 2 shows then that $\mathcal{O}_x = \mathcal{O}_\omega$ is strictly ergodic.

Conversely, if $\mathcal{O}_x$ is strictly ergodic, then x is strictly transitive (because ω is strictly transitive) and Lemma 3 shows that x is a Morse sequence. ⌋

§ 4. The Discrete Spectrum of $\mathbf{O}_x$

Let $x = b^0 \times b^1 \times \cdots$ be a fixed one-sided Morse sequence; $c^t = b^0 \times \cdots \times b^t$; $n_t = n(c^t)$ $(t \in \mathbf{N})$; denote by m_x the unique normalized T-invariant measure on $\mathcal{O}_x$. The purpose of the following two paragraphs is to investigate the spectrum of the operator induced by T on $\mathscr{L}^2(\mathcal{O}_x, m_x) = \mathscr{L}^2_x$. We denote this operator by T also. First of all it is obvious, since x is not periodic, that $\tilde{c^t}$ appears in x for each $t \in \mathbf{N}$; it follows from this that $B_x = B_{\tilde{x}}$ and thus $\mathcal{O}_x$ is invariant with respect to "$\sim$". Moreover, if $d \in B$ and $t \in \mathbf{Z}$, then the measure of the cylinder ${}_t[d]$ is given by

$$m_x({}_t[d]) = r_d(x)\,,$$

and therefore

$$m_x({}_t[\tilde{d}]) = r_{\tilde{d}}(x)\,.$$

By virtue of **I.2.** in the proof of Lemma 3, however, we have

$$r_d(x) = r_{\tilde{d}}(x)\,,$$

so that the mapping "$\sim$" preserves measures of cylinders. Since the set of cylinders is intersection-stable and generates the complete σ-field, we have $m_x = \tilde{m}_x$. The following theorem is a collection of the statements above.

Theorem 4. *Let x be a Morse sequence and $\mathcal{O}_x$ the corresponding strictly ergodic subset of Ω. Then $\mathcal{O}_x$ is mirror-invariant ("$\sim$") and mirroring preserves the unique invariant measure m_x.*

As a consequence (see the end of § 1), the space of m_x-square-integrable functions on $\mathcal{O}_x$ (which we denote by $\mathscr{L}^2_x$) is the direct sum of the two T-invariant subspaces

$$\mathscr{D}_x = \{f \in \mathscr{L}^2_x \mid f = \tilde{f}\}$$

and

$$\mathscr{C}_x = \{f \in \mathscr{L}^2_x \mid f = -\tilde{f}\}\,.$$

Since we are assuming that x is non-periodic, we may also assume (by grouping the b^i into new products in a suitable manner) that each of the four blocks of length two occur in each b^i $(i \in \mathbf{N})$. (See end of proof to Lemma 4.)

Now let t be fixed and choose an $\omega \in \mathcal{O}_x$ with $x = (\omega_0, \omega_1, \omega_2, \ldots)$. We define the sets $D_0, D_1, \ldots, D_{n_t-1}$ as follows:

$$D_k := \overline{\{T^{k+in_t}\omega \mid i \in \mathbf{N}\}} \quad (0 \leqq k < n_t)\,.$$

Lemma 5. $D_0 = \left[\bigcup_{i=0}^{n(b^{t+1})-1} T^{in_t}({}_0[c^{t+1}] \cup {}_0[\widetilde{c^{t+1}}])\right] \cap \mathcal{O}_x$.

Proof. Call the set on the right-hand side D'. Then D' is closed as a finite union of cylinders, and since

$$(\omega_0, \omega_1, \ldots) = x = c^{t+1} \times (b^{t+2} \times \cdots)$$

consists of a sum of c^t and $\widetilde{c^t}$, we have that $T^{in_t}\omega \in D'$ for all $i \in \mathbf{N}$. Thus $D_0 \subseteqq D'$. To prove $D' \subseteqq D_0$ it suffices to prove that

$$({}_0[c^{t+1}] \cup {}_0[\widetilde{c^{t+1}}]) \cap \mathcal{O}_x \subseteqq D_0,$$

since $T^{in_t} D_0 \subseteqq D_0$ $(i \in \mathbf{N})$. Suppose that $\eta \in \mathcal{O}_x \cap {}_0[c^{t+1}]$ (the case where $\eta \in \mathcal{O}_x \cap {}_0[\widetilde{c^{t+1}}]$ is handled analogously), and let $k_i \in \mathbf{N}$ be a sequence such that $T^{k_i}\omega \to \eta$. Since ${}_0[c^{t+1}]$ is open, we may assume that $T^{k_i}\omega \in {}_0[c^{t+1}]$ for all i. Now b^{t+1} contains the blocks (00) and (11) and therefore either the block (001) or the block (110). It follows that either $c^t + c^t + \widetilde{c^t}$ or $\widetilde{c^t} + \widetilde{c^t} + c^t$ occurs in c^{t+1}. But each of the blocks $c^t + c^t$, $c^t + \widetilde{c^t}$, $\widetilde{c^t} + c^t$, $\widetilde{c^t} + \widetilde{c^t}$ occur in each of the blocks $c^t + c^t + \widetilde{c^t}$, $\widetilde{c^t} + \widetilde{c^t} + c^t$ at most one time, namely either at the beginning or at the end, because no block is equal to its mirror image. Since $T^{k_i}\omega \in {}_0[c^{t+1}]$ and since $(\omega_0, \omega_1, \ldots) = c^t \times (b^{t+1} \times b^{t+2} \times \cdots)$, we conclude that $k_i \equiv 0 \bmod n_t$ for all i. This implies that $T^{k_i}\omega \in D_0$ and hence $\eta \in D_0$. ┘

Note that we have also proved that if $k_i \in \mathbf{N}$ is a sequence such that $T^{k_i}\omega$ converges to a point in D_0, then $k_i \equiv 0 \bmod n_t$ for sufficiently large i. In particular, if $T^k\omega \in D_0$, then $k \equiv 0 \bmod n_t$, a fact we shall need in the proof of the next lemma.

Lemma 6. *$\{D_0, D_1, \ldots, D_{n_t-1}\}$ is a partition of $\mathcal{O}_x$ into open and closed subsets of $\mathcal{O}_x$ such that*

$$TD_{k-1} = D_k \quad (1 \leqq k < n_t)$$
$$TD_{n_t-1} = D_0.$$

Proof. Since T is a homeomorphism and since

$$T^k D_0 = D_k \quad (1 \leqq k < n_t),$$

Lemma 5 shows indeed that the sets $D_0, \ldots, D_{n_t-1}$ are open and closed. Also

$$\bigcup_{k=0}^{n_t-1} D_k \subseteqq \mathcal{O}_x,$$

and since the union on the left is closed and contains $T^j\omega$ $(j \in \mathbf{N})$, we have

$$\bigcup_{k=0}^{n_t-1} D_k = \mathcal{O}_x.$$

Now if $D_0 \cap D_k \neq \emptyset$ for some k, then, since $D_0 \cap D_k$ is open, there exists an $i \in \mathbf{N}$ such that $T^{k+in_t}\omega \in D_0$. The remark after the proof of Lemma 5 shows that $k = 0$. Hence $D_0, D_1, \ldots, D_{n_t-1}$ are pairwise disjoint. Finally, since $TD_{n_t-1} \subseteqq D_0$ and since T is a homeomorphism, $TD_{n_t-1} = D_0$. ┘

Theorem 5. *For each $t \in \mathbf{N}$, $\zeta = \exp(2\pi i/n_t)$ is an eigenvalue of T corresponding to a continuous eigenfunction belonging to $\mathscr{D}_x$.*

Proof. Let t be fixed and construct the sets $D_0, \ldots, D_{n_t-1}$ as above. Set

$$f_t := \sum_{k=0}^{n_t-1} \zeta^k 1_{D_k}.$$

Then f_t is a continuous eigenfunction with eigenvalue ζ, and since $\mathcal{O}_x$ is mirror-invariant and $\tilde{D}_k = D_k$ (Lemma 5), we obtain $f_t \in \mathscr{D}_x$.

Theorem 6. *T has pure point spectrum on $\mathscr{D}_x$ and the eigenvalue group of T on $\mathscr{D}_x$ is given by*

$$\mathscr{G}_x = \{\exp(2\pi k i/n_t) \mid t \in \mathbf{N},\ 0 \leqq k < n_t\}.$$

Proof. Let $\mathscr{D}$ denote the closed subspace of $\mathscr{L}_x^2$ spanned by the functions f_t^k for $t \in \mathbf{N}$ and $0 \leqq k < n_t$, the functions f_t being those defined in the proof of Theorem 5. It suffices to show that $\mathscr{D}_x = \mathscr{D}$. According to Theorem 5, $\mathscr{D} \subseteqq \mathscr{D}_x$. For fixed t,

$$1_{D_0} = \frac{1}{n_t} \sum_{k=0}^{n_t-1} f_t^k,$$

so that $1_{D_k} \in \mathscr{D}$ $(0 \leqq k < n_t)$. For any block $b \in B$, we set

$$E_b := {}_0[b] \cup {}_0[\tilde{b}].$$

Since T is an isometry on the T-invariant subspace $\mathscr{D}_x$ of $\mathscr{L}_x^2$ and since the set

$$\{T^k 1_{E_b} \mid b \in B,\ k \in \mathbf{Z}\}$$

spans $\mathscr{D}_x$, it suffices to show that $1_{E_b} \in \mathscr{D}$ for $b \in B$.

Now let b be fixed and choose t such that

$$|m_x(E_b) - [r_b(c^t) + r_{\tilde{b}}(c^t)]| < \varepsilon.$$

This is possible because $\mathcal{O}_x$ is strictly ergodic and therefore

$$r_b(c^t) + r_{\tilde{b}}(c^t) \to m_x(E_b) \quad \text{as} \quad t \to \infty.$$

If b occurs in c^t or $\tilde{c}^t$ at k, then $0 \leqq k < n_t - n(b)$, and according to Lemma 5, we have $T^k D_0 = D_k \subseteqq E_b$. Let E_b' be the union of all D_k such that b occurs in c^t or $\tilde{c}^t$ at k. Because of Lemma 6, $m_x(D_k) = 1/n_t$ for $0 \leqq k < n_t$, and thus

$$m_x(E_b') = r_b(c^t) + r_{\tilde{b}}(c^t).$$

But $E_b' \subseteqq E_b$ and $1_{E'_b} \in \mathscr{D}$. It follows that

$$\int (1_{E_b} - 1_{E'_b})\, dm_x = |m_x(E_b) - [r_b(c^t) + r_{\tilde{b}}(c^t)]| < \varepsilon.$$

Since ε was arbitrary, $1_{E_b} \in \mathscr{D}$. ∎

It remains to be noted that Theorem 5 and Theorem 6 are true when we remove the restriction, made at the beginning of this paragraph, that each block of length two occurs in each b^i $(i \in \mathbf{N})$. This is true because the process of grouping the b^i does not change the group $\mathscr{G}_x$ itself, but only the generators.

§ 5. The Continuous Spectrum of O_x

As in the preceding paragraph, let $x = b^0 \times b^1 \times b^2 \times \cdots$ be a fixed Morse sequence; $c^t = b^0 \times \cdots \times b^t$ $(t \in \mathbb{N})$; $n_t = n(c^t)$; m_x the unique T-invariant probability measure on $\mathcal{O}_x$. Our goal is now to determine a necessary and sufficient condition for T to have continuous spectrum on the subspace $\mathscr{C}_x$ of $\mathscr{L}_x^2$.

First of all, we prove a general theorem on powers of strictly ergodic transformations which we shall need later; although the proof is simple, this theorem does not seem to have appeared in the literature yet.

Theorem 7. *Suppose that Ω_0 is a strictly (uniquely) ergodic set under the homeomorphism T_0; denote by m_0 the unique T_0-invariant probability measure on Ω_0. Then for any $n \geqq 2$, Ω_0 is strictly (uniquely) ergodic under T_0^n if and only if Ω_0 is ergodic under T_0^n with respect to m_0.*

Proof. Since $T_0^n m_0 = m_0$, strict (unique) ergodicity implies m_0-ergodicity. Now suppose that T_0^n is m_0-ergodic on Ω_0, and let ν be a probability measure on Ω_0 with $T_0^n \nu = \nu$ and such that T_0^n is ν-ergodic. We want to show that $\nu = m_0$. Set

$$\nu_0 := \frac{\nu + T_0 \nu + \cdots + T_0^{n-1} \nu}{n}.$$

Then $T_0 \nu_0 = \nu_0$, and since Ω_0 is strictly (uniquely) ergodic under T_0, we have $\nu_0 = m_0$. This implies that ν is absolutely continuous with respect to $m_0 = \nu_0$; since T_0^n is both ν-ergodic and m_0-ergodic, and since different ergodic measures are mutually singular, we conclude that $\nu = m_0$; i.e. Ω_0 is uniquely ergodic under T_0^n. Now suppose that Ω_0 is strictly ergodic under T_0; in the presence of unique ergodicity, strict ergodicity is equivalent with the statement that all open sets have positive measure (see e.g. OXTOBY [10]). But this remains true, no matter whether we are considering T_0 or T_0^n; thus Ω_0 is strictly ergodic under T_0^n. ⌋

We now set $\lambda_t = n(b^t)$ $(t \in \mathbb{N})$; obviously λ_t and n_t are connected by

$$n_t = \prod_{t'=0}^{t} \lambda_t \qquad (t \in \mathbb{N}).$$

Oddly enough, a necessary condition that $\mathscr{C}_x$ have some point spectrum is that almost all λ_t be odd; this is shown by the following lemma.

Lemma 7. *If $\{t \mid \lambda_t \text{ even}\}$ is infinite, then T has continuous spectrum on $\mathscr{C}_x$.*

Proof. Suppose that $f \in \mathscr{C}_x$ is an eigenfunction of T with eigenvalue θ:

$$T f = \theta f.$$

Then $f^2 \in \mathscr{D}_x$ and $T f^2 = \theta^2 f^2$ or $\theta^2 \in \mathscr{G}_x$, $\mathscr{G}_x$ being the group of eigenvalues of T on $\mathscr{D}_x$ as defined in Theorem 6. Now multiplication of f by an arbitrary eigenfunction g of $\mathscr{D}_x$ produces another eigenfunction of $\mathscr{C}_x$; choosing g suitably and replacing f by fg, we may assume that

$$\theta = \exp\left(\frac{2\pi i}{2 n_t}\right)$$

for suitable $t \in \mathbb{N}$ ($\theta = 1$ is out of question since T is ergodic on $\mathcal{O}_x$ and the constants do not belong to $\mathscr{C}_x$). But if $t' > t$ is chosen such that $\lambda_{t'}$ is even, then

$$\theta = \left[\exp\left(\frac{2\pi i}{n_{t'}}\right)\right]^{\lambda_{t+1}\lambda_{t+2}\cdots\lambda_{t'-1}(\lambda_{t'}/2)} \in \mathscr{G}_x$$

is a contradiction to the simplicity of the eigenvalue θ (which is a consequence of the ergodicity of T). ⌋

We remark that if $\mathscr{C}_x$ possesses any point spectrum, then the above proof shows that for some $t \in \mathbf{N}$,

$$\theta = \exp\left(\frac{2\pi i}{2 n_t}\right)$$

is an eigenvalue of T on $\mathscr{C}_x$, and then $\lambda_{t'}$ is odd for all $t' > t$. Let us take a look at the Morse sequence

$$y = b^{t+1} \times b^{t+2} \times \cdots$$

and the subset

$$D_0 = \overline{\{T^{i n_t} \omega \mid i \in \mathbf{N}\}}$$

of $\mathscr{O}_x$ defined in § 3, ω being a point of $\mathscr{O}_x$ with $x = (\omega_0, \omega_1, \ldots)$. The transformation T^{n_t} maps D_0 into D_0, and if f is an eigenfunction of $\mathscr{C}_x$ with $Tf = \theta f$, then the restriction f_0 of f to the subset D_0 fulfills $T^{n_t} f_0 = -f_0$. We define a mapping $\psi: \mathscr{O}_y \to D_0$ by setting, for $\eta \in \mathscr{O}_y$,

$$[\psi(\eta)](i n_t, n_t - 1) = \begin{cases} c^t & \text{if} \quad \eta_i = 0 \\ \tilde{c}^t & \text{if} \quad \eta_i = 1\,. \end{cases}$$

What this mapping does is to replace the zeroes in η by copies of c^t and the ones in η by blocks $\tilde{c}^t$.

Lemma 8. *ψ is a measure-preserving homeomorphism between the strictly ergodic set $\mathscr{O}_y$ with its unique probability measure and the set D_0 with the normalized measure given by the restriction of $n_t m_x$ to D_0; the transformation T on $\mathscr{O}_y$ is carried by ψ into the transformation T^{nt} on D_0; ψ commutes with mirroring.*

Proof. The proof is straightforward and we leave the details to the reader. ⌋

The above lemma, in addition to showing that the subset D_0 of $\mathscr{O}_x$ is strictly ergodic under T^{n_t}, has reduced our problem of finding out when T has continuous spectrum on $\mathscr{C}_x$ to the case where all $n(b^i)$ are odd; indeed, as remarked above, if f is an eigenfunction of $\mathscr{C}_x$, then f_0 is an eigenfunction of (D_0, T^{n_t}) with eigenvalue -1, ψ carries f_0 into an eigenfunction on $(\mathscr{O}_y, T)$ with eigenvalue -1, and this function belongs to $\mathscr{C}_y$ since ψ preserves mirroring, moreover all $n(b^{t'})$ are odd for $t' > t$. Conversely, given a t such that $n(b^{t'})$ is odd for $t' > t$ and an eigenfunction belonging to $\mathscr{C}_y$ with eigenvalue -1, then the process is easily reversed to construct an eigenfunction $f \in \mathscr{C}_x$ with eigenvalue θ. Therefore we restrict our attention to sequences for which all $n(b^i)$ are odd.

Lemma 9. *Let $x = b^0 \times b^1 \times b^2 \times \cdots$ be a Morse sequence, $n(b^i)$ being odd for all $i \in \mathbf{N}$. Then T has continuous spectrum on $\mathscr{C}_x$ if and only if T^2 is ergodic on $\mathscr{O}_x$ with respect to the probability measure m_x.*

Proof. According to the remarks after Lemma 7, if T has an eigenvalue on $\mathscr{C}_x$, then for some $t \in \mathbf{N}$,

$$\theta = \exp\left(\frac{2\pi i}{2 n_t}\right)$$

is an eigenvalue. Since n_t is odd, $\theta^{n_t} = -1$ is also a $\mathscr{C}_x$-eigenvalue and T^2 cannot in this case be ergodic. Suppose conversely that T^2 is not ergodic. Then there exists a T^2-invariant measurable partition $\{A, A'\}$ of $\mathcal{O}_x$ such that $m_x(A) > 0$, $m_x(A') > 0$. But $A \cap TA$ and $A' \cap TA'$ are T-invariant and T is ergodic; therefore $m_x(A \cap TA) = m_x(A' \cap TA') = 0$ and $A' = TA$. The function $f = 1_A - 1_{A'}$ is an eigenfunction with eigenvalue -1 and must belong to $\mathscr{C}_x$ because of the orthogonality of eigenfunctions and because $-1 \notin \mathscr{G}_x$ (all n_t are odd). ┘

Our next task is to determine a necessary and sufficient condition for the strict ergodicity of $\mathcal{O}_x$ under the transformation T^2, all $n(b^i)$ being odd. The method used is analogous to the one for determining the strict ergodicity of $\mathcal{O}_x$ under T, and we content ourselves to give a sketch of the proof of the following theorem, referring the reader to Lemmas 2.3 and Theorem 3.

Denote now for a block $b = (b_0, b_1, \ldots, b_{n-1}) \in B$ the relative frequencies of even zeroes and ones, odd zeroes and ones, by

$$e_0(b) := \frac{1}{n} \sum_{\substack{k=0 \\ k \text{ even}}}^{n-1} \tilde{b}_k, \quad e_1(b) := \frac{1}{n} \sum_{\substack{k=0 \\ k \text{ even}}}^{n-1} b_k,$$

$$0_0(b) := \frac{1}{n} \sum_{\substack{k=0 \\ k \text{ odd}}}^{n-1} \tilde{b}_k, \quad 0_1(b) := \frac{1}{n} \sum_{\substack{k=0 \\ k \text{ odd}}}^{n-1} b_k,$$

and set

$$w(b) := \min\{e_0(b) + 0_1(b), e_1(b) + 0_0(b)\}.$$

Theorem 8. *Let $x = b^0 \times b^1 \times b^2 \times \cdots$ be a Morse sequence, $n(b^i)$ being odd for all $i \in \mathbb{N}$. Then $\mathcal{O}_x$ is strictly ergodic under T^2 if and only if*

$$\sum_{t=0}^{\infty} w(b^t)$$

diverges.

Sketch of proof. Corresponding to the function s_0 defined in § 2 to measure the balance of zeroes and ones in blocks, we define here

$$s_0'(b) := |1 - 2w(b)|.$$

Then it is also easy to show that

$$s_0'(b \times c) = s_0'(b) s_0'(c).$$

Now suppose that $\sum w(b^t)$ diverges. Then according to the product convergence criterion,

$$s_0(b^0 \times \cdots \times b^t) = \prod_{j=0}^{t} s_0(b^j) \underset{t \to \infty}{\longrightarrow} 0,$$

which implies that

$$\lim_t [e_0(c^t) + 0_1(c^t)] = \lim_t [e_1(c^t) + 0_0(c^t)] = \frac{1}{2}.$$

Putting this together with the known convergence of

$$e_0(c^t) + 0_0(c^t) = r_0(c^t), \quad 0_0(c^t) + 0_1(c^t) = \frac{1}{2} - \frac{1}{2n_t}, \quad \text{etc.},$$

one obtains easily the convergence of each of $e_0(c^t)$, $e_1(c^t)$, $0_0(c^t)$, $0_1(c^t)$ to $\frac{1}{4}$. By an approximation argument similar to the one used in Lemma 3, it is easy to see that the relative frequencies of odd and even occurrences of any fixed block in x exist uniformly and are equal to half of the m_x-measure of the corresponding cylinder. In this manner the set $\mathcal{O}_x$ can be shown to be strictly ergodic under the transformation T^2 (actually only the ergodicity is necessary because of Theorem 7).

Conversely, if $\sum w(b^t)$ converges, then

$$\lim_t \prod_{j=0}^{t} s_0'(b^j) > 0 , \tag{4}$$

since each $s_0'(b^i) > 0$ (none of the $w(b^i)$ are equal to $\frac{1}{2}$ because the $n(b^i)$ are all odd). However, if $\mathcal{O}_x$ were strictly ergodic under T^2, then $e_0(c^t)$, $e_1(c^t)$, $0_0(c^t)$, $0_1(c^t)$ would all have to converge to $\frac{1}{4}$, and this contradicts (4). ┘

We collect the results of this paragraph in the following theorem; note that Theorem 7 is needed to put Lemma 9 and Theorem 8 together. Also, according to the remarks after Lemma 8, the condition in Theorem 8 for T to have continuous spectrum on $\mathscr{C}_x$ is correct if finitely many $n(b^t)$ are even.

Theorem 9. *Let $x = b^0 \times b^1 \times b^2 \times \cdots$ be a Morse sequence and set $\lambda_t = n(b^t)$ $(t \in \mathbb{N})$. T possesses continuous spectrum on $\mathscr{C}_x$ if and only if either*

1) *there exist infinitely many even λ_t, or*

2) $\sum_{t=0}^{\infty} w(b^t)$ *diverges.*

A Morse sequence for which one of these conditions is satisfied is called *continuous*.

§ 6. Entropy

In this paragraph we prove that if x is a Morse sequence, then the entropy of T on the strictly ergodic set $\mathcal{O}_x$ is zero. Let x be a Morse sequence and denote by B_x^n the set of all blocks of length n which occur in x.

Lemma 10. *For each t, the number of blocks in $B_x^{n_t}$ is less than $4n_t$.*

Proof. Since x is an infinite sum of the blocks c^t and $\tilde{c}^t$ (of length n_t), any block of length n_t occurring in x must occur in one of the four blocks $c^t + c^t$, $c^t + \tilde{c}^t$, $\tilde{c}^t + c^t$, $\tilde{c}^t + c^t$; the maximal number of blocks of length n_t in these four blocks is less than $4n_t$.

Theorem 10. *If x is a Morse sequence, then T has entropy zero on the strictly ergodic set $\mathcal{O}_x$.*

Proof. The partition of Ω into the cylinders $_0[0]$ and $_0[1]$ generates the σ-field $\mathscr{B}$ of Ω. Thus the entropy of T on $\mathcal{O}_x$ is given by the mean entropy of this partition. But according to Lemma 10, the number of cylinders of length n_t with positive m_x-measure is less than $4n_t$, since a cylinder must have m_x-measure zero if its corresponding block does not occur at all in x. We conclude that the mean entropy of the said partition is less than

$$\frac{\log 4n_t}{n_t}$$

for each t, and this expression converges to zero as t goes to infinity. ┘

§ 7. Examples and Problems

1. Our first example is the one which led us to consider more general sequences, the original Morse sequence. Set $b^0 = b^1 = \cdots = (01)$; then $r_0(b^t) = \frac{1}{2}$, $n(b^t) = 2$ $(t \in \mathrm{N})$, and $x = b^0 \times b^1 \times \cdots$ is in our terminology a continuous Morse sequence, its eigenvalue group being the group of all 2^k-th roots of unity. These results were announced without proof by KAKUTANI in [4]; it was also stated in [4] that the spectrum of T on $\mathscr{C}_x$ for this sequence is singular with respect to Lebesgue spectrum, and other sequences were given for which these results are also true. In our terminology, those sequences may be constructed by varying the blocks b^i, setting them either equal to (00) or (01) according to another fixed sequence of zeroes and ones, which could be given for instance by the binary representation of a number from the unit interval.

2. Let $\mathscr{G}$ be an arbitrary infinite subgroup of the group of all (complex) roots of unity. Then there exists a strictly ergodic subset of Ω with partly continuous spectrum whose eigenvalue group is exactly $\mathscr{G}$. To see this, choose a sequence $2 \leqq n_0 < n_1 < n_2 < \cdots$ of integers such that n_t divides n_{t+1} $(t \in \mathrm{N})$ and such that $\mathscr{G}$ is generated by $\{\exp(2\pi i/n_t) \mid t \in \mathrm{N}\}$ (note that $\mathscr{G}$ is countable). Then set

$$\lambda_0 := n_0\,, \quad \lambda_t := \frac{n_t}{n_{t-1}}\,(t > 0)\,,$$

and choose blocks $b^0, b^1, \ldots$ with $n(b^t) = \lambda_t$ $(t \in \mathrm{N})$ and such that $x = b^0 \times b^1 \times b^2 \times \cdots$ is a continuous Morse sequence. For instance, it would suffice to set the first $[\lambda_t/2]$ components of b^t equal to zero and the rest equal to one. Then Theorem 3 shows that $\mathscr{O}_x$ is strictly ergodic, Theorem 9 says that T has continuous spectrum on $\mathscr{C}_x$, and Theorem 6 provides $\mathscr{G}_x = \mathscr{G}$.

3. Set $b^0 = b^1 = \cdots = (001)$; then $r_0(b^t) = \frac{2}{3}$, $w(b^t) = \frac{1}{3}$, and thus the conditions are fulfilled for $x = b^0 \times b^1 \times \cdots$ to be a continuous Morse sequence. The eigenvalue group $\mathscr{G}_x$ of x is the group of all 3^k-th roots of unity.

4. Suppose now that $b^0, b^1, \ldots$ are chosen such that

$$\sum_{k=0}^{\infty} \min(r_0(b^k), r_1(b^k)) < \infty\,.$$

This is easily attained; for instance set $n(b^k) = 2^{k+1}$ and $b^k = (00\ldots001)$. The above sum then reduces to

$$\sum_{k=0}^{\infty} \frac{1}{2^{k+1}} = 1\,.$$

It follows that $x = b^0 \times b^1 \times \cdots$ is not a Morse sequence. Lemma 2 implies that $\mathscr{O}_x$ is a minimal set, but Theorem 3 shows that $\mathscr{O}_x$ is not strictly ergodic. In this manner we can construct a multitude of minimal sets which are not strictly ergodic. The first example of a minimal but not strictly ergodic set was given by A. MARKOV (see [9], [10]); it cannot be constructed in the above manner.

5. For some continuous Morse sequences $x = b^0 \times b^1 \times \cdots$ we have been able to show the singularity of the spectrum of T on $\mathscr{C}_x$, e.g. 1. and 3. above. The proofs unfortunately make use of recursion formulas for relative frequencies derivable because of $b^0 = b^1 = \cdots$; the idea for this method was divulged to the author by KAKUTANI. It would be interesting to determine whether the spectrum of T on $\mathscr{C}_x$ is singular for every continuous Morse sequence.

6. The problem of isomorphy between strictly ergodic systems of the type discussed here is connected with the problem in 5. as well as with the problem of representation of a given (continuous) Morse sequence $x = b^0 \times b^1 \times \cdots$ as a product of other blocks than the b^i. How can an isomorphy statement of the type given in [4] be generalized and proved?

7. Set $b^0 = b^1 = \cdots = (010)$; then $x = b^0 \times b^1 \times \cdots = (0, 1, 0, 1, \ldots)$ is a periodic sequence of period 2. This example induced us to prove Lemma 1, which says roughly that this is the "only type" of periodicity which can occur if the b^i are not finally all zeroes. Looking at 2. above and trying to find a strictly ergodic system with a finite eigenvalue group, we have only been able to produce examples of a periodic nature. Is it possible to construct a strictly ergodic system with a finite eigenvalue group? with continuous spectrum? What is the eigenvalue group of the strictly ergodic system given in [3]? Is there a connection between this problem and entropy?

8. One can consider other spaces than Ω; the idea of considering a state space of n elements (instead of 2) is inviting, and produces more assymmetry in that there are more permutations of the state space than in the case of two elements, where the only permutations are the identity and mirroring.

9. It is practically trivial that the strictly ergodic sets given here do not exhaust the strictly ergodic subsets of Ω — it is only a beginning on the classification of these subsets.

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