Z. Wahrscheinlichkeitstheorie verw. Gebiete
65, 563–572 (1984)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Approximating Ito Integrals of Differential Forms and Geodesic Deviation

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Summary. Suppose X is a semimartingale on a differential manifold M with a linear connection Γ. The main purpose of this paper is to show that the „Ito integral" (with respect to Γ) of a differential form along the path of X is the limit in probability of certain Riemann sums, constructed in a natural way using the exponential map in differential geometry. For this, we study the deviation between the stochastic development of X in the tangent space at some point, and the image of X under the inverse of the exponential map at the point.

Approximating Ito Integrals of Differential Forms: Motivation

The stochastic integral of a continuous real-valued process K with respect to a continuous semimartingale X can be obtained as the limit in probability of Riemann sums of the form $\sum_j K(t_j)(X(t_{j+1})-X(t_j))$, as is well known. Suppose now that X is a continuous semimartingale on a manifold M with a linear connection Γ, and η is a first-order differential form on M. A natural way to construct a stochastic integral of η along the path of X is as follows. Let v be a bounded stopping-time and $(0=v(0)<v(1)<\ldots<v(q)=v)$ a partition of the interval $[0, v]$ by increasing stopping-times, chosen according to certain technical criteria. Write X_j instead of $X(v(j))$. Let γ_j be the geodesic on M (assumed unique) with $\gamma_j(0)=X_j$, $\gamma_j(1)=X_{j+1}$. The derivative of γ_j at zero is an element of the tangent space at X_j, and is usually denoted

$$\exp_{X_j}^{-1}(X_{j+1}).$$

We approximate the Ito integral of η along X from 0 to v, with respect to the connection Γ, by the Riemann sum

$$\sum_j \eta(X_j)(\exp_{X_j}^{-1}(X_{j+1})).$$

Our purpose is to prove that as the mesh of the partition tends to zero, such Riemann sums converge in probability to the Ito integral previously defined by Bismut [1] and Meyer [9], p. 59.

§ 1. Basic Stochastic Notations

Let us fix a complete right-continuous probabilized stochastic basis $(\Omega, (\mathscr{F}_t)_{t \geqq 0}, P)$. We follow the notation and terminology of Metivier and Pellaumail [8] for stochastic integration theory, except that

$$\int_0^t Y_s \circ dZ_s$$

will denote a Stratonovich integral. We would also like to mention:

1. The tensor quadratic variation of a Hilbert-valued semimartingale is defined in [8, § 3.6]. For a continuous $\mathbf{E}$- valued semimartingale X (here $\mathbf{E} = \mathbf{R}^n$) the tensor quadratic variation process $\int (dX \otimes dX)$ may be characterized as the $(\mathbf{E} \otimes \mathbf{E})$-valued process such that: if b and c are in $\mathbf{E}^*$, and $b \otimes c$ is their tensor product regarded as an element of $(\mathbf{E} \otimes \mathbf{E})^*$, then

$$\int_0^t b \otimes c (dX \otimes dX)_s = \langle b(X), c(X) \rangle_t$$

where the expression on the right is the angle-brackets process of the real-valued continuous semimartingales $b(X)$ and $c(X)$. When H is an $(\mathbf{E} \otimes \mathbf{E})^*$-valued process, it is more convenient in some cases to write $H_s(dX_s, dX_s)$ instead of $H_s(dX \otimes dX)_s$.

2. Suppose that M is a smooth manifold modelled on $\mathbf{E}$. *All processes on M will have continuous trajectories.* A process X on M will be called a *semimartingale* if for all f in $C^2(M)$, the image process $f \circ X$ is a real-valued (continuous) semimartingale. The definition is from L. Schwartz [10].

3. Suppose X is a semimartingale on M, and let (W, φ) be a chart for M. Suppose J and H are locally bounded previsible processes taking values in $L(\mathbf{E}: \mathbf{G})$ and $L(\mathbf{E}, \mathbf{E}; \mathbf{G})$ respectively, for some Euclidean space G, such that up to modification

$$J = J 1_{\{X \in W\}}, \qquad H = H 1_{\{X \in W\}}$$

meaning that J_t and H_t are almost surely zero when X_t is not in W. Then the stochastic integrals

$$\int J d(\varphi X), \qquad \int H(d(\varphi X) \otimes d(\varphi X))$$

are well-defined. We shorten these expressions to

$$\int J dX^\varphi, \qquad \int H(dX \otimes dX)^\varphi.$$

If $(X) = (X^1, \ldots, X^n)$ with respect to a basis for $\mathbf{E}$, Meyer [9] would write:

$$\int J^i_j dX^i, \qquad \int H^i_{jk} d\langle X^j, X^k \rangle.$$

§ 2. Stochastic Integrals with Moving Frames

A *linear frame* u at x in M means a linear isomorphism from $\mathbf{E}$ to $T_x M$. We shall deal frequently in the sequel with processes U such that U_t is a linear frame at X_t for all t, where X is a semimartingale on M. We would like to explain the appropriate notation in great detail so that probabilists will not be confused.

Let $\mathbf{G}$ be a Euclidean space and let η be a $\mathbf{G}$-valued 1-form on M. Then for all t,

$$\eta(X_t) \circ U_t \quad \text{('}\circ\text{' denotes composition)}$$

is an $L(\mathbf{E}; \mathbf{G})$-valued random variable, since U_t maps linearly from $\mathbf{E}$ to $T_{X_t} M$ and $\eta(X_t)$ maps linearly from $T_{X_t} M$ to $\mathbf{G}$. Hence it makes sense to say that

$$\eta(X) \circ U \quad \text{is an } L(\mathbf{E}; \mathbf{G})\text{-valued process.}$$

Consequently if Z is an $\mathbf{E}$-valued process, one can speak of the $\mathbf{G}$-valued process

$$L = \int (\eta(X) \circ U)\, dZ. \tag{1}$$

If $(e_1, \ldots, e_n)$ is a basis for $\mathbf{E}$, and $Z = (Z^1, \ldots, Z^n)$ with respect to this basis, we could write out (1) in full as

$$L_t = \int_0^t (\eta(X_s) \circ U_s(e_i))\, dZ_s^i. \tag{1'}$$

Suppose now that M has a connection Γ, yielding a covariant derivative ∇; thus $\nabla\eta(x) \in L(T_x M, T_x M \colon \mathbf{G})$. Consequently for all x in M and all linear frames u at x, we have a bilinear map

$$\nabla\eta(x)(u(\,.\,), u(\,.\,)) \in L(\mathbf{E}, \mathbf{E}; \mathbf{G})$$

therefore we can speak of the stochastic integral

$$J = \int \nabla\eta(X)(U(\,.\,), U(\,.\,))(dZ \otimes dZ)$$

which is commonly abbreviated to

$$J = \int \nabla\eta(X)(U\, dZ \otimes U\, dZ). \tag{2}$$

Corresponding to (1′) we have

$$J_t = \int_0^t \nabla\eta(X_s)(U_s(e_i), U_s(e_j))\, d\langle Z^i, Z^j \rangle_s. \tag{2'}$$

§ 3. Constructions and Formulae from Stochastic Differential Geometry

A good reference for this section is Meyer [9], but the notation is more in the spirit of Darling [2].

1. Let M be a smooth n-dimensional manifold; for brevity, we shall often denote the model space by $\mathbf{E}$ instead of $\mathbf{R}^n$. Let η be a first-order differential

form ('1-form') on M. If (W, φ) is a chart for M, then η has the local representation $(\varphi(x), a(x)) \in \mathbf{E} \times \mathbf{E}^*$ at each x in W, where $\eta = \varphi^* a$ on W (meaning that $\eta(x)(v) = a(X)(T_x \varphi(v))$ for v in $T_x M$). On the random set $\{X \in W\}$, the following differential makes sense and is intrinsic (for the notation, see § 1):

$$a(X_t)\, dX_t + \tfrac{1}{2} Da(X_t)(dX \otimes dX)_t.$$

Consequently there is a unique real-valued semimartingale Y with $Y_0 = 0$ such that dY_t equals the last expression on $\{X \in W\}$, for each chart (W, φ). The process Y is called the *Stratonovitch integral* of the 1-form η along the semimartingale X, and we usually write

$$Y = (S) \int_X \eta, \quad Y_t = (S) \int_{X_0^t} \eta \qquad \text{(Meyer would omit the symbol } (S).)$$

2. Continue the notations of the previous paragraph, and let Γ be a smooth linear connection for M. Γ is specified in the chart (W, φ) by the local connector $\Gamma(\cdot)\colon W \to L(\mathbf{E}, \mathbf{E}; \mathbf{E})$ defined as follows: if vector fields Y and Z have local representations $(\varphi(x), y(x)) = T_x \varphi(Y(x))$ and $(\varphi(x), z(x))$ respectively at x in W, then the covariant derivative of η is given by $\nabla\eta(x)(Y, Z) = Da(x)(y(x), z(x)) - a(x)(\Gamma(x)(y(x), z(x)))$. (In relation to the Christoffel symbols and a basis $e_1, \ldots, e_n$ for $\mathbf{E}$, the k^{th} component of $\Gamma(x)(e_i, e_j)$ is $\Gamma_{ij}^k(x)$.)

On the random set $\{X \in W\}$, the following differential makes sense and is intrinsic (for the notation, see § 1)

$$a(X_t)(dX_t + \tfrac{1}{2} \Gamma(X_t)(dX \otimes dX)_t).$$

Consequently there is a unique real-valued semimartingale N with $N_0 = 0$ such that dN_t equals the last expression on $\{X \in W\}$, for each chart (W, φ). The process N is called the *Ito integral* of the 1-form η along the semimartingale X, with respect to the connection Γ, and we usually write

$$N = (\Gamma) \int_X \eta, \qquad N_t = (\Gamma) \int_{X_0^t} \eta. \tag{1}$$

3. Let $p\colon L(M) \to M$ denote the linear frame bundle of M. Hence for each x in M and each u in $p^{-1}(x)$, u is a linear isomorphism from $\mathbf{E}$ into $T_x M$. Let ω be a connection 1-form on $L(M)$. Various authors have shown (Meyer [9, p. 80], Darling [2, pp. 30–34], Shigekawa [11]) that given a semimartingale on M and an initial frame U_0 in $p^{-1}(X_0)$, there is a unique semimartingale U on $L(M)$, called the *horizontal lift of X to $L(M)$ through ω*, satisfying the equations:

$$(S) \int_U \omega = 0, \qquad p(U_t) = X_t. \tag{2}$$

For a given connection 1-form ω and a given initial frame U_0, the *stochastic development of X into E* is the $\mathbf{E}$-valued semimartingale Z defined by:

$$Z = (S) \int_U \theta \tag{3}$$

where θ is the canonical 1-form on $L(M)$, namely the **E**-valued 1-form defined by:

$$\theta(u)(\xi)=u^{-1}(T_u p(\xi)), \qquad u\in P,\ \xi\in T_u P. \tag{4}$$

The usefulness of the horizontal lift and the stochastic development is that they allow formulas related to the process X on M to be written down in 'absolute' terms, without reference to any system of co-ordinates on M. For example, let $f\colon M\to\mathbf{R}$ be a smooth function. Then $df(X_s)\in T^*_{X_s}M$. Let $(e_1,\dots,e_n)$ be an orthonormal basis for **E**, and write $Z_t=(Z_t^1,\dots,Z_t^n)$ with respect to this basis. Then $U_s(e_i)$ is a tangent vector at X_s for each i, and so $df(X_s)(U_s(e_i))$ is real-valued. Likewise $\nabla df(X_s)(U_s(e_j),U_s(e_j))$ is real-valued, where ∇ is the covariant derivative induced by ω. Versions of the following 'Ito formula' have been given by many authors (e.g., Meyer [9], Bismut [1]) but we give the version appearing in Darling [2, p. 24];

$$f(X_t)-f(X_0)=\int_0^t (df(X_s)\circ U_s(e_i))\,dZ_s^i+\tfrac{1}{2}\int_0^t \nabla df(X_s)(U_s(e_i),U_s(e_j))\,d\langle Z^i,Z^j\rangle_s. \tag{5}$$

In fact the use of a basis for **E** is not necessary, and we may abbreviate to:

$$f(X_t)-f(X_0)=\int_0^t (df(X_s)\circ U_s)\,dZ_s+\tfrac{1}{2}\int_0^t \nabla df(X_s)(U\,dZ\otimes U\,dZ)_s. \tag{6}$$

A still more general formula is needed:

Lemma 1. (An extended Ito formula.) *Suppose G is a Euclidean space, and u and v are bounded stopping-times such that $u\leq v$. Let $F\colon\Omega\times M\to\mathbf{G}$ be a map such that:*

(a) *for all x in M, $F(\cdot,x)$ is an $\mathcal{F}_u$-measurable random variable with values in* **G**.

(b) *for all ω in Ω, $F(\omega,\cdot)$ is a C^2 map from M to* **G**. *Then*

$$F(X_v)-F(X_u)=\int_u^v \{(dF(X)\circ U)\,dZ+\tfrac{1}{2}\nabla dF(X)(U\,dZ\otimes U\,dZ)\}.$$

Proof. When $M=\mathbf{E}$, the assertion is a special case of a theorem of Kunita [6, p. 119]. The geometric constructions which are used to prove formula (6) apply equally to the case where the deterministic function f is replaced by the random function F, so the result follows. □

4. Another useful application of the ideas mentioned above is to give a co-ordinate-free expression for the Ito integral defined in paragraph 2 above. Continuing the previous notations, it is not difficult to prove from formula (5) that

$$(\Gamma)\int_{X_0^t}\eta=\int_0^t (\eta(X_s)\circ U_s)\,dZ_s. \tag{7}$$

A detailed derivation is found in Darling [2, p. 21].

§ 4. The Exponential Map in Differential Geometry

Let M be a smooth manifold, modelled on $\mathbf{E}$, and denote its tangent bundle by $\tau: TM \to M$. Let Γ be a smooth connection for M, and let $\exp_x$ denote the resulting exponential map at $x \in M$. Let Δ denote the diagonal $\{(x,x): x \in M\}$, a closed submanifold of $M \times M$. Define $\zeta: \Delta \to TM$ by $\zeta(x,x) = 0_x$, the zero vector in $T_x M$. From the tubular neighbourhood theorem (see Lang [7, IV § 5]) it follows that there exists an open neighbourhood Q of $\zeta(\Delta)$ in TM, and an open neighbourhood V_0 of Δ in $M \times M$, such that the map $f: Q \to V_0$ given by $v \to (\tau(v), \exp_{\tau(v)}(v))$ is a diffeomorphism. When (x, x') belongs to $V_0, f^{-1}(x,x')$ will be written as $\exp_x^{-1}(x')$, which is a tangent vector at x. Finally take V to be an open neighbourhood of Δ in $M \times M$ such that $\bar{V} \subset V_0$.

Let (W, φ) be a chart at some x in M. Let W' be the set of x' in W such that (x, x') belongs to the set V defined above. Let u be a linear frame at x, that is, a linear isomorphism from $\mathbf{E}$ to $T_x M$. Define a map $\psi: \varphi(W') \to \mathbf{E}$ by

$$\psi = (\varphi \circ \exp_x \circ u)^{-1}.$$

Since the connection Γ is smooth, ψ is a smooth map and $D\psi$ and $D^2\psi$ are well-defined on the open set $\varphi(W')$. The following elementary results will be needed later:

Lemma 2. (*Derivatives of the exponential map.*)

(i) $D\psi^{-1}(0) = T_x \varphi \circ u$ *or in intrinsic notation*, $d(u^{-1} \circ \exp_x^{-1})(u(e)) = e, e \in \mathbf{E}$.

(ii) $D^2\psi(\varphi x)(e,e') - D\psi(\varphi x)(\Gamma(x)(e,e')) = 0, e, e' \in \mathbf{E}$ *or in intrinsic notation*, $\nabla d(u^{-1} \circ \exp_x^{-1})(x) = 0$.

Proof. Equation (i) is Corollary 3.1 in Eliasson [3, p. 180]. As for (ii), let $\Gamma(\cdot)$ denote the local connector associated with the chart (W', θ) at x, where θ is the map $(\exp_x \circ u)^{-1}$. The usual transformation formula for local connectors (see for example Eliasson [3, p. 172]) reads

$$D\psi(\varphi x')(\Gamma(x')(e,e')) = D^2\psi(\varphi x')(e,e') + \Gamma(x')(D\psi(\varphi x')(e), D\psi(\varphi x')(e'))$$

for all x' in W'. Put $x' = x$. By the well-known property of normal co-ordinates, $\Gamma(x) = 0$. Equation (ii) follows. □

§ 5. Geodesic Deviation

Let M and Γ and V be as in § 4, and let X be a semimartingale on M. Let U be a horizontal lift of X to $L(M)$ through Γ, and let Z be the corresponding stochastic development of X into $\mathbf{E}$, as in § 3.

Define a map $H: \Omega \times R^+ \to R^+$, depending on Γ and X, as follows:

$$H(\cdot, s) = \sup\{t \geqq s; (X_s, X_u) \in V \text{ for all } s \leqq u \leqq t\}. \tag{1}$$

If u and u' are finite-valued stopping-times such that $u \leqq u'$, we shall say that u and u' *satisfy the exponential map condition for* X if

$$u(\omega) \leqq u'(\omega) \leqq H(\omega, u(\omega)) \quad \text{a.s.}$$

This means that

$$\exp_{X_u}^{-1}(X_t) \quad \text{is well-defined,} \quad u \leqq t \leqq u'.$$

For the remainder of this section, u and u' are assumed to satisfy this condition.

Definition. The geodesic deviation of X from time u to time u', using the horizontal lift U, is the $\mathbf{E}$-valued random variable

$$G(u,u') = Z_{u'} - Z_u - (U_u^{-1} \circ \exp_{X_u}^{-1})(X_{u'}).$$

The main aim of this section is to express $G(u,u')$ as a stochastic integral with respect to Z and the tensor quadratic variation of Z.

Lemma 3. (*Geodesic deviation formula*)

$$G(u,u') = \int_u^{u'} \{[I - d\theta_u(X_t) \circ U_t]\, dZ_t - \tfrac{1}{2} \nabla d\theta_u(X_t)(U\,dZ \otimes U\,dZ)_t\} \tag{2}$$

where $\theta_u = (U_u^{-1} \circ \exp_{X_u}^{-1})$. *If* $(e_1, \ldots, e_n)$ *is a basis for* $\mathbf{E}$, *and* $Z_t = (Z_t^1, \ldots, Z_t^n)$ *with respect to this basis, we could also write*

$$\begin{aligned} G(u,u') = \int_u^{u'} \{ & dZ_t - d\theta_u(X_t)(U_t(e_p))\, dZ_t^p \\ & - \tfrac{1}{2} \nabla d\theta_u(X_t)(U_t(e_p), U_t(e_q))\, d\langle Z^p, Z^q \rangle_t\} \end{aligned} \tag{2'}$$

If we abbreviate (2) *to*

$$G(u,u') = \int_u^{u'} \{R(u,t)\, dZ_t + Q(u,t)(dZ \otimes dZ)_t\} \tag{3}$$

then $R(u,t)$ *and* $Q(u,t)$ *are continuous in* t *on the stochastic interval* $[u,u']$, *and*

$$R(u,u) = 0, \qquad Q(u,u) = 0. \tag{4}$$

Proof. Regard the stopping-time u as fixed. There exists a map $F\colon \Omega \times M \to \mathbf{E}$, satisfying the conditions of Lemma 1, such that for each ω, $F(\omega,x) = \theta_{u(\omega)}(\omega,x)$ for all x in a suitable neighbourhood $W(\omega)$ of $X_{u(\omega)}(\omega)$. The fact that $F(\omega,\cdot)$ is C^2 follows from the fact that the map $(x,x') \to \exp_x^{-1}(x')$ is C^2 on the domain V. Formula (2) is now immediate from the definition (1), and Lemma 1.

The continuity in t of $R(u,t)$ and $Q(u,t)$ follows from the continuity (for each ω) of $d\theta_u$ and $\nabla d\theta_u$, and the fact that X is a continuous process. Assertion (4) follows from Lemma 2. □

§6. Approximation Theorem for Ito Integrals of 1-Forms

The data consists of a semimartingale X on M and a smooth linear connection Γ for M, with covariant derivative ∇ and exponential map 'exp'.

We choose a bounded stopping-time v, and for each natural number n we assume that we are given an increasing sequence of bounded stopping-times $(v(n,m): m-0,1,2,\ldots)$ satisfying conditions (*1), (*2), (*3) for X, namely:

*1. The mesh tends to zero almost surely, i.e.,

$$\lim_{n\to\infty} \sup_m (v(n,m+1)-v(n,m))=0 \qquad \text{a.s.}$$

*2. $\lim_{m\to\infty} P(v(n,m)<v)=0$ for each n

*3. $(X^t_{n,n}, X^t_{n,m+1})\in V$ for all n,m and all t, where V is the subset of $M\times M$ defined in §4., and $X^t_{n,m}$ is short for $X(v(n,m)\wedge t)$. In other words, $\exp^{-1}_{X^t_{n,m}}(X^t_{n,m+1})$ is a well-defined random vector in the tangent space at $X^t_{n,m}$ for all n,m and t.

When these three conditions hold, we shall say that the stopping-times satisfy condition (*) for X.

Theorem A. (Approximation in probability.) *Let M be a smooth manifold with a linear connection Γ. Let X be a semimartingale on M, and let v be a bounded stopping-time. Suppose that $(v(n,m))$ is a family of stopping-times satisfying condition (*) (above) for X. Then for all 1-forms η on M,*

$$(\Gamma)\int_{X^v_0}\eta=\lim_{n\to\infty}\operatorname{prob}\sum_m \eta(X^v_{n,m})(\exp^{-1}_{X^v_{n,m}}(X^v_{n,m+1})) \tag{1}$$

where $X^v_{n,m}=X(v(n,m)\wedge v)$.

Remarks. 1. By assumption *2. on the $(v(n,m))$, $P(\lim_{m\to\infty} v(n,m)<v)=0$, for every n, so the sum on the right side of (1) has only finitely many terms almost surely.

2. For a Riemannian manifold (M,g), an L^2 convergence result similar to (1) can be proved when η has compact support. Since the assumptions on the semimartingale X are a little cumbersome, we omit the details.

§8. Proofs

Step 1. We introduce the following notations:

$$\begin{aligned} T_n(v)&=(\Gamma)\int_{X^v_0}\eta-\sum_m \eta(X^v_{n,m})(\exp^{-1}_{X^v_{n,m}}(X^v_{n,m+1}))\\ G_{n,m+1}&=G(v(n,m)\wedge v,\, v(n,m+1)\wedge v)\end{aligned} \tag{1}$$

where the last expression is the geodesic deviation with respect to a chosen horizontal lift U

$$\begin{aligned} F_{n,m}&=]v(n,m)\wedge v,\, v(n,m+1)\wedge v]\\ a_t&=\eta(X_t)\circ U_t,\; a_{n,m}=a_{v(n,m)\wedge v}.\end{aligned} \tag{2}$$

Notice that the process (a_t) takes values in $\mathbf{E}^*$. Formula (7) of § 3 says that

$$(\Gamma)\int_{X_0^v}\eta=\int_0^v a_t\,dZ_t.$$

The definition of geodesic deviation in § 5, (1) shows that

$$\eta(X^v_{n,m})(\exp^{-1}_{X^v_{n,m}}(X^v_{n,m+1}))=a_{n,m}(Z_{n,m+1}-Z_{n,m})-a_{n,m}(G_{n,m+1})$$

where $Z_{n,m}=Z_{v(n,m)\wedge v}$. Consequently (1) gives

$$T_n(v)=\int_0^v J_n(t)\,dZ_t+\sum_m a_{n,m}(G_{n,m+1})$$

where

$$J_n(t)=\sum_m 1_{F_{n,m}}(t)\,(a_t-a_{n,m}). \tag{3}$$

The geodesic deviation formula, § 5 (2), shows that

$$\sum_m a_{n,m}(G_{n,m+1})=\int_0^v\{R_n(t)\,dZ_t-Q_n(t)\,(dZ\otimes dZ)_t\}$$

where

$$R_n(t)=\sum_m 1_{F_{n,m}}(t)\,a_{n,m}(I-d\theta_{n,m}(X_t)\circ U_t) \tag{4}$$

$$Q_n(t)=\tfrac{1}{2}\sum_m 1_{F_{n,m}}(t)\,a_{n,m}(\nabla d\theta_{n,m}(X_t)(U_t(\cdot),U_t(\cdot))). \tag{5}$$

$$\theta_t=(U_t^{-1}\circ\exp^{-1}_{X_t})\quad\text{and}\quad\theta_{n,m}=\theta_{v(n,m)\wedge v}.$$

We arrive at the formula:

$$T_n(v)=\int_0^v\{(J_n+R_n)\,dZ-Q_n(dZ\otimes dZ)\}. \tag{6}$$

Step 2. We need to construct four new sequences of stopping-times. First, let $(u_1(k), k=1,2,\ldots)$ be a localizing sequence of stopping-times (see [8, § 2.9]) for both the semimartingale Z (the stochastic development of X into $\mathbf{E}$, explained in § 3, (3)) and its tensor quadratic variation $\int dZ\otimes dZ$. Next, recall the definition of $H(\cdot,s)$ in § 5, (1), and set

$$u_2(k)=\inf\{s\colon \|I-d\theta_s(X_t)\circ U_t\|>k \text{ for some } t \text{ in } [s, H(s,\cdot)\wedge(s+1)]\}$$

$$u_3(k)=\inf\{s\colon \|\nabla d\theta_s(X_t)(U_t(\cdot),U_t(\cdot))\|>k \text{ for some } t \text{ in } [s, H(s,\cdot)\wedge(s+1)]\}.$$

Finally, let

$$u_4(k)=\inf\{t\colon |a_t|>k\}.$$

Now let

$$u(k)=\min(u_1(k), u_2(k), u_3(k), u_4(k))$$

which is also a localizing sequence for Z and its tensor quadratic variation process.

Observe that each of the sequences (J_n), (R_n) and (Q_n) are uniformly bounded on each $[0, u(k)]$. The almost sure continuity of (a_t) ensures that $J_n(t)$ tends to zero almost surely for all t. Part (4) of Lemma 3 shows that $R_n(t)$ and $Q_n(t)$ tend to zero almost surely for all t. We now apply Metivier and Pellaumail's stochastic dominated convergence theorem [8, § 2.11] to the right side of (6) to deduce that $T_n(v)$ converges to zero in probability as n tends to infinity, as desired. □

Acknowledgements. The author thanks his Ph.D. supervisor K.D. Elworthy for pressing him to study this problem and for his careful reading of the typescript. This research was supported by a Research Studentship from the Science and Engineering Research Council of Great Britain, and by the F.S. Dynamische Systeme, Universitaét Bremen.

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Received December 1982; in revised form October 20, 1983