Z. Wahrscheinlichkeitstheorie verw. Geb. 6, 1—9 (1966)

On Invariant Measures for Operators

DAVID W. DEAN and LOUIS SUCHESTON*

Received September 26, 1965

1. Introduction

Let $(X, \mathscr{A}, p)$ be a measure space with $p(X) = 1$. All sets introduced are assumed in $\mathscr{A}$, all functions measurable with respect to $\mathscr{A}$. Many relations hold only modulo sets of p-measure zero; the words "almost everywhere" are omitted. No distinction is made between equivalence classes of functions and their representatives.

If T is a positive linear operator on $L_1 = L_1(X, \mathscr{A}, p)$ and $\|T\| \leqq 1$, then the problem of existence of positive fixed-points in L_1 is of interest, in particular in connection with the individual ergodic theorem. It has indeed been proved implicitly by HOPF that if there exists a positive fixed-point, i.e., a function $f_0 \in L_1$ with $0 < f_0 = T f_0$, then for each $f \in L_1$, $T^n f$ converges Cesàro to a finite limit (this also follows from the CHACON-ORNSTEIN ergodic theorem), and for each $g \in L_\infty = L_\infty(X, \mathscr{A}, p)$, $T^{*n} g$ converges Cesàro to a finite limit, where T^* is the adjoint of the operator T (cf. NEVEU [16], p. 190). Conversely, if $T^{*n} 1_A$ converges Cesàro for each set A (1_A is the characteristic function of A), and — a mild assumption, explained below — if the operator T is conservative, then there exists a positive fixed-point under T (Theorem 4 below). The principal results of the paper are given in terms of measures π_n defined by

$$\pi_n(A) = \int_A T^n 1\, dp = \int_X T^{*n} 1_A\, dp \quad A \in \mathscr{A}. \tag{1}$$

Theorem 1 asserts that there are positive fixed-points if and only if $p(A) > 0$ implies $F[\pi_n(A)] > 0$, where F is a functional defined on bounded sequences of real numbers (x_n); the smallest F we obtain is $\inf x_n$; the largest is $M(x_n)$, the maximal value of Banach limits on the sequence (x_n), given by

$$M(x_n) = \lim_n \left(\sup_j n^{-1} \sum_{i=0}^{n-1} x_{i+j} \right). \tag{2}$$

If there are no positive fixed-points and the operator T is conservative and ergodic, then there are large "bad" sets: there exists for each $\varepsilon > 0$ a set B with $p(B) > 1 - \varepsilon$ and $M[\pi_n(B)] = 0$ (Theorem 2). Further (Theorem 3), if T is conservative, then there exists a positive fixed-point if and only if for each set A, all Banach limits on the sequence $\pi_n(A)$ coincide; their common value is a finite measure equivalent to p and invariant under the operator Λ which we will now define.

---

* The research of this author was supported by the National Science Foundation, Grant NSF-GP1458. Both authors wish to acknowledge helpful information they obtained from Professors B. BAJSANSKI, R. CHACON and U. KRENGEL.

By the Radon-Nikodým theorem the Banach space L_1 is isomorphic to the space Φ of finite, p-continuous, signed measures. Under this isomorphism there is, corresponding to the operator T, an operator Λ on Φ, which ascribes to a signed measure $\varphi \in \Phi$ a signed measure $\Lambda\varphi \in \Phi$, defined by

$$\Lambda\varphi(A) = \int_A T \frac{d\varphi}{dp}\, dp \qquad A \in \mathscr{A} \tag{3}$$

where $d\varphi/dp$ is the Radon-Nikodým derivative of φ with respect to p. In terms of operator Λ the measures π_n given by (1) are simply images of p under Λ^n, $\pi_n = \Lambda^n p$, and our problem is that of finding a measure φ_0 equivalent with p, and invariant under Λ; φ_0 is such a measure if and only if $d\varphi_0/dp$ is a positive fixed-point under T.

The operator Λ may be generated by a measurable point-transformation τ mapping X to X, by the relation

$$\Lambda\varphi(A) = \varphi(\tau^{-1}A) \qquad \varphi \in \Phi, \quad A \in \mathscr{A}$$

provided that $p(\tau^{-1}A) = 0$ if $p(A) = 0$, which ensures that $\Lambda\varphi$ is p-continuous, and thus our problem may be seen to include the classical problem of existence of finite equivalent measures invariant under a point-transformation, first studied by HOPF in 1932 [*12*]. Or Λ may be induced by a Markov process $P(x, A)$[1] by the relation

$$\Lambda\varphi(A) = \int_X P(x, A)\, d\varphi \quad \varphi \in \Phi, \quad A \in \mathscr{A}$$

provided that $p(A) = 0$ implies $P(x, A) = 0$, which again ensures that $\Lambda\varphi$ is p-continuous; thus our problem includes the problem of existence of finite equivalent measures invariant under a Markov process.

The present paper extends to abstract operators results already known for point-transformations and, in part, for Markov processes. Theorem 1 was obtained for point-transformations by Mrs. DOWKER [*5*] and CALDERÓN [*1*] with $F = \liminf$; also by CALDERÓN [*1*] with $F = \liminf$ of Cesàro averages, and again by Mrs. DOWKER [*6*] with $F = \limsup$ of Cesàro averages (see also HAJIAN and KAKUTANI [*10*]). $F = M$ and Theorem 2 were given by SUCHESTON [*19*]. Theorem 3 was proved for point-transformations by SUCHESTON [*18*] and Theorem 4 by Mrs. DOWKER ([*4*], p. 607). The results of Mrs. DOWKER [*6*] were extended to Markov processes by ITO [*14*]; most arguments of Ito carry over to abstract operators. FELDMAN [*8*] studied the problem of existence of σ-finite invariant measures for operators essentially of the type considered in this paper. (The conditions which he imposes on his L_∞ operators are satisfied if and only if they are adjoints of positive linear contractions in L_1.) Our Theorem 4 is contained in FELDMAN's Theorem 7.1 ([*8*], p. 89).

Theorem 1 has also been proved by NEVEU [*17*], whose note was not available to us at the writing of this paper. NEVEU's proof, introducing "weakly wandering" functions, has considerable independent interest. It seems to us that our argument,

[1] A Markov process $P(x, A)$ is a function of two variables which for each fixed $x \in X$ is a probability measure in A; for each fixed $A \in \mathscr{A}$, a measurable function in x.

based on a rather general "maximal ergodic lemma" (Proposition 1), also has some independent interest. The proof of Proposition 1 imitates the elegant proof by Garsia of Hopf's maximal ergodic theorem [*9*]. The idea of applying a maximal ergodic theorem in this context is due to Mrs. Dowker [*6*].

2. The Main Theorem

X is an abstract set, $\mathscr{A}$ a σ-field of subsets of X, p a measure on $\mathscr{A}$, i.e. a non-negative, countably additive set-function, and $p(X) = 1$. L_1 is the space of integrable functions with norm

$$\|f\| = \int_X |f|\, dp \qquad f \in L_1 .$$

Theorem 1. *Let T be a positive linear operator on L_1 with $\|T\| \leqq 1$ and for each positive integer n let*

$$\pi_n(A) = \int_A T^n 1\, dp \qquad A \in \mathscr{A} .$$

The following conditions are equivalent.

(0) *There exists a function $f_0 \in L_1$ with $f_0 > 0$ and $T f_0 = f_0$.*

(i) $p(A) > 0$ *implies* $\inf_n \pi_n(A) > 0$, $A \in \mathscr{A}$.

(ii) $p(A) > 0$ *implies* $M[\pi_n(A)] = \lim_n [\sup_j n^{-1} \sum_{i=0}^{n-1} \pi_{i+j}(A)] > 0$, $A \in \mathscr{A}$.

Proof. For the purpose of proof we introduce a third condition:

(iii) $p(A) > 0$ implies $\liminf n^{-1} \sum_{i=0}^{n-1} \pi_i(A) > 0$, $A \in \mathscr{A}$.

The proof follows the scheme (0) ⇒ (i) ⇒ (ii) ⇒ (iii) ⇒ (0). The implication (i) ⇒ (ii) is obvious since $\inf_n \pi_n(A) \leqq M[\pi_n(A)]$.

Part I: (0) ⇒ (i). T^*, the adjoint of T, is defined by

$$\int_X T f \cdot g\, dp = \int_X f \cdot T^* g\, dp \qquad f \in L_1, \quad g \in L_\infty ;$$

T^* is a positive linear operator on L_∞ and $\|T^*\| = \|T\| \leqq 1$. Assume there is an $f_0 \in L_1$ with $0 < f_0 = T f_0$ and let F_N be the set of all $x \in X$ such that $f_0(x) > N$, in symbols $F_N = \{f_0 > N\}$. Let $A \in \mathscr{A}$, $p(A) > 0$. Then

$$0 < \delta \overset{\text{def}}{=} \int_A f_0\, dp = \int_X T^n f_0 \cdot 1_A\, dp = \int_X f_0 \cdot T^{*n} 1_A\, dp$$
$$\leqq N \int_X T^{*n} 1_A\, dp + \int_{F_N} f_0 \cdot T^{*n} 1_A\, dp .$$

Since $T^{*n} 1_A \leqq 1$, we may choose a fixed N so large that the last integral is less than $\frac{1}{2}\delta$. Then for each n, $\pi_n(A) \geqq \delta / 2N$ which proves (i).

Part II: (ii) ⇒ (iii). The following result is stated with more generality than needed for applications in this paper. In particular, it contains Hopf's maximal ergodic lemma.

Proposition 1. *Let K be a set of real-valued functions on X and assume that K is a linear space and a lattice under pointwise operations. Let V be a positive linear* *operator on K, N a positive integer and set for an $f \in K$*

$$g_N = \max_{1 \leqq n \leqq N} \sum_0^{n-1} V^i f, \quad A_N = \{g_N > 0\}.$$

Then there is a non-negative function $h_N \in K$, such that

$$f \cdot 1_{A_N} \geqq h_N - V h_N. \tag{4}$$

Proof. Set $h_N = \max(0, g_N)$, then

$V h_N \geqq \max(V 0 = 0, V g_N) \geqq \max(0, Vf, Vf + V^2 f, \ldots, Vf + \cdots + V^N f) =$
$= g_{N+1} - f \geqq g_N - f$.

On A_N, $g_N(x) = h_N(x)$ while on A_N^c, $h_N(x) = 0$ and $V h_N(x) \geqq 0$ (since $V h_N(x) \geqq 0$ everywhere). Hence (4) holds for each $x \in X$.

We now define a functional m on bounded sequences of real numbers (x_n) by

$$m(x_n) = \lim_n \left(\inf_j n^{-1} \sum_{i=0}^{n-1} x_{i+j}\right). \tag{5}$$

The following simple property of m, and of M defined by (2), will be referred to as "telescoping":

$$|m(x_n - x_{n+1})| = |-M(x_{n+1} - x_n)| = |\lim_n [\inf_j n^{-1}(x_j - x_{j+n})]| \leqq$$
$$\leqq \lim_n 2 n^{-1} \sup_k |x_k| = 0.$$

Proposition 2. *Let U be a positive linear operator in L_∞ with $\|U\| \leqq 1$. Let*

$$f \in L_\infty, \; g_N = \max_{1 \leqq n \leqq N} \sum_0^{n-1} U^i f, \; A_N = \{g_N > 0\}.$$

Then

$$m[U^n(f \cdot 1_{A_N})(x)] \geqq 0 \qquad x \in X.$$

This proposition generalizes a result of ITO ([*14*], p. 168).

Proof. Assume $K = L_\infty$, set $V = U$ in (4) and apply U^n to both sides:

$$U^n(f \cdot 1_{A_N})(x) \geqq U^n h_N(x) - U^{n+1} h_N(x), \qquad x \in X.$$

Now apply m and note telescoping to the right.

Lemma 1. *Let U be a positive linear operator on L_∞ such that $\|U\| \leqq 1$ and $U1 = 1$. Set $\pi_n(A) = \int_X U^n 1_A \, dp$. Let A be a set with $p(A) > 0$ and*

$$\liminf n^{-1} \sum_0^{n-1} \pi_i(A) = 0,$$

then for each $\varepsilon > 0$ there is a set $B \subset A$ with $p(B) > p(A) - \varepsilon$ and $M[\pi_n(B)] = 0$.

Proof. We introduce the following notation

$$U_m(f)(x) = m[U^n f(x)], \; U_M f(x) = M[U^n f(x)], \; U_* f(x) = \liminf n^{-1} \sum_0^{n-1} U^i f(x).$$

By Fatou's lemma

$$\int_X U_* 1_A \, dp \leqq \liminf n^{-1} \sum_0^{n-1} \pi_i(A)$$

hence $U_* 1_A = 0$. For a fixed $k > 0$ let

$$A_N = \{\max_{1 \leq n \leq N} n^{-1} \sum_{i=0}^{n-1} U^i (k^{-1} - 1_A) > 0\} = \{\min_{1 \leq n \leq N} n^{-1} \sum_{i=0}^{n-1} U^i 1_A < k^{-1}\}.$$

Let $f_N^k = 1_A \cdot 1_{A_N} + k^{-1} 1_{A_N^c}$. Since on A_N^c, $1_A(x) \geqq k^{-1}$, one has $0 \leqq f_N^k \leqq 1_A$ and $k^{-1} - f_N^k = (k^{-1} - 1_A) 1_{A_N}$. Therefore by Proposition 2, $U_m(k^{-1} - f_N^k) \geqq 0$ hence $U_M(f_N^k) = - U_m(- f_N^k) \leqq k^{-1}$. Moreover, $A_N \to X$ and $f_N^k \xrightarrow{N} 1_A$; therefore by Egorov's theorem there is for each positive integer k an integer N_k and a set $B_k \subset A$ such that $p(B_k) > p(A) - 2^{-k}\varepsilon$ and $f_{N_k} \geqq \frac{1}{2} 1_{B_k}$, hence

$$U_M(1_{B_k}) \leqq 2\, U_M(f_{N_k}^k) \leqq 2 k^{-1}.$$

Let $B = \bigcap_k B_k$, then $p(B) > p(A) - \varepsilon$ and $U_M(1_B) = 0$.

That $M[\pi_n(B)] = 0$ now follows from the relation

$$\int_X (\sup_j n^{-1} \sum_{i=0}^{n-1} U^{i+j} 1_B)\, dp \geqq \sup_j \int_X n^{-1} \sum_{i=0}^{n-1} U^{i+j} 1_B\, dp = \sup_j n^{-1} \sum_{i=0}^{n-1} \pi_{i+j}(B)$$

on letting n go to infinity and applying the bounded convergence theorem.

The implication (ii) $\Rightarrow$ (iii) will follow from Lemma 1 if we show that under the assumption (ii), $T^* 1 = 1$. Since $\| T^* \| \leqq 1$, $T^* 1 \leqq 1$. Assume that $1 - T^* 1 = g \neq 0$. Then there is a set E with $p(E) > 0$ and a positive constant c such that $g(x) > 1/c$ on E, hence $cg \geqq 1_E$. Because of the telescoping property of M

$$M(\int_X c\, T^{*n} g\, dp) = 0$$

hence $M[\pi^n(E)] = 0$ which contradicts (ii).

Part III: (iii) $\Rightarrow$ (0). The dual space of $L_1 = L_1(X, \mathscr{A}, p)$ is the space $L_\infty = L_\infty(X, \mathscr{A}, p)$ of essentially bounded functions with essential supremum norm, and in turn the dual of L_∞ is the space $\Psi (= ba(X, \mathscr{A}, p)$, cf. [7], p. 296) of signed finite finitely additive measures (signed charges), vanishing on p-null sets, with norm $\| \psi \|$ = total variation of ψ, $\psi \in \Psi$. If T is a positive linear operator on L_1, then T^*, the adjoint of T, is a positive linear operator on L_∞ and $\| T^* \| = \| T \|$. In turn T^* admits an adjoint $(T^*)^* = T^{**}$ mapping Ψ to Ψ and T^{**} is a positive linear operator with $\| T^{**} \| = \| T^* \|$. Under the natural embedding of the Banach space L_1 in its second conjugate Ψ, L_1 is mapped on Φ, the space of finite signed p-continuous measures, already mentioned in the Introduction, and T^{**} coincides on Φ with Λ defined by (3). Now let L be a Banach limit and set

$$\nu(A) = L\left[n^{-1} \sum_0^{n-1} \pi_i(A)\right] \qquad A \in \mathscr{A}. \tag{6}$$

It easily follows from properties of Banach limits that $\nu \in \Psi$ and $\nu \geqq 0$ (see e.g. [7], p. 73 or LORENTZ [14]). Therefore ν defines a positive linear functional on L_∞, and in fact this functional is simply given by

$$\nu(g) = \int_X g\, d\nu = L\left(n^{-1} \sum_0^{n-1} \int_X T^{*i} g\, dp\right), \qquad g \in L_\infty.$$

Note that ν is invariant under T^{**}. Indeed, for each $g \in L_\infty$, $(T^{**}\nu) g = \nu(T^* g) = \nu(g)$, the last equality holding, as can be shown by a simple computation, because of the invariance of Banach limits under shifts on sequences. We now recall that every charge may be uniquely decomposed into a measure and a pure charge: if $\nu \in \Psi$, $\nu \geqq 0$, then $\nu = \nu_m + \nu_c$, where $\nu_m \geqq 0$ is a measure and $\nu_c \geqq 0$ is a pure charge; i.e., ν_c does not dominate any non-trivial measure (YOSIDA and HEWITT [20], p. 52 or [7], p. 163). If $\nu = \nu_m + \nu_c$ is defined by (6), then $\nu \geqq \nu_m$ implies $T^{**}\nu = \nu \geqq T^{**}\nu_m$. Thus $T^{**}\nu_m - \nu_m \leqq \nu_c$, and $(T^{**}\nu_m - \nu_m)^+ \leqq \nu_c$. Because ν_c is a pure charge, $T^{**}\nu_m \leqq \nu_m$ and hence $T^{**}\nu_c \geqq \nu_c$. If $T^{**}\nu_c \neq \nu_c$, then there exists a set A with $T^{**}\nu_c(A) > \nu_c(A)$, and since $T^{**}\nu_c(A^c) \geqq \nu_c(A^c)$, it follows that $T^{**}\nu_c(X) > \nu_c(X)$ and $\|T^{**}\| > 1$. Assume $\|T\| \leqq 1$, then $\|T^{**}\| = \|T\| \leqq 1$, hence $T^{**}\nu_c = \nu_c$ and $T^{**}\nu_m = \Lambda\nu_m = \nu_m$. Thus ν_m is invariant under Λ and ν_m will be shown to be an equivalent invariant measure if we prove that $\nu_m(A) = 0$ implies $p(A) = 0$. Assume the contrary: there is a set E with $\nu_m(E) = 0$ and $p(E) > 0$. Restricted to the σ-field of subsets of E in $\mathscr{A}$, $\nu = \nu_c$ is a pure charge and p is a non-trivial measure. It has been proved by YOSIDA and HEWITT ([20], p. 50) that a pure charge is nearly orthogonal to every measure; as applied to ν and p this means that for each ε with $0 < \varepsilon < p(E)$, there is a set $A \subset E$ with $\nu(A) = 0$ and $p(A) > p(E) - \varepsilon > 0$ hence, assuming (iii), $\nu(A) > 0$ because $L(x_n) \geqq \liminf x_n$ for every bounded sequence (x_n). This is a contradiction.

Remark. The proof of Theorem 1 uses the analytical form of M and of m, but not their identification, obtained in [19] as, respectively, the maximal and the minimal value of Banach limits. A proof based on this identification was given by NEVEU [17].

3. Conservative Operators

We assume that T is a fixed positive linear operator on L_1 with $\|T\| \leqq 1$. First we state some known results, many of which appear in the fundamental work of HOPF [13]; see also NEVEU ([16], p. 178, ff.)[2].

The space X can be uniquely decomposed into two parts, $X = C + D$, where C is called the conservative part, D the dissipative part. We denote by $T_\infty f$ the sum $\sum_0^\infty T^n f$. For each non-negative function $f \in L_1$, $T_\infty f(x) = 0$ or $+\infty$ on C; $T_\infty f(x) < \infty$ on D. The notions of invariance and ergodicity can be defined only on C which in the words of HOPF "is the vital part as far as ergodic theory is concerned". A satisfactory separation of C and D is possible: if a function f vanishes on D, so does Tf; conversely, the influence of D on C is also negligible, even though only asymptotically (cf. HOPF [13], p. 44; CHACON [2]). Consequently, the assumption frequently made in this section that $X = C$, in words: T is conservative, is not a severe loss of generality. This assumption is satisfied if there is a positive fixed-point, but conversely the assumption $X = C$ is not sufficient for the existence of a positive fixed-point even in the particular case of a T induced by a point-transformation (cf. HALMOS [11], p. 85).

[2] Our Tf and T^*g are in NEVEU's book, respectively fT and Tg. Neveu's notation would be confusing in the present paper, because if τ is a measurable and null-preserving, but not measure-preserving, point-transformation, then the operator ascribing to f the function $f\tau$ is not an L_1 contraction, but the adjoint of such a contraction.

In the following discussion of invariant sets we assume that $X = C$. The σ-field $\mathscr{I}$ of invariant sets may be characterized by either of the two equivalent definitions:

a) $A \in \mathscr{I}$ if and only if $T^* 1_A = 1_A$.

b) $A \in \mathscr{I}$ if and only if there is a non-negative function $f \in L_1$, such that $A = \{T_\infty f = \infty\}$.

A function $g \in L_\infty$ is called invariant if $T^* g = g$. Invariant functions are exactly those measurable with respect to $\mathscr{I}$. If for a $g \in L_\infty$, $T^* g \leqq g$, then g is invariant: hence in particular constants are invariant. T is called ergodic if $\mathscr{I} = \{0, X\}$.

The following theorem may be considered as a quantitative strengthening of Theorem 1.

Theorem 2. *If the operator T is conservative and ergodic and there exists no positive fixed-point, then for each $\varepsilon > 0$ there is a set B with $p(B) > 1 - \varepsilon$ and*

$$\lim_{n \to \infty} n^{-1} \sum_{i=0}^{n-1} \pi_{i+j}(B) = 0$$

uniformly in j.

Proof. We revert to Part III of the proof of Theorem 1. If no positive fixed-point exists, then there is a set E with $p(E) > 0$ and $\nu_m(E) = 0$. Since ν_m is invariant under Λ, $d\nu_m | dp = f_0$ is a fixed-point. Set $F = \{T_\infty f_0 = 0\}$, then $f_0(x) = 0$ on E implies $E \subset F$; therefore $p(F) > 0$. F is invariant since it is the complement of the set $\{T_\infty f_0 = \infty\}$, and because T is ergodic, $F = X$. Applying the argument from the end of Part III with X instead of E, we obtain that for each $\varepsilon > 0$, there is a set A with $p(A) > 1 - \varepsilon$ and $\liminf n^{-1} \sum_0^{n-1} \pi_i(A) = 0$. The proof is now completed by application of Lemma 1 with $U = T^*$.

The following Proposition is preparatory for Theorem 3.

Proposition 3. *Let $0 \leqq g \in L_1$ and set*

$$\gamma_n(A) = \int_A T^n g \, dp, \quad A \in \mathscr{A}. \tag{7}$$

If there is a positive fixed-point, then the measures γ_n are uniformly p-continuous.

Proof. Assume that there is a function $f_0 \in L_1$ with $0 < f_0 = T f_0$. We are to show that given an $\varepsilon > 0$, there is a $\delta > 0$ such that $p(A) < \delta$ implies $\gamma_n(A) < \varepsilon$ for all n. First select a function g', $0 \leqq g' \in L_\infty$, such that $\|g - g'\|_{L_1} < \varepsilon/3$. Let $d = ess \sup g'$, $F_\alpha = \{f_0 > \alpha\}$. Then for each $\alpha > 0$

$$\int_{F_\alpha^c} g' T^{*n} 1_A \, dp \leqq d \int_{F_\alpha^c} 1 \, dp = c p \, (F_\alpha^c). \tag{8}$$

Choose an α fixed so small that $dp(F_\alpha^c) < \varepsilon/3$. Now let δ be so small that if $p(A) < \delta$, then

$$\frac{\varepsilon}{3} > \frac{d}{\alpha} \int_A f_0 \, dp = \frac{d}{\alpha} \int_X T^{*n} 1_A \cdot f_0 \, dp \geqq \int_{F_\alpha} g' T^{*n} 1_A \, dp. \tag{9}$$

(8) and (9) together yield

$$\int_X g' T^{*n} 1_A \, dp = \int_A T^n g' \, dp < \frac{2}{3} \varepsilon,$$

hence

$$\gamma_n(A) \leqq \int_A T^n g' \, dp + \int_X |T^n(g - g')| \, dp < \varepsilon .$$

Theorem 3. *If there exists a positive fixed-point, then for each set A, all Banach limits on the sequence $\pi_n(A)$ coincide. If $\lambda(A)$ is their common value, then λ is a measure and*

$$\lambda(A) = \lim_{n\to\infty} n^{-1} \sum_{i=0}^{n-1} \pi_{i+j}(A) \quad \textit{uniformly in } j; \tag{10}$$

λ coincides with p on invariant sets and $d\lambda/dp$ is a positive fixed-point. Conversely, if for each A all Banach limits on the sequence $\pi_n(A)$ coincide and if T is conservative, then there is a positive fixed-point.

Proof. Assume that there is a positive fixed-point f_0; then T is conservative since $\{T_\infty f_0 = \infty\} = X$. Let L be a Banach limit and set

$$\lambda(A) = L[\pi_n(A)], \quad A \in \mathscr{A}. \tag{11}$$

It is easy to see that λ is a charge and λ vanishes on p-null sets. From Proposition 3 applied with $g = 1$ it follows that given an $\varepsilon > 0$, there is a $\delta > 0$ such that if $p(A) < \delta$, then $\pi_n(A) < \varepsilon$ for all n, hence $\lambda(A) < \varepsilon$. Thus λ is p-continuous and given a sequence of sets A_n with $A_n \downarrow 0$, one has $p(A_n) \downarrow 0$ and therefore $\lambda(A_n) \downarrow 0$. This implies that λ is a measure and hence $T^{**}\lambda = \Lambda\lambda$. Since

$$T^{**}\lambda(1_A) = \lambda(T^* 1_A) = \lambda(1_A),$$

λ is invariant under Λ and $d\lambda/dp$ is a positive fixed point (cf. the proof of invariance of ν under T^{**} in part III of the proof of Theorem 1). Let $F_1 = \{f_1 = 0\}$, then $F_1 = \{T_\infty f_1 = \infty\}^c$ is an invariant set and since p coincides with each π_n and hence with λ on invariant sets, $p(F_1) = \lambda(F_1) = 0$, i.e., $f_1 > 0$. Also, $E(f_1/\mathscr{I}) = 1$; if now f_2 is another fixed-point with $E(f_2/\mathscr{I}) = 1$, then from the ergodic theorem of Hopf ([*16*], p. 190) it follows that

$$f_2 = \lim n^{-1} \sum_0^{n-1} T^i f_2 = \frac{f_1}{E(f_1/\mathscr{I})} E(f_2/\mathscr{I}) = f_1 .$$

Consequently λ defined by (11) does not depend upon the choice of the Banach limit L; i.e., for each A, all Banach limits on the sequence $\pi_n(A)$ coincide. Now $M[\pi_n(A)] = m[\pi_n(A)]$ proves (10) (see the remark following the proof of Theorem 1, and also Lorentz [*15*]) and the first part of the theorem. To prove the second part, assume that T is conservative and that for each set A, all Banach limits on the sequence $\pi_n(A)$ agree. $M[\pi_n(A)] = m[\pi_n(A)]$ implies that $\pi_n(A)$ converges Cesàro, and by the Vitali-Hahn-Saks theorem (see [*7*] p. 160, Corollary 4, or [*16*] p. 111) the limit, say $\mu(A)$, is a measure. It is easy to see that μ is invariant under the operator Λ defined by (3) and μ agrees with p on invariant sets; proceeding as in the first part of the proof, one shows that $d\mu/dp$ is a positive fixed-point. The same argument proves, after integration of (12), the following.

Theorem 4. *If the operator T is conservative and for each set A the limit*

$$\lim n^{-1} \sum_0^{n-1} T^{*i} 1_A = f_A \tag{12}$$

exists, then $\mu(A) = \int_X f_A \, dp$ is a measure agreeing with p on invariant sets and $d\mu/dp$ is a positive fixed-point (under T).

Finally, considering Banach limits on sequences of Cesàro averages of $\pi_n(A)$ and proceeding as in the first part of the proof of Theorem 3, one obtains the following partial converse of Proposition 3.

Proposition 4. *If T is conservative and the measures $\mu_n = n^{-1} \sum_0^{n-1} \pi_i$ are uniformly p-continuous (hence* a fortiori *if the measures π_n are uniformly p-continuous), then there exists a positive fixed-point.*

Bibliography

[1] CALDERÓN, ALBERTO: Sur les mesures invariantes. C. r. Acad. Sci. Paris **240**, 1960–1962 (1955).

[2] CHACON, R. V.: The influence of the dissipative part of a general Markov process. Proc. Amer. math. Soc. **11**, 957–961 (1960).

[3] —, and D. S. ORNSTEIN: A general ergodic theorem. Illinois J. Math. **4** 153–160 (1960).

[4] DOWKER, Y. N.: Finite and σ-finite measures. Ann. of Math., II. Ser. **54**, 595–608 (1951).

[5] — On measurable transformations in finite measure spaces. Ann. of Math. **62**, 504–516 (1955).

[6] — Sur les applications mesurables. C. r. Acad. Sci. Paris **242**, 329–331 (1956).

[7] DUNFORD, NELSON, and JACOB T. SCHWARTZ: Linear operators I. New York: 1958.

[8] FELDMAN, J.: Subinvariant measures for Markoff operators. Duke math. J. **29**, 71–98 (1962).

[9] GARSIA, A.: A simple proof of E. Hopf's maximal ergodic theorem. J. Math. Mech. **14**, 381–382 (1965).

[10] HAJIAN, A., and S. KAKUTANI: Weakly wandering sets and invariant measures. Trans. Amer. math. Soc. **110**, 136–151 (1964).

[11] HALMOS, P. R.: Lectures on ergodic theory. Tokyo: The Mathematical Society of Japan 1956.

[12] HOPF, E.: Theory of measure and invariant integrals. Trans. Amer. math. Soc. **34**, 373–393 (1932).

[13] — The general temporally discrete Markoff process. J. rat. Mech. Analysis **3**, 13–45 (1954).

[14] ITO, Y.: Invariant measures for Markov processes. Trans. Amer. math. Soc. **110**, 152–184 (1964).

[15] LORENTZ, G. G.: A contribution to the theory of divergent sequences. Acta math. **80**, 167–190 (1948).

[16] NEVEU, J.: Bases mathématiques du calcul des probabilités. Paris: Masson 1964.

[17] — Sur l'existence de mesures invariantes en theorie ergodique. C. r. Acad. Sci. Paris **260**, 393 – 396 (1965).

[18] SUCHESTON, L.: An ergodic application of almost convergent sequences. Duke math. J. **30**, 417–422 (1963).

[19] — On existence of finite invariant measures. Math. Z. **86**, 327–336 (1964).

[20] YOSIDA, K., and E. HEWITT: Finitely additive measures. Trans. Amer. math. Soc. **72**, 46–66 (1952).

Department of Mathematics
The Ohio State University
Columbus, Ohio 43210, USA