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An Almost Sure Invariance Principle for Triangular Arrays of Banach Space Valued Random Variables

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Summary. We give a simpler proof of the probability invariance principle for triangular arrays of independent identically distributed random variables with values in a separable Banach space, recently proved by de Acosta [1], and improve this result to an almost sure invariance principle.

1. Introduction

The behavior of triangular arrays of row-wise independent identically distributed random variables differs markedly from that of sequences of independent identically distributed random variables. The following theorem shows that triangular arrays can be approximated with probability 1, a property not shared by sequences of random variables. (See Remark 5 below.)

Let B be a separable Banach space and let $\mathscr{P}(B)$ be the class of all probability measures on B. We denote the n-th convolution power of $\mu \in \mathscr{P}(B)$ by μ^n and the p-th root of an infinitely divisible measure $\mu \in \mathscr{P}(B)$ by $\mu^{1/p}$. The purpose of this note is to give a much simpler proof of the following improvement of a recent theorem of de Acosta [1] who proved the same result but with almost sure convergence replaced by convergence in probability.

Theorem. *Let $\mu_n \in \mathscr{P}(B)$ be such that $\mu_n^{k_n}$ converges weakly to μ. Here k_n is a sequence of integers tending to infinity. As is well-known μ is an infinitely divisible law. Then there exists a probability space and two row-wise independent triangular arrays of B-valued random variables $\{x_{nj}, 1 \leqq j \leqq k_n\}$ and $\{y_{nj}, 1 \leqq j \leqq k_n\}$ with partial sums $S_{nk} = \sum_{j \leqq k} x_{nj}$ and $T_{nk} = \sum_{j \leqq k} y_{nj}$ such that*

$$\mathscr{L}(x_{nj}) = \mu_n, \quad \mathscr{L}(y_{nj}) = \mu^{1/k_n}, \quad (1 \leqq j \leqq k_n) \tag{1}$$

and

$$\max_{k \leqq k_n} \|S_{nk} - T_{nk}\| \to 0 \quad \text{a.s.} \tag{2}$$

2. Some Lemmas

Let $D_B[0,1]$ be the set of mappings $f\colon [0,1]\to B$ which are right-continuous and have left limits. Let $\xi\in D_B[0,1]$ with probability 1 be a process with independent increments and let

$$\Delta^P(c,\delta)=\sup\min(P\{\|\xi(t)-\xi(t_1)\|>\delta\};\ P\{\|\xi(t_2)-\xi(t)\|>\delta\})$$
$$\Delta(c)=\sup\min(\|\xi(t)-\xi(t_1)\|;\ \|\xi(t_2)-\xi(t)\|)$$

the suprema being extended over all (t,t_1,t_2) with $0\leqq t\leqq 1$ and

$$t-c\leqq t_1<t<t_2\leqq t+c.$$

Lemma 1 (Skorohod [12]). *Let $0<c\leqq 1$ be such that $\Delta^P(c,\delta/20)\leqq\frac{1}{4}$. Then for any positive integer $l\geqq 3/c$*

$$P\{\Delta(1/l)>\delta\}\leqq 10^3\Delta^P(3/l,\delta/12)/c.$$

Lemma 2. *Let ρ denote the Prohorov distance and δ_0 the point mass at 0. Then*

$$\lim_{c\to 0}\limsup_{n\to\infty}\sup_{k\leqq ck_n}\rho(\mu_n^k,\delta_0)=0.$$

Proof. If the lemma were not true we could find a sequence $\{j_n,\ n\geqq 1\}$ of integers such that $j_n/k_n\to 0$, but $\mu_n^{j_n}\not\to\delta_0$. Let $\alpha_n=\mu_n^{j_n}$ and $\beta_n=\mu_n^{k_n-j_n}$. Then $\alpha_n*\beta_n\to\mu$ and hence by Theorem 2.2 of Parthasarathy [7] $\{\alpha_n\}$ is shift-compact. For $f\in B^*$ define $\varphi_n(f)=\mu_n(f)=\int\exp(ix)\ \mu_n\circ f^{-1}(dx)$. These are characteristic functions of real-valued random variables and thus by standard arguments there exists a neighborhood of the zero functional on which $(\varphi_n(f))^{j_n}\to 1$ uniformly. From this and the shift-compactness of $\{\alpha_n\}$ we conclude arguing as in the proof of Theorem 4.5 of Parthasarathy [7] that $\alpha_n\to\delta_0$, a contradiction.

Let $\{x_{nj},\ 1\leqq j\leqq k_n\}$ be a row-wise independent triangular array satisfying (1). Set $\xi(t)=\xi_n(t)=\sum_{i\leqq tk_n}x_{ni}$ and let $\Delta^P(c,\delta)=\Delta^P(c,\delta,n)$ and $\Delta(c)=\Delta(c,n)$ be defined as above.

Corollary 1. *Let $\varepsilon>0$. Then*

$$\lim_{c\to 0}\limsup_{n\to\infty}\Delta^P(c,\varepsilon,n)=0.$$

Lemma 3. *Let $S,S_1,S_2,\ldots$ be a sequence of Polish spaces and let H_n be distributions on $S\times S_n$, $n=1,2,\ldots$ such that the first marginals of H_n are all the same. Then there exists a sequence of random variables $X,X_1,X_2,\ldots$ such that $\mathscr{L}((X,X_n))=H_n$, $n=1,2,\ldots$.*

Proof. By [3], Lemma A1, p. 53 there exists a law Φ_2 defined on $S\times S_1\times S_2$ with (two-dimensional) marginals H_j on $S\times S_j$, $j=1,2$. Suppose that we already have constructed a consistent family of laws Φ_k, $2\leqq k\leqq m$ defined on $S\times S_1\times\ldots\times S_k$ with the property that the marginals of Φ_k on $S\times S_j$ are H_j, $1\leqq j\leqq k$. Applying [3], Lemma A1 to Φ_m and H_{m+1} we obtain a law Φ_{m+1} whose marginals on $S\times S_j$ are H_j, $1\leqq j\leqq m+1$. We thus have constructed inductively a consistent system $\{\Phi_n, n \geqq 1\}$ of laws Φ_n on $S \times S_1 \times \ldots \times S_n$ with the property that the marginals of Φ_n on $S \times S_j$ are H_j, $1 \leqq j \leqq n$. We apply Kolmogorov's theorem and obtain the result.

Lemma 4. *For each* $\varepsilon > 0$

$$d(\varepsilon) = \sup_{n \geqq 1} k_n \mu_n(x: \|x\| > \varepsilon) < \infty.$$

This lemma is well-known. It follows for instance from the necessary conditions for the central limit theorem in Banach space. (See Theorem 5.9 on p. 129 of [2]: We thank A. de Acosta for this remark.)

Lemma 5. *Let* $\{z_j, 1 \leqq j \leqq n\}$ *be a finite sequence of independent identically distributed random variables with sum* $S = \sum_{j \leqq n} z_j$. *Assume that the distribution function of* $\|z_1\|$ *is continuous. Then* L, *defined by* $\|z_L\| = \max_{1 \leqq j \leqq n} \|z_j\|$, *is with probability one, a well-defined random variable that is independent of* S *and has uniform distribution on* $\{1, 2, \ldots, n\}$.

Proof. Let A be a Borel set. Then for each $1 \leqq m \leqq n$

$$P(S \in A) = \sum_{j \leqq n} P(S \in A, L = j) = n P(S \in A, L = m)$$

$$= P(S \in A, L = m)/P(L = m).$$

3. Proof of Theorem

For the proof of the theorem it is enough to show the ostensibly weaker statement: Given $\varepsilon > 0$ there exist two triangular arrays of row-wise independent random variables x_{nj} and y_{nj} satisfying (1) such that

$$\limsup_{n \to \infty} P\{\max_{k \leqq k_n} \|S_{nk} - T_{nk}\| > \varepsilon\} < \varepsilon. \tag{3}$$

Indeed, for each $m \geqq 1$ we then can find two triangular arrays $\{x_{nj}^{(m)}, j \leqq k_n\}$ and $\{y_{nj}^{(m)}, j \leqq k_n\}$ such that

$$P\{\max_{k \leqq k_n} \|S_{kn}^{(m)} - T_{kn}^{(m)}\| > 1/m\} < 1/m$$

for all $n \geqq n_m$. We assume that for different m's the triangular arrays $\{(x_{nj}^{(m)}, y_{nj}^{(m)}), 1 \leqq j \leqq k_n\}$ are independent. Then the arrays $\{x_{nj}, j \leqq k_n\}$ and $\{y_{nj}, j \leqq k_n\}$ defined by

$$x_{nj} = x_{nj}^{(m)}, \quad y_{nj} = y_{nj}^{(m)} \quad \text{if } n_m \leqq n < n_{m+1}$$

obviously satisfy (1) and

$$\max_{k \leqq k_n} \|S_{nk} - T_{nk}\| \to 0 \quad \text{in probability,} \tag{4}$$

the result proved by de Acosta [1].

We now deduce (2) from (4) using a well-known theorem of Skorohod [11]. (See also Dudley [4], Theorem 19.1 for a more recent presentation.) We define two sequences $\{X_n, n \geqq 1\}$ and $\{Y_n, n \geqq 1\}$ of stochastic processes with values in $C_B[0,1]$ by setting

$$X_n(t) = S_{nk}, \quad Y_n(t) = T_{nk} \quad \text{if } t = k/k_n,\ 0 \leqq k \leqq k_n$$
$$= \text{linear in between.} \tag{5}$$

Put

$$Z_n = X_n - Y_n, \quad n \geqq 1.$$

By (5) $Z_n \in C_B[0,1]$ and

$$Z_n \to 0 \quad \text{in distribution.}$$

Hence by Skorohod's theorem there exists a sequence $Z' = \{Z'_n, n \geqq 1\}$ of random variables with values in $C_B[0,1]$ such that

$$Z'_n \to 0 \text{ a.s.} \quad \text{and} \quad \mathscr{L}(Z'_n) = \mathscr{L}(Z_n), \quad n \geqq 1. \tag{6}$$

We apply [3], Lemma A1 to $F = F_n = \mathscr{L}((X_n, Y_n), Z_n)$ and $G = G_n = \mathscr{L}(Z'_n, Z')$. Then F is defined on $\bigtimes_{i=1}^{3} C_B[0,1]$ and G on $C_B[0,1] \times \bigtimes_{i=1}^{\infty} C_B[0,1]$. For each $n \geqq 1$ we obtain a probability space and random variables (X_n^*, Y_n^*), Z_n^*, Z^* such that $\mathscr{L}(X_n^*, Y_n^*, Z_n^*) = F_n$ and $\mathscr{L}(Z_n^*, Z^*) = G_n$, $n \geqq 1$. Consequently

$$Z_n^* = X_n^* - Y_n^* \quad \text{and} \quad Z^* = \{Z_n^*, n \geqq 1\} \quad \text{a.s.} \tag{7}$$

We need to have these random variables defined on the same probability soace without changing their joint law. This is achieved by an application of Lemma 3 with $S = \bigtimes_{i=1}^{\infty} C_B[0,1]$, the range space of Z^*, $S_n = \bigtimes_{i=1}^{3} C_B[0,1]$, the range space of (X_n^*, Y_n^*, Z_n^*) and $H_n = \mathscr{L}(Z^*, (X_n^*, Y_n^*, Z_n^*))$, $n \geqq 1$. We then obtain random variables Z^{**}, $(X_n^{**}, Y_n^{**}, Z_n^{**})$ such that $\mathscr{L}(Z^{**}, (X_n^{**}, Y_n^{**}, Z_n^{**})) = H_n$, $n \geqq 1$. Thus by (7)

$$Z_n^{**} = X_n^{**} - Y_n^{**}, \quad \mathscr{L}(X_n^{**}) = \mathscr{L}(X_n), \quad \mathscr{L}(Y_n^{**}) = \mathscr{L}(Y_n), \quad n \geqq 1$$

and by (6)

$$Z_n^{**} \to 0 \quad \text{a.s.}$$

In view of (5) this proves (2).

We now finish the proof of the theorem by establishing (3). First we observe that we can assume without loss of generality that both $\mu_n(x\colon \|x\| \leqq t)$ and $\mu(x\colon \|x\| \leqq t)$ are continuous functions in t. To see this we consider the separable Banach space $B \times \mathbb{R}$ with norm $\|\cdot\| + |\cdot|$ where $(\mathbb{R}, |\cdot|)$ denotes the real line. On this space we define laws $Q_n = \mu_n \times \Phi^{1/k_n}$ and $Q = \mu \times \Phi$ where Φ is the standard normal law. Now both functions

$$Q_n\{(u, v) \in B \times \mathbb{R}\colon \|u\| + |v| \leqq t\} \quad \text{and} \quad Q\{(u, v) \in B \times \mathbb{R}\colon \|u\| + |v| \leqq t\}$$

are continuous in t. Thus if we can prove (3) under this extra continuity assumption we obtain two row-wise independent triangular arrays

$$\{(x_{ni}, s_{ni}),\ 1 \leqq i \leqq k_n\} \quad \text{and} \quad \{(y_{ni}, t_{ni}),\ 1 \leqq i \leqq k_n\}$$

with common laws Q_n and Q respectively and satisfying the properly modified relation (3). We drop the real components and obtain (3) in the desired form.

By Lemmas 1, 4 and Corollary 1 there exists $r = r(\varepsilon) \geqq 1$ such that $(d(\varepsilon))^2\, 2^{-r+1} < \varepsilon$ and

$$P\{\Delta(2^{-r}, n) > \varepsilon\} \leqq \varepsilon, \qquad n \geqq n_0(\varepsilon). \tag{8}$$

We now define blocks H_{nj} in the following way

$$H_{nj} = \{i\colon\ j\,2^{-r} < i/k_n \leqq (j+1)\,2^{-r}\}, \qquad 0 \leqq j < 2^r \tag{9}$$

and

$$t_{nj} = \min H_{nj}, \qquad p_{nj} = \operatorname{card} H_{nj} = t_{nj+1} - t_{nj} \sim 2^{-r} k_n. \tag{10}$$

Then arguing as in the proof of Lemma 1 of Kuelbs [6] we conclude that $\mu_n^{p_{nj}} \to \mu^{2^{-r}}$ for all $0 \leqq j < 2^r$. Hence there exists n_1 such that for all $n \geqq n_1$

$$\rho(\mu_n^{p_{nj}}, \mu^{2^{-r}}) < \varepsilon\, 2^{-r-1}, \qquad 0 \leqq j < 2^r. \tag{11}$$

By (9) and (10) $|p_{nj} - k_n 2^{-r}| \leqq 1$ and hence by Lemma 2 there exists n_2 such that for all $n \geqq n_2$

$$\rho(\mu^{p_{nj}/k_n}, \mu^{2^{-r}}) < \varepsilon\, 2^{-r-1}, \qquad 0 \leqq j < 2^r. \tag{12}$$

Relations (11) and (12) imply that for all $n \geqq \max(n_1, n_2)$

$$\rho(\mu_n^{p_{nj}}, \mu^{p_{nj}/k_n}) < \varepsilon\, 2^{-r}, \qquad 0 \leqq j < 2^r. \tag{13}$$

Applying Theorem 3 of [8] for each $n \geqq 1$ we obtain two finite sequences $\{X_{nj}, 0 \leqq j < 2^r\}$ and $\{Y_{nj}, 0 \leqq j < 2^r\}$ with the following properties: Both $\{X_{nj}, 0 \leqq j < 2^r\}$ and $\{Y_{nj}, 0 \leqq j < 2^r\}$ are sequences of independent random variables with $\mathscr{L}(X_{nj}) = \mu_n^{p_{nj}}$ and $\mathscr{L}(Y_{nj}) = \mu^{p_{nj}/k_n}$, and if $n \geqq \max(n_1, n_2)$

$$P\{\|X_{nj} - Y_{nj}\| > \varepsilon\, 2^{-r}\} < \varepsilon\, 2^{-r}, \qquad 0 \leqq j < 2^r. \tag{14}$$

After possibly enlarging the probability space again, we can define a sequence of independent random variables $\{L_{nj}, 0 \leqq j < 2^r\}$ having the following additional properties:

i) $\{L_{nj}, 0 \leqq j < 2^r\}$ is independent of $\{X_{nj}, 0 \leqq j < 2^r\}$ and $\{Y_{nj}, 0 \leqq j < 2^r\}$

ii) L_{nj} is uniformly distributed over H_{nj}, $0 \leqq j < 2^r$.

Now for each $n \geqq 1$ let $\{x_{ni}, 1 \leqq i \leqq k_n\}$ be a sequence of independent random variables satisfying (1). Put $X_{nj}^* = \sum_{i \in H_{nj}} x_{ni}$ and let L_{nj}^* be the location of the largest $\|x_{ni}\|$, $i \in H_{nj}$. It follows easily from Lemma 5 and independence that $(\{X_{nj}^*, 0 \leqq j < 2^r\}, \{L_{nj}^*, 0 \leqq j < 2^r\})$ has the same joint distribution as $(\{X_{nj}, 0 \leqq j < 2^r\}, \{L_{nj}, 0 \leqq j < 2^r\})$. We apply Lemma A1 in [3, p. 53] to the joint law F of the sequences $\{x_{ni}, i \leqq k_n\}$ and $\{(X_{nj}^*, L_{nj}^*), 0 \leqq j < 2^r\}$ and the joint law G of the sequences $\{(X_{nj}, L_{nj}), 0 \leqq j < 2^r\}$ and $\{(Y_{nj}, L_{nj}), 0 \leqq j < 2^r\}$ and the spaces S_1

$=B^{k_n}$ and $S_2=S_3=(B\times\mathbb{R})^{2^r}$. We then obtain a law Q on $S_1\times S_2\times S_3$ with marginals F on $S_1\times S_2$ and G on $S_2\times S_3$ respectively. We realize Q on some probability space Ω_n. Hence keeping the same notation we can set $X^*_{nj}=X_{nj}$ and $L^*_{nj}=L_{nj}$.

(For the reader unfamiliar with this kind of argument we shall explain the last step in some detail. Let $Z\colon \Omega_n\to S_1\times S_2\times S_3=B^{k_n}\times(B\times\mathbb{R})^{2^r}\times(B\times\mathbb{R})^{2^r}$ be a random variable with law Q. Denote the one dimensional projections of Z onto B and $\mathbb{R}$ by x'_{ni} $(1\leqq i\leqq k_n)$; (X'_{nj}, L'_{nj}), $(0\leqq j<2^r)$; (Y'_{nj}, L''_{nj}), $(0\leqq j<2^r)$ in the obvious order. Since the projection of Z onto $B^{k_n}\times(B\times\mathbb{R})^{2^r}$ has distribution F we know that $X'_{nj}=\sum_{i\in H_{nj}} x'_{ni}$ with probability 1 and that L'_{nj} is the location of $\max\|x'_{ni}\|$ in H_{nj} with probability 1. Since the projection of Z onto $(B\times\mathbb{R})^{2^r}\times(B\times\mathbb{R})^{2^r}$ has distribution G we conclude that $L'_{nj}=L''_{nj}$ and that $\{(X'_{nj}, L'_{nj}, Y'_{nj}),\ 0\leqq j<2^r\}$ has the same (joint) law as $\{(X_{nj}, L_{nj}, Y_{nj}),\ 0\leqq j<2^r\}$. We are only interested in the properties of the joint distribution. So we may replace $\{(X_{nj}, L_{nj}, Y_{nj}),\ 0\leqq j<2^r\}$ by $\{(X'_{nj}, L'_{nj}, Y'_{nj}),\ 0\leqq j<2^r\}$. For notational convenience we drop the primes' and obtain random variables x_{ni}, X_{nj}, L_{nj}, Y_{nj} satisfying (14), (i), (ii) and $X_{nj}=\sum_{i\in H_{nj}} x_{ni}$ and $L_{nj}=$location of $\max\|x_{ni}\|$ in H_{nj} with probability 1.)

Likewise we obtain a sequence $\{y_{ni},\ 1\leqq i\leqq k_n\}$ of independent random variables with analogous properties.

(Again we shall explain this in more detail: Let $\{y_{ni},\ i\leqq k_n\}$ be a sequence of independent identically distributed random variables satisfying (1). Put $Y^*_{nj}=\sum_{i\in H_{nj}} y_{ni}$ and let L^{**}_{nj} be the location of the largest $\|y_{ni}\|$, $i\in H_{nj}$. By Lemma 5 and independence $(\{Y^*_{nj},\ 0\leqq j<2^r\}, \{L^{**}_{nj},\ 0\leqq j<2^r\})$ has the same joint distribution as $(\{Y_{nj},\ 0\leqq j<2^r\}, \{L_{nj},\ 0\leqq j<2^r\})$. We apply Lemma A1 in [3, p. 53] again but this time to the joint law F' of the sequences $\{x_{ni},\ i\leqq k_n\}$, $\{(X_{nj}, L_{nj}),\ 0\leqq j<2^r\}$, $\{(Y_{nj}, L_{nj}),\ 0\leqq j<2^r\}$ and the joint law G' of the sequences $\{(Y^*_{nj}, L^{**}_{nj}),\ 0\leqq j<2^r\}$, $\{y_{ni},\ i\leqq k_n\}$ and the spaces $S'_1=B^{k_n}\times(B\times\mathbb{R})^{2^r}$, $S'_2=(B\times\mathbb{R})^{2^r}$ and $S'_3=B^{k_n}$. We obtain a joint law Q' with marginals F' and G'. Again we realize Q' on some probability space Ω'_n. Hence keeping the same notation we can set $Y^*_{nj}=Y_{nj}$ and $L^{**}_{nj}=L_{nj}$. This shows in particular that the locations of the maxima of $\|x_{ni}\|$, $i\in H_{nj}$ and of $\|y_{ni}\|$, $i\in H_{nj}$ are the same.) To have all random variables defined on the same probability space we redefine the sequences $\{x_{ni},\ i\leqq k_n\}$ and $\{y_{ni},\ i\leqq k_n\}$ on $\bigtimes_{n=1}^{\infty}\Omega_n$ without changing their joint law $\mathscr{L}(\{x_{ni},\ i\leqq k_n\}, \{y_{ni},\ i\leqq k_n\})$ on Ω_n.

In summary, we have constructed two triangular arrays

$$\{x_{ni},\ 1\leqq i\leqq k_n,\ n\geqq 1\} \quad\text{and}\quad \{y_{ni},\ 1\leqq i\leqq k_n,\ n\geqq 1\}$$

of row-wise independent identically distributed random variables and a triangular array $\{L_{nj},\ 0\leqq j<2^r,\ n\geqq 1\}$ with the following properties:

$$\mathscr{L}(x_{ni})=\mu_n, \qquad \mathscr{L}(y_{ni})=\mu^{1/k_n}$$

$$X_{nj}=\sum_{i\in H_{nj}} x_{ni}, \qquad Y_{nj}=\sum_{i\in H_{nj}} y_{ni},\ 0\leqq j<2^r \tag{15}$$

and (14) holds for all $n \geqq \max(n_1, n_2)$. Moreover, L_{nj} is with probability one the location of $\max_{i \in H_{nj}} \|x_{ni}\|$ and $\max_{i \in H_{nj}} \|y_{ni}\|$. Alternative proofs can be based on Lemma 2.11 of Dudley and Philipp [5] which shows that we can keep all along the random variables we started with or on a generalized Vorobev theorem (R. Shortt [10, Theorem 2.6]) amounting to a two fold application of [3, Lemma A1] or a combination of these three.

Write

$$S(m) = \sum_{i \leqq m} x_{ni}, \qquad T(m) = \sum_{i \leqq m} y_{ni}.$$

From (8), (10) and (14) we conclude that for $n \geqq \max(n_0, n_1, n_2)$

$$\max_{j < 2^r, t_{nj} < m \leqq t_{n,j+1}} \min(\|S(m) - S(t_{nj})\|, \|S(m) - S(t_{n,j+1})\|) < \varepsilon \tag{16}$$

$$\max_{j < 2^r, t_{nj} < m \leqq t_{n,j+1}} \min(\|T(m) - T(t_{nj})\|, \|T(m) - T(t_{n,j+1})\|) < \varepsilon \tag{17}$$

and

$$\sum_{k < 2^r} \|X_{nk} - Y_{nk}\| < \varepsilon, \qquad 0 \leqq j < 2^r \tag{18}$$

except on a set E of probability $< 3\varepsilon$.

Now by Lemma 4 and (8)

$$\sum_{0 \leqq j \leqq 2^r} \sum_{i,k \in H_{nj}, i \neq k} P(\min(\|x_{ni}\|, \|x_{nk}\|) > 2\varepsilon) \leqq (d(\varepsilon))^2 \cdot 2^r \cdot (p_{nj}/k_n)^2 \leqq (d(\varepsilon))^2\, 2^{-r+1} < \varepsilon.$$

Thus, we can discard the set $F(x)$ on which, for some j, at least two of $\|x_{ni}\|$ within the same block H_{nj} exceed 2ε. Likewise we can discard $F(y)$ defined similarly and thus we can discard the set $F = F(x) \cup F(y)$.

Let $\omega \in E^c \cap F^c$ and $m \leqq k_n$ be given. Choose k such that $t_{nk} < m \leqq t_{n,k+1}$. We have to show that

$$\|S(m) - T(m)\| < 8\varepsilon. \tag{19}$$

Suppose first that $\|X_{nk}(\omega)\| \leqq 5\varepsilon$, where X_{nk} is defined in (15). Recall that both (16) and (17) hold on E^c. Hence $\|S(m) - S(t_{nk})\| < \|X_{nk}\| + \varepsilon \leqq 6\varepsilon$. If $\|T(m) - T(t_{nk})\| < \varepsilon$ then by (18)

$$\|S(m) - T(m)\| \leqq \sum_{j \leqq k} \|X_{nj} - Y_{nj}\| + \|S(m) - S(t_{nk})\| + \|T(m) - T(t_{nk})\| < 8\varepsilon.$$

If $\|T(m) - T(t_{n,k+1})\| < \varepsilon$ then (19) follows similarly. Hence it remains to prove (19) under the assumption

$$\|X_{nk}\| > 5\varepsilon. \tag{20}$$

Now in view of (20) and (16) within each block H_{nj} exactly one $\|x_{ni}\|$ exceeds 2ε and this happens for $i = L_{nj}$; the same is true for $\|y_{ni}\|$ at $i = L_{nj}$. Hence on $E^c \cap F^c$ and for all $0 \leqq j < 2^r$

$$\|S(t_{nj}) - S(t_{nj} + h)\| < \varepsilon \qquad \text{if } 1 \leqq h < L_{nj}$$

and

$$\|S(t_{n,j+1}) - S(t_{nj} + h)\| < \varepsilon \qquad \text{if } L_{nj} \leqq h < p_{nj}.$$

Analogous inequalities hold for $T(t_{nj}+h)$. Consequently, and by (18) we have for each m with $1 \leqq m \leqq k_n$ (and suitably chosen j)

$$\|S(m)-T(m)\| = \|S(t_{nj}+h)-T(t_{nj}+h)\| < 3\varepsilon.$$

We conclude with a few remarks.

Remark 1. If μ is Gaussian the proof of the theorem can be simplified considerably because in essence the sample paths of Brownian motion are continuous. Since the standard maximal inequalities suffice there is no need to pair off the largest jumps of the x- and y-processes.

Remark 2. The proof of Lemma 2 is adapted from Kuelbs [6]. It shows that his conditions (3.1) and (3.2) actually imply his condition (3.3), thus rendering it redundant.

Remark 3. Let $\{z_j,\ j \geqq 1\}$ be a sequence of independent identically distributed random variables in the domain of partial attraction to an infinitely divisible law μ. Then by definition there exist sequences $k_n \to \infty$ and $a(n)$ such that

$$a(n)^{-1} \sum_{j \leqq k_n} z_j \to \mu \qquad \text{in distribution.}$$

Applying the theorem to the probability measures $\mu_n = \mathscr{L}(a(n)^{-1} z_j)$ $1 \leqq j \leqq k_n$, $n \geqq 1$ we obtain two row-wise independent triangular arrays $\{x_{nj},\ 1 \leqq j \leqq k_n,\ n \geqq 1\}$ and $\{y_{nj},\ 1 \leqq j \leqq k_n,\ n \geqq 1\}$ with $\mathscr{L}(x_{nj}) = \mu_n$, $\mathscr{L}(y_{nj}) = \mu^{1/k_n}$, $1 \leqq j \leqq k_n$, $n \geqq 1$ such that

$$\max_{k \leqq k_n} \|\sum_{j \leqq k} (x_{nj}-y_{nj})\| \to 0 \quad \text{a.s.} \tag{21}$$

Remark 4. On the other hand we were unable to construct two single sequences $\{x_j,\ j \geqq 1\}$ and $\{y_j,\ j \geqq 1\}$ of independent identically distributed random variables such that their properly normalized partial sums would satisfy (21). This is perhaps not surprising since in Theorem 1 of [9] it is in general impossible to obtain almost sure convergence to zero. (See the discussion of relations (1.2) and (1.3) in [9], p. 69.)

Remark 5. The theorem cannot be generalized to triangular arrays of independent but not necessarily identically distributed random variables: Let $\{u_j,\ j \geqq 1\}$ and $\{v_j,\ j \geqq 1\}$ be two sequences of independent identically distributed random variables having $N(0,1)$ and Cauchy distribution respectively. Suppose that the two sequences are independent. We define a triangular array by setting

$$\begin{aligned} x_{nj} &= n^{-\frac{1}{2}} u_j, &\quad 1 \leqq j \leqq n \\ &= n^{-1} v_{j-n}, &\quad n < j \leqq 2n. \end{aligned}$$

Then $\sum_{j \leqq 2n} x_{nj}$ converges in distribution to $u_1 + v_1$. But it is obviously impossible to approximate the partial sum process $\sum_{j \leqq k} x_{nj}$ by the partial sum process of a row-wise independent triangular array of independent identically distributed random variables.

Remark 6. In his paper [1], de Acosta acknowledges that the Proposition in the Addendum to his paper is due to us. Our proof of this fact which is identical with the proof given in the original preprint of our paper is very similar to de Acosta's proof.

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