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On the Completion of Probabilistic Metric Spaces*

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It is well-known that every metric space has a completion which is unique up to isometry. The proof of this theorem depends strongly on the completeness of the metric space of non-negative real numbers, the uniform continuity of the distance function, and the continuity of the binary operation, addition. In the preceding paper B. SCHWEIZER has shown that under suitable conditions the distance function in a probabilistic metric space $(S, \mathscr{F})$ is a uniformly continuous function from $S \times S$ into Δ, the Lévy space of distribution functions. This latter space is a complete metric space. Thus it is to be expected that under hypotheses akin to those of the aforementioned continuity theorem, every PM space will have a completion which will again be unique up to isometry. The purpose of this paper is to show that this is indeed the case.

In addition to the definitions, theorems and notations given in the preceding paper [2], the following will also be needed.

Definition 1. Let $(S, \mathscr{F})$ be a PM space. Then

(a) A sequence of points $\{p_n\}$ in S is a *Cauchy sequence* if $F_{p_n p_m} \to H$ (pointwise) as $n, m \to \infty$. Here H is the distribution function defined by

$$H(x) = \begin{cases} 0, & x \leqq 0, \\ 1, & x > 0. \end{cases}$$

(b) The space $(S, \mathscr{F})$ is *complete* if every Cauchy sequence in S is convergent.

(c) The PM spaces $(S, \mathscr{F})$ and $(S', \mathscr{F}')$ are *isometric* if there is a one-to-one mapping φ of S onto S' such that for any $p, q \in S$,

$$\mathscr{F}(p,q) = \mathscr{F}'(\varphi(p), \varphi(q)).$$

As usual the mapping φ will be called an *isometry*.

(d) A complete PM space $(S^*, \mathscr{F}^*)$ is a *completion* of $(S, \mathscr{F})$ if $(S, \mathscr{F})$ is isometric to a dense subset of $(S^*, \mathscr{F}^*)$.

Definition 2. Let $(S, \mathscr{F}, T)$ be a Menger space. The Menger space $(S^*, \mathscr{F}^*, T^*)$ is a completion of $(S, \mathscr{F}, T)$ if $(S^*, \mathscr{F}^*)$ is a completion of $(S, \mathscr{F})$ and $T = T^*$.

Theorem. *Every Menger space with a continuous t-norm has a completion which is unique up to isometry.*

Proof. Let $(S, \mathscr{F}, T)$ be the space in question. The proof will be divided into the following steps:

(i) Partition the set of all Cauchy sequences in S into equivalence classes and define S^* as the collection of these equivalence classes.

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(ii) Define the probabilistic distance $\mathscr{F}^*$ on S^*.
(iii) Show that $(S^*, \mathscr{F}^*, T)$ is a Menger space.
(iv) Embed $(S, \mathscr{F})$ isometrically into $(S^*, \mathscr{F}^*)$.
(v) Show that the isometric image of $(S, \mathscr{F})$ is dense in $(S^*, \mathscr{F}^*)$.
(vi) Show that $(S^*, \mathscr{F}^*)$ is complete.
(vii) Show that any two completions of $(S, \mathscr{F})$ are isometric.

The reader will note that these steps are precisely those used in the usual proof of the corresponding theorem for metric spaces (see, e.g., [*1*]). Moreover, the proofs of some of these steps amount to nothing more than a paraphrasing of the metric space proof. Whenever this is the case, details will be omitted. The proof here differs from the metric space proof primarily because addition is a group operation while t-norms are semigroup operations.

(i) Two Cauchy sequences $\{p_n\}$ and $\{q_n\}$ will be called *equivalent*

$$(\{p_n\} \sim \{q_n\}) \quad \text{if} \quad \{F_{p_n q_n}\} \to H.$$

This relation is clearly reflexive and symmetric. Next let $\{p_n\}$, $\{q_n\}$ and $\{r_n\}$ be Cauchy sequences in S such that $\{p_n\} \sim \{q_n\}$ and $\{q_n\} \sim \{r_n\}$. Then for any $x > 0$,

$$1 \geqq \lim_{n\to\infty} F_{p_n r_n}(2x) \geqq \lim_{n\to\infty} T(F_{p_n q_n}(x), F_{q_n r_n}(x)) = 1,$$

from which the transitivity follows. Thus "$\sim$" is an equivalence relation on S. Let S^* be the collection of equivalence classes determined by this relation.

(ii) Let p^*, $q^* \in S^*$; let $\{p_n\} \in p^*$ and $\{q_n\} \in q^*$; and let $\varepsilon > 0$ be given. Choose λ so that $T(a, b) > a - \varepsilon/2$ whenever $0 \leqq a \leqq 1$ and $1 - \lambda < b \leqq 1$. Since $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences there exist integers N_1 and N_2 such that

$$F_{p_n p_m}(\varepsilon/2) > 1 - \lambda, \quad \text{whenever } m, n > N_1,$$

and

$$F_{q_n q_m}(\varepsilon/2) > 1 - \lambda, \quad \text{whenever } m, n > N_2.$$

Thus if $m, n > \max(N_1, N_2)$ then for any x,

$$\begin{aligned} F_{p_n q_n}(x + \varepsilon) &\geqq T(F_{p_n q_m}(x + \varepsilon/2), F_{q_m q_n}(\varepsilon/2)) > F_{p_n q_m}(x + \varepsilon/2) - \varepsilon/2 \\ &\geqq T(F_{p_n p_m}(\varepsilon/2), F_{p_m q_m}(x)) - \varepsilon/2 > F_{p_m q_m}(x) - \varepsilon, \end{aligned}$$

and similarly,

$$F_{p_m q_m}(x) > F_{p_n q_n}(x - \varepsilon) - \varepsilon.$$

Combining the above inequalities yields

$$F_{p_n q_n}(x - \varepsilon) - \varepsilon < F_{p_m q_m}(x) < F_{p_n q_n}(x + \varepsilon) + \varepsilon,$$

i.e., $L(F_{p_n q_n}, F_{p_m q_m}) < \varepsilon$. It follows that $\{\mathscr{F}(p_n, q_n)\}$ is a Cauchy sequence in the complete metric space (Δ, L). Consequently $\lim_{n\to\infty} \mathscr{F}(p_n, q_n)$ exists. An argument similar to the one used above shows that this limit is independent of the particular representatives chosen from p^* and q^*. Thus the probabilistic metric $\mathscr{F}^*$ can be defined via

$$\mathscr{F}^*(p^*, q^*) = \lim_{n\to\infty} (\mathscr{F}(p_n, q_n)),$$

for any p^*, $q^* \in S^*$.

(iii) It is easy to show that $(S^*, \mathscr{F}^*)$ satisfies conditions (I), (II) and (III) of [2]. To show that $(S^*, \mathscr{F}^*)$ is a Menger space under T, let p^*, q^* and r^* be elements of S^* and let $\{p_n\} \in p^*$, $\{q_n\} \in q^*$ and $\{r_n\} \in r^*$. Then for any $x, y > 0$ such that $x + y$ is a point of continuity of $F^*_{p^*r^*}$

$$F^*_{p^*r^*}(x+y) = \lim_{n\to\infty} F_{p_n r_n}(x+y) \geqq \lim_{n\to\infty} T(F_{p_n q_n}(x), F_{q_n r_n}(y))$$
$$= T(\lim_{n\to\infty} F_{p_n q_n}(x), \lim_{n\to\infty} F_{q_n r_n}(y)) \geqq T(F^*_{p^*q^*}(x), F^*_{q^*r^*}(y)).$$

Suppose $x + y$ is not a point of continuity of $F^*_{p^*r^*}$. Let $\{y_k\}$ be an increasing sequence of real numbers with limit y such that for every positive integer k, $x + y_k$ is a point of continuity of $F^*_{p^*r^*}$. Then

$$\begin{aligned} F^*_{p^*r^*}(x+y) &= \lim_{k\to\infty} F^*_{p^*r^*}(x+y_k) \\ &\geqq \lim_{k\to\infty} T(F^*_{p^*q^*}(x), F^*_{q^*r^*}(y_k)) \\ &= T(F^*_{p^*q^*}(x), F^*_{q^*r^*}(y)). \end{aligned}$$

It should be noted that the verification of the triangle inequality requires the continuity of the t-norm T. This is to be expected. For at this stage of the corresponding metric space proof, the continuity of addition is used.

(iv) Let each point p in S correspond to the equivalence class of Cauchy sequences determined by the constant sequence of value p. It follows from the continuity of $\mathscr{F}$ and the definition of $\mathscr{F}^*$ that this correspondence, say ψ, is an isometric embedding of S into S^*.

(v) Let $p^* \in S^*$ and $\{p_n\} \in p^*$. Then the sequence $\{\psi(p_n)\}$ is a Cauchy sequence in S^* which converges to p^*. Therefore $\psi(S)$ is dense in S^*.

(vi) Let $\{p_n^*\}$ be a Cauchy sequence in S^*. There exists a Cauchy sequence $\{p_n\}$ in S such that $\{\psi(p_n)\} \sim \{p_n^*\}$. Also there is an element p^* in S^* such that $\{p_n\} \in p^*$; and it is readily shown that the sequence $\{p_n^*\}$ converges to p^*. Thus $(S^*, \mathscr{F}^*)$ is complete.

(vii) Let $(S', \mathscr{F}')$ and $(S'', \mathscr{F}'')$ be completions of $(S, \mathscr{F})$ and let ψ and ψ'' be the isometric embeddings of S into S' and S'', respectively. Let p' be any point of S'. Since the image of S under ψ' is dense in S', there is a sequence $\{p_n\}$ in S such that $\{\psi'(p_n)\} \to p'$. Since $\{p_n\}$ and consequently $\{\psi''(p_n)\}$ are Cauchy sequences, there is a point $p'' \in S''$ such that $\{\psi''(p_n)\} \to p''$. The function φ defined by $\varphi(p') = p''$ is the desired isometry from S' onto S''. This completes the proof.

References

[1] Kolmogorov, A. N., and S. V. Fomin: Elements of the theory of functions and functional analysis, vol. 1. Rochester: Graylock Press 1957.

[2] Schweizer, B.: On the uniform continuity of the probabilistic distance. Z. Wahrscheinlichkeitstheorie verw. Geb. 5, 357—360 (1966).

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