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On the Fourier Series of a Stationary Stochastic Process*

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1. Introduction

Let $X(t) = X(t, \omega)$, $t \in R = (-\infty, \infty)$, be a real-valued continuous stationary process in the wide sense satisfying

$$EX^2(t) < \infty, \quad \text{for} \quad t \in R, \tag{1.1}$$

$$EX(t) = 0, \quad \text{for} \quad t \in R, \tag{1.2}$$

$$EX(s)X(t) = \varrho(s-t), \quad \text{for} \quad s, t \in R, \tag{1.3}$$

where $\varrho(u)$ is a continuous even function of a single variable u.

Suppose that $X(t)$ is separable and measurable in $R \times \Omega$. Then $\int_b^a X^2(t)\, dt$ exists, with probability one, for every real numbers a, b. $\varrho(u)$ has the form

$$\varrho(u) = \int_{-\infty}^{\infty} e^{ixu}\, dF(u) = 2\int_0^{\infty} \cos xu\, dF(u), \tag{1.4}$$

where $F(u)$ is the spectral distribution function of the process; $F(u)$ is a bounded non-decreasing function with $F(x) = 1 - F(-x+0)$, $\varrho(0) = F(+\infty) - - F(-\infty) = EX^2(t)$.

We shall now consider the Fourier series of $X(t)$ over $(0, T)$, where T is a arbitrary positive number. The restriction of $X(t)$ to $(0, T)$ is called a sample of the process $X(t)$. Then

$$X(t) \sim A_0/2 + \sum_{n=1}^{\infty} \left(A_n \cos \frac{2\pi nt}{T} + B_n \sin \frac{2\pi nt}{T}\right), \tag{1.5}$$

where

$$A_n = \frac{2}{T}\int_0^T X(t) \cos \frac{2\pi nt}{T}\, dt, \quad n = 0, 1, 2, \ldots, \tag{1.6}$$

$$B_n = \frac{2}{T}\int_0^T X(t) \sin \frac{2\pi nt}{T}\, dt, \quad n = 1, 2, \ldots \tag{1.7}$$

It is known that as T increases indefinitely, the covariance of two Fourier coefficients with different amplitudes approaches zero. But in the engineering literature it is customary to assume that

$$\begin{aligned} EA_m A_n &= EB_m B_n = 0, \quad &&\text{for} \quad n \neq m, \\ EA_m B_n &= 0, &&\text{for all } m, n, \end{aligned} \tag{1.8}$$

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R. C. DAVIS [2] has shown that if (1.8) holds for one pair of integers n, m $(n \neq m)$ and for all $T > 0$, then the stationary process reduces to the trivial process in which $\varrho(u) = 1$ for every u.

In connection with this statement, it is of interest to obtain the more precise information concerning the order of the Fourier coefficients as they approach zero. In fact, W. L. ROOT and T. S. PITCHER [10] proved that if $\varrho(u) \in L_1(-\infty, \infty)$ and $\int_{-\infty}^{\infty} \varrho(u) du \neq 0$, then $E|C_n|^2 = O(T^{-1})$, $2C_n = A_n - iB_n$ and $R_{nm} = E C_n \bar{C}_m / (E|C_n|^2 \; E|C_m|^2)^{\frac{1}{2}} \to 0$. They also discussed the order of R_{nm} when $T \to \infty$. The author [6] has shown that if $u\varrho(u) \in L_1(-\infty, \infty)$, then $E A_m A_n = O(T^{-2})$, $E B_m B_n = O(T^{-2})$ for $m \neq n$, $E A_m B_n = O(T^{-2})$ for all m, n and

$$E A_m^2 \sim \frac{2}{T} \int_{-\infty}^{\infty} \varrho(u)\, du, \quad E B_m^2 \sim \frac{2}{T} \int_{-\infty}^{\infty} \varrho(u)\, du.$$

He has also shown that under the additional condition $\int_{-\infty}^{\infty} \varrho(u) du \neq 0$, the correlation coefficients between A_m and A_n, B_m and $B_n (m \neq n)$ and between A_m and B_n are of $O(T^{-1})$ as $T \to \infty$. In **2** and **3**, we shall give more precise estimations of the covariances under some conditions on the spectral distribution function or the spectral density function.

The convergence problems of the Fourier series of $X(t)$ are discussed in **5** and **6**. H. B. MANN [8] gave a sufficient condition that a stochastic process (not necessarily stationary) with continuous covariance function $r(s, t)$ should have the Fourier series which converges in L_2-mean (regarding Ω) to the given process in $(0, T)$. The author [5] has shown that the Fourier series (1.5) of a continuous stationary process $X(t)$ is always convergent in L_2-mean (in Ω) to $X(t)$ for every $0 < t < T$. A brief proof of this statement is given in **5**.

It is quite natural to ask under what conditions the Fourier series converges to $X(t)$ almost everywhere or absolutely in $(0, T)$ with probability one. The analogues of the following well known theorems concerning ordinary Fourier series produce some answers to these problems.

Theorem A (KOLMOGOROV-SELIVERSTOV). *If*

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1.9}$$

is the Fourier series of a function $f(x)$ of $L_2(0, 2\pi)$ and

$$\sum_{n=1}^{\infty} (a_n^2 + b_n^2) \log n < \infty,$$

then (1.9) *converges almost everywhere to $f(x)$ in* $(0, 2\pi)$ (see [12] p. 252).

Theorem B (S. BERNSTEIN). *If $f(x) \in$ Lip α, with $\alpha > \frac{1}{2}$, then the Fourier series of $f(x)$ converges absolutely* (see [12] p. 135).

This is a basic theorem on the absolute convergence of a Fourier series. Generalizations are known (see [12]) but their stochastic analoguet are not considered in this paper.

The absolute convergence of a Fourier series implies its uniform convergence. Hence the conditions for absolute convergence with probability one of the Fourier series of a stochastic process implies the continuity of the process with probability one (sample continuity). This, incidentally, is a generalization of known results on sample continuity.

In the theory of random noise (see for ex. [9]) it is usually assumed that the process is a Gaussian process, so that the joint distribution of Fourier coefficients in $(0, T)$, T being fixed, is also Gaussian. But the random noise process formulated as ([4] p. 433—434)

$$X(t) = \int_{-\infty}^{\infty} \varphi(t-s)\, dY_{\alpha}(s) \tag{1.10}$$

is not a Gaussian process, where $\varphi(u)$ is a bounded continuous function of $L_2(0, \infty)$ and $Y_{\alpha}(s)$ is another stochastic process whose sample function is constant between events which occur in accordance with a Poisson process with parameter α, and increases at the j-th event by the amount U_j, where $\{U_j\}$ is a sequence of non-negative independent identically distributed random variables with finite variance.

However it is known that with suitable normalization, the joint distribution of $X(t)$ at any finite number of points t converges to the Gaussian distribution as $\alpha \to \infty$.

We shall consider the more general linear process

$$X(t) = \int_{-\infty}^{\infty} \varphi(t-s)\, dY(s), \tag{1.11}$$

where $Y(s)$ is any stochastic process having independent increments with

$$E\,|dY(s)|^2 = ds. \quad E\,|dY(s)|^3 = O(ds). \tag{1.12}$$

(1.12) is satisfied by the random noise process when the right hand side is replaced by a constant multiple of ds.

We shall show in 7 that the joint distribution of Fourier coefficients (1.6) and (1.7) with normalizing factor $T^{1/2}$ converges to the joint distribution of independent normal variables as $T \to \infty$. Hence when T is large enough the Fourier coefficients behave as independent and normally distributed random variables.

2. Covariances of Fourier Coefficients I

Let $X(t)$ be the real valued continuous stationary process considered in **1.** We are going to discuss the asymptotic behavior of the covariances EA_mA_n, EB_mB_n and EA_mB_n of Fourier coefficients (1.6) and (1.7) of $X(t)$ in $(0, T)$ as $T \to \infty$. m and n are fixed.

Let $F(\lambda)$ be the spectral distribution function of $X(t)$. We see that

$$\begin{aligned} EA_mA_n &= E\,\frac{4}{T^2}\int_0^T X(u)\cos\frac{2m\pi u}{T}\,du \int_0^T X(v)\cos\frac{2n\pi v}{T}\,dv \\ &= \frac{4}{T^2}\int_0^T dv \int_0^T du\; \varrho(u-v)\cos\frac{2m\pi u}{T}\cos\frac{2n\pi v}{T}. \end{aligned}$$

Inserting (1.4) in here, we have, after some simple manipulations,

$$EA_mA_n = J_1 + J_2, \tag{2.1}$$

where

(2.2) $$J_1 = \int_{-\infty}^{\infty} \varphi(\lambda T; m, n) \sin^2 \lambda T dF(\lambda),$$

(2.3) $$J_2 = \int_{-\infty}^{\infty} \varphi(\lambda T; m, n) (1 - \cos \lambda T)^2 dF(\lambda)$$

with

(2.4) $$\varphi(u; m, n) = \frac{4u^2}{[u^2 - (2m\pi)^2][u^2 - (2n\pi)^2]}.$$

Also, we have

(2.5) $$E B_m B_n = K_1 + K_2,$$

where

(2.6) $$K_1 = \int_{-\infty}^{\infty} \psi(\lambda T; m, n) \sin^2 \lambda T dF(\lambda),$$

(2.7) $$K_2 = \int_{-\infty}^{\infty} \psi(\lambda T; m, n) (1 - \cos \lambda T)^2 dF(\lambda)$$

with

(2.8) $$\psi(u; m, n) = \frac{4m\pi \cdot 4n\pi}{[u^2 - (2m\pi)^2][u^2 - (2n\pi)^2]}.$$

Furthermore, we have

(2.9) $$E A_m B_n = 0, \quad \text{for every pair } (m, n).$$

Noting that

(2.10) $$\lim_{T\to\infty} \varphi(\lambda T; m, n) \sin^2 \lambda T = 0, \quad -\infty < \lambda < \infty, \quad m^2 + n^2 \neq 0,$$

we obtain

(2.11) $$E A_m A_n \to 0, \quad \text{as} \quad T \to \infty, \quad \text{if} \quad m^2 + n^2 \neq 0.$$

Similarly we have

(2.12) $$E B_m B_n \to 0, \quad \text{as} \quad T \to \infty, \quad m, n = 1, 2, \ldots,$$

since we may verify that

$$\lim_{T\to\infty} \psi(\lambda T; m, n) \sin^2 \lambda T = 0, \quad -\infty < \lambda < \infty$$

for every pair (m, n) of positive integers.

If $m = n = 0$, then $\varphi(\lambda T; 0, 0) = 4/(\lambda T)^2$ and hence

$$J_1 = 4 \int_{-\infty}^{\infty} \frac{\sin^2 \lambda T}{\lambda^2 T^2} dF(\lambda), J_2 = 4 \int_{-\infty}^{\infty} \frac{\sin^4 (\lambda T/2)}{\lambda^2 T^2/4} dF(\lambda).$$

The kernel $\sin^2 \lambda T/(\lambda T)^2$ converges to zero as $T \to \infty$ if $\lambda \neq 0$, and equals 1 if $\lambda = 0$. Hence J_1 converges to $4[F(0+) - F(0-)]$. On the other hand J_2 converges to zero for every λ. We therefore get

(2.13) $$E A_0^2 \to 4[F(0+) - F(0-)] \quad \text{as} \quad T \to \infty.$$

Under the above assumption, the behavior described in (2.9), (2.11), (2.12) and (2.13) is all that can be claimed; however, if the process has a spectral density, we can say more.

Theorem 1. *Suppose that a continuous stationary process $X(t)$ has the spectral density $f(\lambda)$ which is continuous at $\lambda = 0$. Then we have*

$$\lim_{T\to\infty} T\cdot EA_m^2 = 4\pi f(0), \quad m = 1, 2, \ldots, \tag{2.14}$$

$$\lim_{T\to\infty} T\cdot EA_0^2 = 8\pi f(0), \tag{2.15}$$

$$\lim_{T\to\infty} T\cdot EB_m^2 = 4\pi f(0), \quad m = 1, 2, \tag{2.16}$$

and

$$\lim_{T\to\infty} T\cdot EA_m A_n = 0, \quad m \neq n, \tag{2.17}$$

$$\lim_{T\to\infty} T\cdot EB_m B_n = 0, \quad m \neq n. \tag{2.18}$$

This is known [*6*], [*10*] under the condition that $\varrho(u) \in L_1(-\infty, \infty)$ which implies the existence of the continuous spectral density.

In order to show this theorem, we shall use the following standard summability theorem (See [*1*] and [*11*] p. 28).

Theorem C. *Let $f(x)$ be a function on $(-\infty, \infty)$, continuous at a point c and be such that $f(x)/(1+|x|) \in L_1(-\infty, \infty)$. Let $K(x)$ be a function of $L_1(-\infty, \infty)$ satisfying*

$$xK(x) = O(1), \qquad \textit{for large } |x|. \tag{2.19}$$

Then we have

$$\lim_{\lambda\to\infty} \lambda \int_{-\infty}^{\infty} f(c + u + \xi_\lambda) K(\lambda u)\, du = f(c) \int_{-\infty}^{\infty} K(x)\, dx, \tag{2.20}$$

where ξ_λ is a number independent of u, converging to zero as $\lambda \to \infty$.

The *proof* of Theorem 1 is a simple application of Theorem C. Setting $m = n$ in (2.1), we have

$$EA_m^2 = J_1 + J_2.$$

Moreover

$$\begin{aligned} J_1 &= \int_{-\infty}^{\infty} \Big(\frac{1}{\lambda T + 2m\pi} + \frac{1}{\lambda T - 2m\pi}\Big)^2 \sin^2 \lambda T f(\lambda)\, d\lambda \\ &= \int_{-\infty}^{\infty} \frac{\sin^2 \lambda T}{(\lambda T + 2m\pi)^2} f(\lambda)\, d\lambda + 2\int_{-\infty}^{\infty} \frac{\sin^2 \lambda T}{(\lambda T)^2 - (2m\pi)^2} f(\lambda)\, d\lambda + \\ &\quad + \int_{-\infty}^{\infty} \frac{\sin^2 \lambda T}{(\lambda T)^2 - (2m\pi)^2} f(\lambda)\, d\lambda \\ &= J_{11} + J_{12} + J_{13}, \end{aligned} \tag{2.21}$$

say. Now

$$T J_{11} = \int_{-\infty}^{\infty} \frac{\sin^2 \mu T}{T \mu^2} f\left(\mu - \frac{2 m \pi}{T}\right) d\mu$$

which is no more than the left hand side of (2.20) with the Fejér kernel for $K(x)$, $c = 0$ and $\xi_T = -\frac{2 m \pi}{T}$. Hence we have

$$\lim_{T \to \infty} T J_{11} = \pi f(0) .$$

In the same way, we have

$$\lim_{T \to \infty} T J_{13} = \pi f(0) .$$

$$T J_{12} = 2 T \int_{-\infty}^{\infty} \frac{\sin^2 \lambda T}{(\lambda T)^2 - (2 m \pi)^2} f(\lambda) d\lambda$$

is again in the form of the left hand side of (2.20) with

$$K(x) = \frac{\sin^2 x}{x^2 - (2 m \pi)^2}$$

which satisfies the conditions of $K(x)$ in Theorem C. We note that

$$\int_{-\infty}^{\infty} K(x) dx = 0 , \quad \text{for} \quad m \neq 0 .$$

Then Theorem C gives us that $T J_{12}$ converges to zero. We have hence proved that J_1 converges to $2 \pi f(0)$.

J_2 is handled in the similar way. As in (2.21) J_2 is written

$$J_2 = J_{21} + J_{22} + J_{23} , \tag{2.22}$$

where

$$J_{21} = \int_{-\infty}^{\infty} \frac{(1 - \cos \lambda T)^2}{(\lambda T + 2 m \pi)^2} f(\lambda) d\lambda = 2 \int_{-\infty}^{\infty} \frac{\sin^4 \mu T}{\mu^2 T^2} f\left(2 \mu - \frac{2 m \pi}{T}\right) d\mu ,$$

$$J_{22} = 8 \int_{-\infty}^{\infty} \frac{\sin^4 (\lambda T/2)}{(\lambda T)^2 - (2 m \pi)^2} f(\lambda) d\lambda$$

and

$$J_{23} = 2 \int_{-\infty}^{\infty} \frac{\sin^4 \mu T}{\mu^2 T^2} f\left(2 \mu + \frac{2 m \pi}{T}\right) d\mu .$$

Again by Theorem C, J_{21} and J_{23} each converge to $\pi f(0)$ as $T \to \infty$. J_{22} converges to zero since

$$\int_{-\infty}^{\infty} \frac{\sin^4 (\lambda T/2)}{(\lambda T)^2 - (2 m \pi)^2} d\lambda = 0 , \quad \text{for} \quad m \neq 0 .$$

Putting the above results together we have (2.14).

The same reasoning also leads to (2.16).

The *proof* of (2.15) is contained in that of (2.14), since, with $m = 0$, $J_1 = 4J_{11}$ and $J_2 = 4J_{21}$.

In order to show (2.17) and (2.18), we may proceed in the same manner and encounter the integrals

$$\int_{-\infty}^{\infty} \frac{\sin^2 \lambda T}{(\lambda \pm 2m\pi)(\lambda \pm 2n\pi)} f(\lambda) d\lambda, \quad m \neq n,$$

and

$$\int_{-\infty}^{\infty} \frac{(1 - \cos \lambda T)^2}{(\lambda \pm 2m\pi)(\lambda \pm 2n\pi)} f(\lambda) d\lambda, \qquad m \neq n.$$

They are handled just as J_{12} or J_{22} and are seen to converge to zero even if they are multiplied by T. This completes the proof of Theorem 1.

3. Covariances of Fourier Coefficients II

In Theorem 1 we have shown that (2.17) and (2.18) hold under the condition that the spectral density is continuous at $x = 0$. Now we are going to show that if we assume more about $f(x)$, then we may find the limit of $T^2 E A_m A_n$ and $T^2 E B_m B_n$ as $T \to \infty$.

Theorem 2. *If the spectral density $f(x)$ has a continuous first derivative in a neighborhood of the origin and*

$$(3.1) \qquad \int_{0+}^{\infty} \frac{f(x) - f(0)}{x^2} dx = \lim_{\varepsilon \to 0+} \int_{\varepsilon}^{\infty} \frac{f(x) - f(0)}{x^2} dx$$

exists, then

$$(3.2) \qquad \lim_{T \to \infty} T^2 E A_m A_n = 16 \int_{0+}^{\infty} \frac{f(\lambda) - f(0)}{\lambda^2} d\lambda, \qquad m \neq n,$$

$$(3.3) \qquad \lim_{T \to \infty} T^2 E B_m B_n = 0, \qquad m \neq n.$$

Since we are assuming that $X(t)$ is real-valued, the spectral density $f(x)$ is an even function and hence $f'(x)$ in a neighborhood of the origin is an odd function. Hence

$$(3.4) \qquad f'(0) = 0.$$

Proof. From (2.1), $E A_m A_n = J_1 + J_2$, where J_1 and J_2 are given by (2.2) and (2.3) respectively. The function $\varphi(\lambda T; m, n)$ in (2.2) and (2.3) is written

$$(3.5) \qquad \left(\frac{1}{\lambda T + 2m\pi} + \frac{1}{\lambda T - 2m\pi}\right)\left(\frac{1}{\lambda T + 2n\pi} + \frac{1}{\lambda T - 2n\pi}\right).$$

Hence J_1 is written as the sum of four integrals each of which involves one of terms in the expansion (3.5). Using the fact that $f(\lambda)$ is an even function we see that

$$J_1 = 2(S_1 + S_2),$$

where

$$S_1 = \int_{-\infty}^{\infty} \frac{\sin^2 \lambda T}{(\lambda T + 2m\pi)(\lambda T + 2n\pi)} f(\lambda) d\lambda, \tag{3.6}$$

$$S_2 = \int_{-\infty}^{\infty} \frac{\sin^2 \lambda T}{(\lambda T + 2m\pi)(\lambda T - 2n\pi)} f(\lambda) d\lambda. \tag{3.7}$$

Since

$$\int_{-\infty}^{\infty} \frac{\sin^2 \lambda T}{(\lambda T + 2m\pi)(\lambda T + 2n\pi)} d\lambda = 0, \qquad m \neq n, \tag{3.8}$$

we have

$$S_1 = \int_{-\infty}^{\infty} \frac{\sin^2 \lambda T}{(\lambda T + 2m\pi)(\lambda T + 2n\pi)} [f(\lambda) - f(0)] d\lambda. \tag{3.9}$$

Take A so large that $A > m, n$ and fix it. Choose δ so small and T so large that $\delta > A/T$ and $f'(x)$ exists and is bounded in $[-\delta, \delta]$. We split S_1 into three parts as follows:

$$\begin{aligned} S_1 &= \int_{|\lambda|<A/T} + \int_{\delta>|\lambda|\geqq A/T} + \int_{|\lambda|\geqq\delta} \\ &= S_{11} + S_{12} + S_{13}, \end{aligned}$$

say. Since the integrand of (3.8) is bounded by 1 and $f'(0) = 0$, we see that

$$|S_{11}| \leqq \int_{|\lambda|<A/T} |f(\lambda) - f(0)| d\lambda = o(1) \int_{|\lambda|<A/T} |\lambda| d\lambda = o(A^2/T^2).$$

Hence,

$$\lim_{T\to\infty} T^2 S_{11} = 0. \tag{3.10}$$

Next

$$\begin{aligned} S_{12} &= \int_{\delta>|\lambda|\geqq A/T} \frac{\sin^2 \lambda T - \frac{1}{2}}{(\lambda T + 2m\pi)(\lambda T + 2n\pi)} [f(\lambda) - f(0)] d\lambda + \\ &\quad + \tfrac{1}{2} \int_{\delta>|\lambda|\geqq A/T} \frac{f(\lambda) - f(0)}{(\lambda T + 2m\pi)(\lambda T + 2n\pi)} d\lambda = S'_{12} + S''_{12}, \end{aligned} \tag{3.11}$$

say. S''_{12} is easily seen to be equal to

$$\begin{aligned} &\frac{1}{2T^2} \int_{\delta>|\lambda|\geqq A/T} \frac{f(\lambda) - f(0)}{\lambda^2} d\lambda \\ &\qquad - \frac{1}{2T^2} \int_{\delta>|\lambda|\geqq A/T} \frac{2\lambda(m+n)\pi/T + 4mn\pi^2/T}{\lambda^2(\lambda + 2m\pi/T)(\lambda + 2n\pi/T)} [f(\lambda) - f(0)] d\lambda \\ &\qquad = \frac{1}{T^2} \int_{\delta>\lambda\geqq A/T} \frac{f(\lambda) - f(0)}{\lambda^2} d\lambda + \frac{1}{2T^2} \cdot O\left(\frac{1}{T}\right) o(1) \int_{\delta>\lambda\geqq A/T} \frac{d\lambda}{\lambda^2} \\ &\qquad = \frac{1}{T^2} \int_{\delta>\lambda\geqq A/T} \frac{f(\lambda) - f(0)}{\lambda^2} d\lambda + o\left(\frac{1}{T^2}\right). \end{aligned} \tag{3.12}$$

The last integral can be made less than ε/T^2, where $\varepsilon > 0$ is arbitrary, by taking $\delta > 0$ sufficiently small. Hence, we have

$$\lim_{T\to\infty} T^2 S_{12}'' = 0\,. \tag{3.13}$$

Also writing S_{12}'' in the form

$$-\tfrac{1}{2}\int\limits_{\delta>|\lambda|\geqq A/T} \frac{\cos 2\lambda T}{(\lambda T + 2m\pi)(\lambda T + 2n\pi)}\,[f(\lambda) - f(0)]\,d\lambda$$

and using the same technique, we find that

$$S_{12}' = -\frac{1}{T^2}\int\limits_{\delta>\lambda\geqq A/T} \frac{\cos 2\lambda T}{\lambda^2}[f(\lambda) - f(0)]\,d\lambda + o\left(\frac{1}{T^2}\right). \tag{3.14}$$

Since $[f(\lambda) - f(0)]/\lambda$ is differential in $[A/T, \delta]$, we see, using the second mean value theorem, that for some ξ, $A/T < \xi < \delta$,

$$\int_{A/T}^{\delta} \frac{\cos 2\lambda T}{\lambda^2}[f(\lambda) - f(0)]\,d\lambda = \frac{T}{A}\left[f\left(\frac{A}{T}\right) - f(0)\right]\int_{A/T}^{\xi}\frac{\cos 2\lambda T}{\lambda}\,d\lambda + $$
$$+ \frac{f(\delta) - f(0)}{\delta}\int_{\xi}^{\delta}\frac{\cos 2\lambda T}{\lambda}\,d\lambda\,.$$

$$\int_{A/T}^{\xi}\frac{\cos 2\lambda T}{\lambda}\,d\lambda = \int_{A}^{T\xi}\frac{\cos 2v}{v}\,dv$$

is bounded for fixed $A > 0$ and the same is true for

$$\int_{\xi}^{\delta}\frac{\cos 2\lambda T}{\lambda}\,d\lambda.$$

Hence noting that $f'(0) = 0$, we may find δ so small that

$$\limsup_{T\to\infty}\left|\int_{A/T}^{\delta}\frac{\cos 2\lambda T}{\lambda^2}[f(\lambda) - f(0)]\,d\lambda\right| < \varepsilon.$$

Then from (3.13) and (3.14),

$$\lim_{T\to\infty} T^2 S_{12} = 0\,. \tag{3.15}$$

Finally, we see, as in (3.11), (3.12) and (3.14) that

$$S_{13} = \frac{1}{T^2}\int\limits_{\lambda>\delta}\frac{f(\lambda) - f(0)}{\lambda^2}\,d\lambda - \frac{1}{T^2}\int\limits_{\lambda>\delta}\frac{\cos 2\lambda T}{\lambda^2}[f(\lambda) - f(0)]\,d\lambda + o\left(\frac{1}{T^2}\right)$$

as $T \to \infty$. The second integral of the right hand side is $o(1/T^2)$ because of the Riemann-Lebesgue lemma. Hence, using (3.10) and (3.15) we obtain

$$\lim_{T\to\infty} T^2 S_{13} = \int\limits_{\lambda>\delta}\frac{f(\lambda) - f(0)}{\lambda^2}\,d\lambda\,. \tag{3.16}$$

Combining (3.10), (3.13), (3.15) and (3.16) we obtain

$$\limsup_{T\to\infty} \left| T^2 S_1 - \int_{\lambda>\delta} \frac{f(\lambda)-f(0)}{\lambda^2}\, d\lambda \right| < C\varepsilon ,$$

C being a constant.

Exactly the same argument yields

$$\lim_{T\to\infty} T^2 S_2 = \int_{0+}^{\infty} \frac{f(\lambda)-f(0)}{\lambda^2}\, d\lambda .$$

Hence we have shown that

$$\lim_{T\to\infty} T^2 J_1 = 4 \int_{0+}^{\infty} \frac{f(\lambda)-f(0)}{\lambda^2}\, d\lambda . \tag{3.17}$$

Now consider J_2. Proceeding in the same way as with J_1 we write

$$J_2 = 2(U_1 + U_2) ,$$

where

$$U_1 = \int_{-\infty}^{\infty} \frac{(1-\cos\lambda T)^2}{(\lambda T + 2m\pi)(\lambda T + 2n\pi)} [f(\lambda) - f(0)]\, d\lambda , \tag{3.18}$$

$$U_2 = \int_{-\infty}^{\infty} \frac{(1-\cos\lambda T)^2}{(\lambda T + 2m\pi)(\lambda T - 2n\pi)} [f(\lambda) - f(0)]\, d\lambda . \tag{3.19}$$

Again U_1 is decomposed into three integrals

$$U_1 = \int_{|\lambda|<A/T} + \int_{\delta>|\lambda|\geqq A/T} + \int_{|\lambda|\geqq\delta} = U_{11} + U_{12} + U_{13} .$$

We note that for a pair of distinct integers m and n

$$\int_{-\infty}^{\infty} \frac{(1-\cos\lambda T)^2}{(\lambda T + 2m\pi)(\lambda T + 2n\pi)}\, d\lambda = 0 .$$

As in (3.10)

$$\lim_{T\to\infty} T^2 U_{11} = 0 . \tag{3.20}$$

If we write $(1-\cos\lambda T)^2 = 1 - 2\cos\lambda T + \cos^2\lambda T$ and consider the corresponding three integrals from U_{12}, we may verify as before that

$$\lim_{T\to\infty} T^2 U_{12} = 0 . \tag{3.21}$$

U_{13} is

$$\int_{|\lambda|>\delta} (1 - 2\lambda\cos\lambda T + \cos^2\lambda T) \frac{f(\lambda)-f(0)}{(\lambda T + 2m\pi)(\lambda T + 2n\pi)}\, d\lambda .$$

The integral which involves the $2\lambda\cos\lambda T$ term of the above integral is $o(1)$ because of the Riemann-Lebesgue lemma and the fact that

$$\int_{|\mu|>\delta T} \frac{\cos\mu}{\mu}\, d\mu = o(1) .$$

The integrals from the first and the last terms in the first bracket in the above integral are, as before,

$$\frac{2}{T^2}\int\limits_{\lambda>\delta}\frac{f(\lambda)-f(0)}{\lambda^2}\,d\lambda+o\left(\frac{1}{T^2}\right)$$

and

$$\frac{1}{T^2}\int\limits_{\lambda>\delta}\frac{f(\lambda)-f(0)}{\lambda^2}\,d\lambda+o\left(\frac{1}{T^2}\right)$$

respectively. Thus we have

(3.22) $$\limsup_{T\to\infty}\left|T^2U_{13}-3\int\limits_{\delta}^{\infty}\frac{f(\lambda)-f(0)}{\lambda^2}\,d\lambda\right|<C\varepsilon,$$

for some constant C. Also as in (3.13) and (3.15)

(3.23) $$\lim_{T\to\infty}T^2U_{13}=0.$$

Hence putting (3.20), (3.22) and (3.23) together, we have

(3.24) $$\lim_{T\to\infty}T^2U_1=3\int\limits_{0+}^{\infty}\frac{f(\lambda)-f(0)}{\lambda^2}\,d\lambda.$$

Similarly we obtain

(3.25) $$\lim_{T\to\infty}T^2U_2=3\int\limits_{0+}^{\infty}\frac{f(\lambda)-f(0)}{\lambda^2}\,d\lambda.$$

Hence

(3.26) $$\lim_{T\to\infty}T^2J_2=12\int\limits_{0+}^{\infty}\frac{f(\lambda)-f(0)}{\lambda^2}\,d\lambda.$$

(3.17) and (3.26) give us (3.2).

It remains to show (3.3). The derivation of (3.3) is analogous to that of (3.2). From (2.5)

$$E\,B_mB_n=K_1+K_2,$$

where K_1 and K_2 are given by (2.6) and (2.7). Using the fact that $f(\lambda)$ is an even function, K_1 can be written in the form

$$K_1=2(V_1-V_2),$$

where

$$V_1=\int\limits_{-\infty}^{\infty}\frac{\sin^2\lambda T}{(\lambda T+2m\pi)(\lambda T+2n\pi)}\,f(\lambda)\,d\lambda,$$

$$V_2=\int\limits_{-\infty}^{\infty}\frac{\sin^2\lambda T}{(\lambda T+2m\pi)(\lambda T-2n\pi)}\,f(\lambda)\,d\lambda.$$

Therefore V_1 and V_2 are the same as S_1 and S_2 respectively. Hence we have

$$\lim_{T\to\infty}T^2K_1=0.$$

Similarly,

$$\lim_{T\to\infty} T^2 K_2 = 0 .$$

This concludes the proof of (3.3).

4. The Behavior of Fourier Coefficients as $n \to \infty$

Let $X(t)$ be a stationary process conditioned as in **1**, except that here we allow $X(t)$ to be complex-valued. We consider the complex Fourier coefficients of $X(t)$, $0 < t < T$,

$$C_n = \frac{1}{2}(A_n - i B_n) = \frac{1}{T}\int_0^T X(t) e^{-2\pi i n t/T} dt . \tag{4.1}$$

We have

$$E C_n = 0 .$$

Now we consider the behavior of C_n when $n \to \infty$. Actually we shall deal with the magnitude of $\sum_{|n|>N} E|C_n|^2$.

It is easy to see that

$$E|C_n|^2 = \int_{-\infty}^{\infty} \frac{\sin^2(\lambda T/2)}{(\lambda T - 2n\pi)^2} dF(\lambda) , \tag{4.2}$$

where $F(\lambda)$ is the spectral distribution function of $X(t)$.

We shall begin with the following theorem.

Theorem 3. *We have*

$$\sum_{|n|>N} E|C_n|^2 = O(N^{-1}) + O(F(\infty) - F(\pi N/T)) + O(F(-\pi N/T)) , \tag{4.3}$$

where O is independent of N and T.

Proof. From (4.2), we have

$$\sum_{n=N}^{\infty} E|C_n|^2 = \sum_{n=N}^{\infty} \int_{-\infty}^{\infty} \frac{\sin^2(\lambda T/2)}{(2n\pi - \lambda T)^2} dF(\lambda)$$

$$= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \sum_{n=N}^{\infty} \frac{\sin^2 \pi\mu}{(n-\mu)^2} dF\left(\frac{2\pi\mu}{T}\right) = \frac{1}{4\pi^2}\left(\int_{-\infty}^{N/2} + \int_{N/2}^{\infty}\right) = \frac{1}{4\pi^2}(I_1 + I_2),$$

say. We then see that

$$|I_1| \leqq \int_{-\infty}^{N/2} \sum_{n=N}^{\infty} \frac{1}{(n - N/2)^2} dF\left(\frac{2\pi\mu}{T}\right) \leqq \sum_{m=[N/2]}^{\infty} \frac{1}{m^2} \int_{-\infty}^{N/2} dF\left(\frac{2\pi\mu}{T}\right)$$

$$\leqq \sum_{m=[N/2]}^{\infty} \frac{1}{m^2} [F(\infty) - F(-\infty)] = O\left(\frac{1}{N}\right).$$

As to I_2, we have

$$|I_2| \leqq \int_{N/2}^{\infty} \sum_{n=N}^{\infty} \frac{\sin^2 \pi(n-\mu)}{(n-\mu)^2} dF\left(\frac{2\pi\mu}{T}\right)$$

$$= \int_{N/2}^{\infty} O(1) dF\left(\frac{2\pi\mu}{T}\right) = O(1)[F(\infty) - F(\pi N/T)] .$$

A similar situation holds for $\sum_{n=-\infty}^{-N}$. This proves the theorem.

Theorem 4. *If*

$$\int_{-\infty}^{\infty} |x|^{\alpha} dF(x) < \infty\,, \qquad 0 \leqq \alpha < 2\,, \tag{4.4}$$

then

$$\sum_{|n|>N} E\,|C_n|^2 = o\left(\frac{1}{N^{\alpha}}\right), \tag{4.5}$$

as $N \to \infty$, T being fixed. If (4.4) *holds for $\alpha = 2$, then*

$$\sum_{|n|>N} E\,|C_n|^2 = O\left(\frac{1}{N^2}\right). \tag{4.6}$$

Proof. We have, for $0 \leqq \alpha \leqq 2$,

$$F(\infty) - F(\pi N/T) = \int_{\pi N/T}^{\infty} dF(\lambda) \leqq \left(\frac{T}{\pi N}\right)^{\alpha} \int_{\pi N/T}^{\infty} \lambda^{\alpha} dF(\lambda)\,, \tag{4.7}$$

$$F(-\pi N/T) = \int_{-\infty}^{-\pi N/T} dF(\lambda) \leqq \left(\frac{T}{\pi N}\right)^{\alpha} \int_{-\infty}^{-\pi N/T} |\lambda|^{\alpha} dF(\lambda)\,. \tag{4.8}$$

Hence if $0 \leqq \alpha < 1$, then (4.5) is a consequence of (4.3).

In order to obtain (4.5) for $1 \leqq \alpha < 2$ and (4.6), it is sufficient to show that I_1 in the proof of Theorem 3 is $o(1/N^{\alpha})$ for $1 < \alpha < 2$, and $O(1/N^2)$ for $\alpha = 2$. Actually we will show that

$$I_1 = \int_{-\infty}^{N/2} \sum_{n=N}^{\infty} \frac{\sin^2 \pi \mu}{(n-\mu)^2}\, dF\left(\frac{2\pi\mu}{T}\right) = O\left(\frac{1}{N^2}\right). \tag{4.9}$$

The corresponding fact for the series $\sum_{n=-\infty}^{-N} E\,|C_n|^2$ is obtained in a similar way.

We write (4.9) in the following way.

$$I_1 = \left(\int_0^{N/2} + \int_{-\infty}^{0}\right) \sum_{n=N}^{\infty} \frac{\sin^2 (n-\mu)\pi}{(n-\mu)^2}\, dF\left(\frac{2\pi\mu}{T}\right) = I_{11} + I_{12}\,,$$

say. Integration by parts shows that

$$\begin{aligned} I_{11} = &- \left[F(\infty) - F\left(\frac{2\pi\mu}{T}\right)\right] \sum_{n=N}^{\infty} \frac{\sin^2(n-\mu)\pi}{(n-\mu)^2}\Bigg|_{\mu=0}^{N/2} \\ &+ 2\int_0^{N/2} \left[F(\infty) - F\left(\frac{2\pi\mu}{T}\right)\right] \left[\sum_{n=N}^{\infty} \frac{\sin 2(n-\mu)\pi}{(n-\mu)^2} + \frac{\sin^2(n-\mu)\pi}{(n-\mu)^3}\right] d\mu\,. \end{aligned}$$

(Termwise differentiation is permitted.) Then

$$|I_{11}| = O\left(F(\infty) - F\left(\frac{\pi N}{T}\right)\right) \cdot O\left(\frac{1}{N}\right) + \int_0^{N/2} \left[F(\infty) - F\left(\frac{2\pi\mu}{T}\right)\right] \cdot O\left(\frac{1}{N^2}\right) d\mu\,, \tag{4.10}$$

since

$$\sum_{n=N}^{\infty} \frac{\sin 2(n-\mu)\pi}{(n-\mu)^2} = O\left(\frac{1}{N^2}\right), \quad \text{for every } \mu < N/2 .$$

The first term of the right hand side of (4.10) is, using (4.7),

$$o(1/N^{\alpha}) \cdot O(1/N) = o(1/N^2), \qquad (1 \leqq \alpha \leqq 2) .$$

Now

$$\int_0^{\infty} \left[F(\infty) - F\left(\frac{2\pi\mu}{T}\right)\right] d\mu = \frac{T}{2\pi} \int_0^{\infty} \lambda dF(\lambda) = O(1) .$$

Hence the second integral of the right hand side of (4.10) is $O(1/N^2)$.

In the similar manner

$$I_{12} = F\left(\frac{2\pi\mu}{T}\right) \sum_{n=N}^{\infty} \frac{\sin^2(n-\mu)\pi}{(n-\mu)^2} \Bigg|_{\mu=-\infty}^{0} - 2\pi \int_{-\infty}^{0} F\left(\frac{2\pi\mu}{T}\right) \times$$
$$\times \left[\sum_{n=N}^{\infty} \frac{\sin 2(n-\mu)\pi}{(n-\mu)^2} + \frac{\sin^2(n-\mu)\pi}{(n-\mu)^3}\right] d\mu .$$

Hence the integrated term vanishes and the last integral is $O(1/N^2)$. This proves (4.9).

5. The Mean Convergence and the Absolute Convergence of the Fourier Series

Let $X(t)$ be a complex-valued stationary process as in 4. Let $S_n(t)$ be the partial sum of the Fourier series of $X(t)$, $0 < t < T$. T is fixed. The following theorem is known [5].

Theorem 5. *For every t, $0 < t < T$,*

$$E\,|S_n(t) - X(t)|^2 \to 0 \tag{5.1}$$

as $n \to \infty$.

For completeness, we shall give a simple proof of (5.1). We note that

$$S_n(t) - X(t) = \frac{1}{T} \int_0^T [X(u) - X(t)]\, D_n(u-t)\, du , \tag{5.2}$$

where

$$D_n(u) = \sin(n + \tfrac{1}{2}) \frac{2\pi u}{T} \Big/ 2 \sin \frac{\pi u}{T} . \tag{5.3}$$

We have

$$\begin{aligned} E\,|S_n(t) - X(t)|^2 &= \frac{1}{T^2} \int_0^T \int_0^T E[X(u) - X(t)]\,[\overline{X(v)} - \overline{X(t)}] \times \\ &\quad \times D_n(u-t)\, D_n(v-t)\, du\, dv \\ &= \int_{-\infty}^{\infty} dF(\lambda) \left| \frac{1}{T} \int_0^T (e^{i(u-t)\lambda} - 1)\, D_n(u-t)\, du \right|^2 . \end{aligned} \tag{5.4}$$

Since it is known that the inner Dirichlet integral is bounded for all t and λ and converges to zero as n goes to infinity, the desired result follows.

Next we shall prove a theorem on the absolute convergence of the Fourier series of $X(t)$ in $0 < t < T$ that is the analogue of Bernstein's theorem B in **1**. Actually by means of a familiar technique (see [*12*] p. 135—136) Theorem 4 leads to the following theorem.

Theorem 6. *If*

$$\int_{-\infty}^{\infty} |x|^{\alpha} dF(x) < \infty \tag{5.5}$$

for some $\alpha > 1$, *then the Fourier series*

$$\sum_{n=-\infty}^{\infty} C_n e^{2in\pi t/T}, \quad 0 < t < T \tag{5.6}$$

converges absolutely with probability one.

Proof. If (5.5) holds for some α, then it also holds for all smaller values of α. Hence we may suppose that $1 < \alpha < 2$.

Consider

$$\sum_{n=1}^{\infty} E|C_n| = \sum_{\nu=1}^{\infty} \sum_{n=2^{\nu-1}+1}^{2^{\nu}} E|C_n| \leqq \sum_{\nu-1}^{\infty} \left(\sum_{n=2^{\nu-1}}^{2^{\nu}} E|C_n|^2 \right)^{1/2} (2^{\nu} - 2^{\nu-1})^{1/2}. \tag{5.7}$$

Because of (4.5), the last expression is not larger than

$$\sum_{\nu=1}^{\infty} O(2^{-\alpha(\nu-1)/2}) \cdot O(2^{\nu/2}) = \sum_{\nu=1}^{\infty} O(2^{-\nu(\alpha-1)/2}) = O(1). \tag{5.8}$$

Similarly we have

$$\sum_{n=-\infty}^{0} E|C_n| < \infty.$$

Therefore we have that $\sum_{n=-\infty}^{\infty} E|C_n| < \infty$. Hence the series $\sum_{-\infty}^{\infty} |C_n|$ converges with probability one. This proves the theorem.

Incidentally Theorem proves the following theorem.

Theorem 7. *If* (5.5) *is true for a given stationary processes* $X(t)$, *then* $X(t)$ *is continuous with probability one.*

When the process is a Gaussian process with condition (5.5), the same result has been shown by Derporte [*3*].

We mention that Theorems 6 and 7 are generalized in the following form.

Theorem 8. *In Theorems* 6 *and* 7, (5.5) *can be replaced by a weaker condition*

$$\int_{-\infty}^{\infty} |x| (\log^{+}|x|)^{\beta} dF(x) < \infty, \quad \beta > 2. \tag{5.9}$$

As in the proof of (4.5) we may show that if (5.7) is true, then

$$\sum_{|n| \geqq N} E|C_n|^2 = o\left(\frac{1}{N \log^{\beta} N}\right). \tag{5.10}$$

If we insert (5.10) in (5.7), then we have $\sum_{\nu=1}^{\infty} O(\nu^{-\beta})$ in place of (5.8), which leads to the same conclusion.

6. The Almost Everywhere Convergence of the Fourier Series

We shall consider the almost everywhere convergence of the Fourier series (5.6) of the stationary process $X(t)$ under consideration. The theorem below is an analogue of the Kolmogorov-Seliverstov theorem A in 1. For its proof we will require the following variant of Theorem 4.

Lemma 1. *If*

$$\int_{-\infty}^{\infty} \log^{+}|\lambda|\, dF(\lambda) < \infty, \tag{6.1}$$

then

$$\sum_{n=-\infty}^{\infty} \log(|n|+1) E|C_n|^2 < \infty, \tag{6.2}$$

where C_n is the Fourier coefficient of $X(t)$.

Proof. Using (4.2) we have

$$I = \sum_{n=2}^{\infty} E[\log n |C_n|^2] = \sum_{n=2}^{\infty} \log n \int_{-\infty}^{\infty} \frac{\sin^2(\lambda T/2\pi)}{(\lambda T - 2n\pi)^2} dF(\lambda)$$

$$= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \sum_{n \geqq \lambda T/2} \frac{\log n \cdot \sin^2(\lambda T/2\pi)}{(n - \lambda T/2\pi)^2} dF(\lambda) +$$

$$+ \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \sum_{2 \leqq n < \lambda T/2\pi} \frac{\log n \cdot \sin^2(\lambda T/2\pi)}{(n - \lambda T/2\pi)} dF(\lambda) = I_1 + I_2,$$

ay. Let us suppose $\lambda > 1$. Then the integrand in I_1 is not greater than

$$\begin{aligned} & O(\log\lambda) + \sum_{n=[\lambda T/2\pi]+1}^{\infty} \frac{\log n}{(n - \lambda T/2\pi)^2} \\ &= O(\log\lambda) + \sum_{k=1}^{\infty} \frac{\log(k + [\lambda T/2\pi] + 1)}{k^2} \\ &= O(\log\lambda) + O\left(\sum_{k=1}^{\infty} \frac{\log k}{k^2}\right) + O(\log\lambda) \sum_{k=1}^{\infty} \frac{1}{k^2} \\ &= O(\log\lambda). \end{aligned}$$

Again, the integrand in I_2 is

$$O(\log\lambda) + \sum_{1 \leqq k \leqq [\lambda T/2\pi]+1} \frac{O(\log([\lambda T/2\pi] + 1 - k))}{k^2}$$

$$= O(\log\lambda) + O(\log\lambda) \cdot \sum_{k=1}^{\infty} \frac{1}{k^2} = O(\log\lambda).$$

If $\lambda < 1$, the same result holds and (6.2) follows.

We now prove the following theorem.

Theorem 9. *If* (6.1) *holds, then the Fourier series* $\sum C_n e^{2\pi int/T}$ *of* $X(t)$, $0 < t < T$, *converges almost everywhere in* $0 < t < T$, *with probability one.*

Needless to say the convergence of the Fourier series means that of $\lim_{N\to\infty} \sum_{n=-N}^{N}$. The theorem is an easy consequence of the Kolmogorov-Seliverstov theorem A if Lemma 1 is applied. In fact, the conclusion of the lemma implies that

$$\sum |C_n|^2 \log(|n|+1) < \infty$$

with probability one, which in turn implies the conclusion of the theorem.

7. The Limit Joint Distribution of Fourier Coefficients

In this section we assume again that the stationary process $X(t)$ is real-valued and conditioned as in 1. A_n and B_n are Fourier cosine and sine coefficients of $X(t)$, $0 < t < T$, given by (1.6) and (1.7).

We shall study the joint limit distribution of the random variable

$$(\tfrac{1}{2}A_0, A_1, \ldots, A_n, B_1, \ldots, B_n) \tag{7.1}$$

when $T \to \infty$.

We further assume that $X(t)$ has a spectral density $f(\lambda)$. It is well known that in this case $X(t)$ can be written as

$$X(t) = \int_{-\infty}^{\infty} C(\lambda - t)\, dy(\lambda), \quad -\infty < t < \infty, \tag{7.2}$$

where $C(u)$ is a numerical function in $L_2(-\infty, \infty)$ and $y(\lambda)$ is a stochastic process with orthogonal increments such that

$$E\,|dy(\lambda)|^2 = d\lambda. \tag{7.3}$$

It is also known that the covariance function can be written in the form

$$\varrho(t) = \int_{-\infty}^{\infty} |\hat{C}(\lambda)|^2 e^{it\lambda} d\lambda, \tag{7.4}$$

(see [4] p. 532) where $\hat{C}$ is the Fourier transform in L_2 $\int_{-\infty}^{\infty} e^{it\lambda} C(t)\, dt$ of $C(t)$. We shall prove the following theorem.

Theorem 10. *Let* $X(t)$ *be the real-valued stationary process of the form* (7.2) *with* $C(u) \in L_2(-\infty, \infty)$, *and with* $y(\lambda)$ *having independent increments and satisfying* (7.3) *and* $E\,|dy(\lambda)|^3 = O(d\lambda)$. *Moreover if* $C(u) \in L_1 \cap L_3(-\infty, \infty)$, *then the joint distribution of the set of the Fourier coefficients of* $X(t)$ *in* $0 < t < T$,

$$(\tfrac{1}{2}A_0 T^{1/2}, A_1 T^{1/2}, \ldots, A_n T^{1/2}, B_1 T^{1/2}, \ldots, B_n T^{1/2}) \tag{7.5}$$

converges to $N(0, \tfrac{1}{2}c^2) * \prod^{2n*} N(0, c^2)$ *as* $T \to \infty$, *where*

$$c = \left|\int_{-\infty}^{\infty} C(w)\, dw\right|,$$

and $\prod^{2n*}$ *means* $2n$*-fold convolution.*

Lemma 2. (i) *If a random variable X is real and is such that $EX = 0$ and*

$$X = \int_{-\infty}^{\infty} C(\lambda)\, dy_\alpha(\lambda), \tag{7.6}$$

where $C(\lambda) \in L_2 \cap L_3(-\infty, \infty)$ and $y_\alpha(\lambda)$ is a stochastic process with independent increments with

$$E|dy_\alpha(\lambda)|^2 = \alpha\, d\lambda, \quad E\, dy_\alpha(\lambda) = 0, \quad E\,|dy_\alpha(\lambda)|^3 = O(\alpha d\lambda) \tag{7.7}$$

$\alpha > 0$ being a constant, then the characteristic function $f_\alpha(u)$ of X is given by

$$f_\alpha(u) = \exp\left(-\frac{\alpha u^2}{2} \int_{-\infty}^{\infty} C^2(\lambda)\, d\lambda + O(\alpha\, u^3)\right) \tag{7.8}$$

for small u.

(ii) *The distribution of $X/\alpha^{1/2}$ converges as $\alpha \to \infty$ to the normal distribution*

$$N(0, \int_{-\infty}^{\infty} C^2(\lambda)\, d\lambda).$$

This is substantially known (e. g. [*4*], [*5*]). For completeness, we shall give its proof.

If $C(\lambda)$ is a step function

$$\begin{aligned} C(\lambda) &= 0, & \lambda &< a_0, \\ &= c_j, & a_{j-1} \leqq \lambda &< a_j, \\ &= 0, & \lambda &\geqq a_n, \end{aligned} \tag{7.9}$$

where $a_0 < a_1 < \cdots < a_n$ are points on the real axis, then, by definition

$$\int_{-\infty}^{\infty} C(\lambda)\, dy_\alpha(\lambda) = \sum_{j=1}^{n} c_j (y_\alpha(a_j - 0) - y_\alpha(a_{j-1} + 0)). \tag{7.10}$$

For any $C(\lambda) \in L_2(-\infty, \infty)$, there is a sequence of step functions which converges to $C(\lambda)$ in L_2-mean and the integral is defined as the limit in $L_2(\Omega)$ of the right hand side of (7.10).

We take a sequence of step functions of the form (7.9) which converges to $C(\lambda)$ in L_2-mean. Then the characteristic function of X is the limit of the characteristic function of the right hand side of (7.10). Hence

$$\begin{aligned} f_\alpha(u) &= E(\exp(i\,u \int_{-\infty}^{\infty} C(\lambda)\, dy_\alpha(\lambda))) \\ &= \lim E(\exp[i\,u \sum_{j=1}^{n} c_j (y_\alpha(a_j - 0) - y_\alpha(a_{j-1} + 0))]) \\ &= \lim \prod_{j=1}^{n} E(\exp[i\,u\, c_j (y_\alpha(a_j - 0) - y_\alpha(a_{j-1} + 0))]) \\ &= \lim \prod_{j=1}^{n} E(1 + i\,u\, c_j (y_\alpha(a_j - 0) - y_\alpha(a_{j-1} + 0)) \\ &\quad - \frac{u^2 c_j^2}{2} (y_\alpha(a_j - 0) - y_\alpha(a_{j-1} + 0))^2 + \\ &\quad + O(c_j^3 u^3 |y_\alpha(a_j - 0) - y_\alpha(a_{j-1} + 0)|^3) \\ &= \lim \prod_{j=1}^{n} \left(1 - \frac{\alpha u^2 c_j^2}{2} (a_j - a_{j-1}) + O(\alpha\, u^3 c_j^3 (a_j - a_{j-1}))\right). \end{aligned}$$

Here we have used (7.7). Since

$$\sum_{j=1}^{n} c_j^2 (a_j - a_{j-1}) \to \int_{-\infty}^{\infty} C^2(\lambda)\, d\lambda,$$

we have (7.8). (ii) is obvious from (7.8).

Lemma 3. *Suppose that a real-valued function* $\psi_\alpha(\lambda) \in L_2(-\infty, \infty)$ *satisfies*

$$\int_{-\infty}^{\infty} |\psi_\alpha(\lambda) - \psi(\lambda)|^2 d\lambda \to 0 \quad \text{as} \quad \alpha \to 0 \tag{7.11}$$

for some $\psi(\lambda) \in L_2 \cap L_3(-\infty, \infty)$. *Let* $y_\alpha(\lambda)$ *be a stochastic process with independent increments, satisfying* (7.7). *Then the characteristic function of*

$$Y_\alpha = \frac{1}{\alpha^{1/2}} \int_{-\infty}^{\infty} \psi_\alpha(\lambda)\, dy_\alpha(\lambda) \tag{7.12}$$

converges uniformly in every finite interval as $\alpha \to 0$ *to the characteristic function of* $N(0, \int_{-\infty}^{\infty} \psi^2(\lambda) d\lambda)$.

Proof. Denoting

$$X_\alpha = \frac{1}{\alpha^{1/2}} \int_{-\infty}^{\infty} \psi(\lambda)\, dy_\alpha(\lambda)$$

(this integral exists), we have

$$|E e^{iX_\alpha u} - E^{iY_\alpha u}| \leqq |u|\, E|X_\alpha - Y_\alpha| \leqq |u|\, (E|X_\alpha - Y_\alpha|^2)^{1/2}$$

which converges to zero, since

$$E|X_\alpha - Y_\alpha|^2 = \int_{-\infty}^{\infty} |\psi_\alpha(\lambda) - \psi(\lambda)|^2 d\lambda.$$

On the other hand $E e^{iX_\alpha u}$ converges to the characteristic function of

$$N(0, \int_{-\infty}^{\infty} \psi^2(\lambda)\, d\lambda)$$

by Lemma 2. This shows Lemma 3.

We are now in a position to prove Theorem 10. The characteristic function of (7.1) is written as

$$\begin{aligned} &f(\sigma_0, \sigma_1, \ldots, \sigma_n, \tau_1, \ldots, \tau_n) \\ &= E\left(\exp\left[\frac{2i}{T^{1/2}} \int_0^T X(t) \left(\frac{1}{2}\sigma_0 + \sum_{j=1}^{n} \left(\sigma_j \cos\frac{2j\pi t}{T} + \tau_j \sin\frac{2j\pi t}{T}\right)\right) dt\right]\right) \\ &= E\left(\exp\left[\frac{2i}{T^{1/2}} \int_0^T X(t)\, \varphi_n(t, T)\, dt\right]\right), \end{aligned} \tag{7.13}$$

where

$$\varphi_n(t, T) = \frac{1}{2}\sigma_0 + \sum_{j=1}^{n}\left(\sigma_j \cos + \frac{2j\pi t}{T} + \tau_j \sin\frac{2j\pi t}{T}\right). \tag{7.14}$$

Hence from (7.6), we have

$$\frac{2}{T^{1/2}}\int_0^T X(t)\,\varphi_n(t, T)\,dt = \frac{2}{T^{1/2}}\int_0^T \varphi_n(t, T)\,dt\int_{-\infty}^{\infty} C(t-\lambda)\,dy(\lambda)$$
$$= \int_{-\infty}^{\infty} dy(\lambda)\,\frac{2}{T^{1/2}}\int_0^T \varphi_n(t, T)\,C(t-\lambda)\,dt.$$

It is easy to see that the interchange of integration is permitted. The above expression is equal to

$$2\,T^{1/2}\int_{-\infty}^{\infty} dy(T v)\int_0^1 \varphi_n(u\,T, T)\,C(u\,T - v\,T)\,du, \tag{7.15}$$

where

$$\begin{aligned}\varphi_n(u\,T, T) &= \tfrac{1}{2}\sigma_o + \sum_{j=1}^{n}(\sigma_j\cos 2\pi j u + \tau_j \sin 2\pi j u)\\ &= \varphi_n(u), \quad 0 \leqq u \leqq 1\end{aligned} \tag{7.16}$$

is independent of T.

Now write

$$Ty(T v) = y_T(v). \tag{7.17}$$

This has the mean square increment given by

$$E\,|\,dy_T(v)\,|^2 = T\,dv, \quad E\,|\,dy_T(v)\,|^3 = O(T dv) \tag{7.18}$$

Now the function $T^{1/2}f(\sigma_0, \sigma_1, \ldots, \sigma_n, \tau_1, \ldots, \tau_n)$ is, from (7.15), the value for $t = 1$ of the characteristic function $g(t)$ of

$$2\,T^{-1/2}\int_{-\infty}^{\infty} dy_T(v)\int_0^1 \varphi_n(u)\,T\,C(T(u-v))\,du. \tag{7.19}$$

Then it is sufficient to show that the characteristic function of (7.19) converges to the product of the normal characteristic function.

(7.19) is of the form (7.12) with $\alpha = T$ and

$$\psi_\alpha(v) = 2\int_0^1 \varphi_n(u)\,T\,C(T(u-v))\,du \equiv \psi(v, T).$$

Take

$$\psi(v) = 2\,\varphi_n(v)\int_{-\infty}^{\infty} C(w)\,dw, \quad 0 \leqq v \leqq 1 = 0, \quad \text{elsewhere.}$$

We are going to show that

$$(7.20)\qquad \int_{-\infty}^{\infty} |\psi(v, T) - \psi(v)|^2 dv \to 0, \quad \text{as} \quad T \to \infty.$$

Since $\varphi_n(u)$ is continuous and $C(w) \in L_1(-\infty, \infty)$, we may see that

$$(7.21)\qquad \psi(v, T) \to \psi(v)$$

for every v except $v = 0$ and $v = 1$. Because we see that

$$(7.22)\qquad \psi(v, T) = 2\int_{-Tv}^{T-Tv} \varphi_n\left(v + \frac{w}{T}\right) C(w)\, dw = 2\int_{-\infty}^{\infty} \varphi_n\left(v + \frac{w}{T}\right) C(w)\, dw,$$

defining $\varphi_n(u) = 0$ outside $0 \leqq u \leqq 1$ and since $\varphi_n(u)$ is bounded the dominated convergence theorem applies to get (7.21).

Furthermore $\psi(v, T)$ is uniformly bounded which is seen from (7.22). Hence in (7.21), the convergence is the bounded convergence. Therefore in order to show (7.20), it is sufficient to show that

$$(7.23)\qquad \int_{|v|>A} |\psi(v, T) - \psi(v)|^2 dv \to 0, \quad \text{as} \quad T \to \infty,$$

for some A.

Take $A > 1$ arbitrarily and fix it. Now

$$\int_{-\infty}^{-A} |\psi(v, T) - \psi(v)|^2 dv = \int_{-\infty}^{-A} |\psi_T(v)|^2 dv \leqq K \int_{-\infty}^{-A} |\psi_T(v)|\, dv$$

for some constant K, since $\psi_T(v)$ is uniformly bounded. Hence the last expression does not exceed

$$\begin{aligned} &K \int_{-\infty}^{-A} dv \int_0^1 |\varphi_n(u)|\, T\, |C(T(u - v))|\, du \\ &\qquad = K \int_0^1 |\varphi_n(u)|\, du\, T \int_{-\infty}^{-A} |C(T(u - v))|\, dv \\ &\qquad = K \int_0^1 |\varphi_n(u)|\, du \int_{A+Tu}^{\infty} |C(w)|\, dw \end{aligned}$$

which converges to zero as $T \to \infty$.

Similarly

$$\begin{aligned} \int_A^{\infty} |\psi(v, T) - \psi(v)|^2 dv &= \int_A^{\infty} |\psi(v, T)|^2 dv \\ &\leqq K \int_A^{\infty} dv \int_0^1 |\varphi_n(u)|\, T\, |C(T(u - v))|\, du \\ &= K \int_0^1 |\varphi_n(u)|\, du \int_{-\infty}^{T(u-A)} |C(w)|\, dw \end{aligned}$$

which converges to zero as $T \to \infty$. Hence we have shown (7.23). Finally it is obvious that $\varphi(v)$ $L_3(-\infty, \infty)$, since it belongs to $L_2(-\infty, \infty)$ and is bounded. Hence from Lemma 3 we obtain that the characteristic function of (7.19) converges to the characteristic function of $N(0, \int_0^1 \psi^2(\lambda)\,d\lambda)$. This shows that $T^{1/2} f(\sigma_0, \sigma_1, \ldots, \sigma_n, \tau_1, \ldots, \tau_n)$ converges to

$$\begin{aligned}\exp(-\tfrac{1}{2}\int_0^1 \psi^2(\lambda)\,d\lambda)& \\ &= \exp(-2\left|\int_{-\infty}^{\infty} C(w)\,dw\right|^2 \int_0^1 \varphi_n^2(v)\,dv) \\ &= \exp(-\left|\int_{-\infty}^{\infty} C(w)\,dw\right|^2 (\tfrac{1}{4}\sigma_0^2 + \tfrac{1}{2}\sum_{j=1}^{n}(\sigma_j^2 + \tau_j^2))) \\ &= e^{-\sigma 20 C^2/4} \prod_{j=1}^{n} e^{-C^2\sigma_j^2/2} \cdot \prod_{k=1}^{n} e^{-C^2\tau_k^2/2},\end{aligned}$$

where $C = \left|\int_{-\infty}^{\infty} C(w)\,dw\right|$. This proves the theorem.

Added in proof: Theorem 7 may be derived also from M. Loèvés lemma, Supplement to P. Lévy, Processus stochastique et mouvement brownien (1948), p. 331. For the stationary Gaussian process, a better result is known, Yu K. Balayev, Continuity and Hölder's conditions for sample functions of stationary Gaussian processes. Fourth Berkeley Symposium vol. 2, 23—33 (1961).

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