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A Theorem about Infinitely Divisible Distributions

Meyer Dwass

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Suppose that $p_0, p_1, \ldots$ are probability masses on $0, 1, \ldots,$ with generating function $P(t) = \sum p_n t^n$. Suppose the distribution is not defective and has first moment not greater than 1. That is $P(1) = 1$ and $P'(1) \leqq 1$. Then $u = u(t) = t/P(t)$ defines a one-to-one mapping of $[0, 1]$ onto itself with inverse $t = h(u)$. Suppose henceforth $p_1 \neq 1$.

Theorem. *$h(u)$ is a probability generating function of an infinitely divisible distribution which assigns its mass to $1, 2, \ldots$ and has first moment equal to $(1 - P'(1))^{-1}$. This is equivalent to saying that*

$$h(u) = u \exp \sum_{n=1}^{\infty} a_n (u^n - 1), \qquad 0 \leqq u \leqq 1,$$

with $a_n \geqq 0$ and $\sum a_n < \infty$. $h'(1) = (1 - P'(1))^{-1}$.

The theorem will be proved with the aid of the following lemma which was proved as Corollary 2, Theorem 1 in [1].

Lemma. *Suppose $X_1, X_2, \ldots$ is a sequence of independent, identically distributed, non-negative, integer-valued random variables. Then*

$$P\left(\sum_{i=1}^{n} X_i < n,\ n = 1, 2, \ldots\right) = \begin{cases} 1 - E(X_1) & \text{if } E(X_1) \leqq 1, \\ 0 & \text{if } E(X_1) > 1. \end{cases}$$

Proof of theorem. For s in $(0, 1)$ define

$$p_n^{(s)} = p_n s^n / P(s), \qquad n = 0, 1, \ldots.$$

The $p_n^{(s)}$ are probabilities which sum to 1 with probability generating function

$$\sum_n p_n^{(s)} t^n = P(st)/P(t), \qquad 0 \leqq t \leqq 1.$$

It is easy to verify that

$$\sum_n n p_n^{(s)} = s P'(s)/P(s) < 1 \quad \text{if} \quad 0 < s < 1. \tag{1}$$

Let $X_1^{(s)}, X_2^{(s)}, \ldots$ be a sequence of independent and identically distributed random variables such that

$$P(X_i^{(s)} = n) = p_n^{(s)}, \qquad n = 0, 1, \ldots,$$

and let

$$T_n^{(s)} = X_1^{(s)} + \cdots + X_n^{(s)}.$$

It follows from the definition of $p_n^{(s)}$ that

$$P(T_n^{(s)} = k) = P(T_n^{(1)} = k) s^k / P^n(s). \tag{2}$$

We now assert the following relationship:

$$1 = [1 - s\,P'(s)/P(s)] + \sum_{n=1}^{\infty} P(T_n^{(s)} = n)\,[1 - s\,P'(s)/P(s)]. \tag{3}$$

This simply says that $T_n^{(s)} < n$ for all $n = 1, 2, \ldots$ or $T_n^{(s)} = n$ only finitely often, with the probability of last occurrence at index n being

$$P(T_n^{(s)} = n)\,[1 - s\,P'(s)/P(s)].$$

This follows from the Lemma with $s\,P'(s)/P(s)$ playing the role of $E(X_1)$. By (2) relationship (3) can be written

$$1 + \sum_{n=1}^{\infty} P(T_n^{(1)} = n)\,[s/P(s)]^n = (1 - s\,P'(s)/P(s))^{-1}. \tag{4}$$

It is easy to verify that $s/P(s)$ defines a one-to-one monotone mapping of $[0, 1]$ onto $[0, 1]$. Let $s = h(u)$ be the inverse of $u = s/P(s)$. It is also easy to verify that h is differentiable in $(0, 1)$ and that

$$[1 - s\,P'(s)/P(s)]^{-1} = u\,h'(u)/h(u)$$

when s is expressed as $h(u)$. Hence (4) is equivalent to

$$1 + \sum_{n=1}^{\infty} P(T_n = n)\,u^n = u\,h'(u)/h(u) \tag{5}$$

(writing $T_n = T_n^{(1)}$ for simplicity).

Integrating (5) gives that

$$h(u) = c\,u \exp \sum_{n=1}^{\infty} u^n\,P(T_n = n)/n, \qquad 0 < u < 1. \tag{6}$$

This already shows that $h(u)$ is the probability generating function of an infinitely divisible distribution, but it is interesting to explicitly determine the constant of integration c. Since $p_1 \neq 1$, $P'(1) \neq 1$ implies that $p_0 > 0$. Since

$$\lim_{u \downarrow 0} h(u)/u = \lim_{s \downarrow 0} P(s) = p_0$$

it follows that $c = p_0 > 0$. On the other hand,

$$\lim_{u \uparrow 1} h(u)/u = \lim_{s \uparrow 1} P(s) = 1 = c \exp \sum_{n=1}^{\infty} P(T_n = n)/n$$

(This implies incidentally that $\sum_n P(T_n = n)/n < \infty$ since $p_0 < 1$.) Hence

$$h(u) = u \exp \sum_{n=1}^{\infty} \frac{P(T_n = n)}{n}\,(u^n - 1).$$

That $h'(1) = (1 - P'(1))^{-1}$ follows from (4).

Example. Suppose $P(t) = 1/2 + (1/2)t^2$. Then an elementary computation shows that

$$h(u) = [1 - (1 - u^2)^{1/2}]/u.$$

h is the generating function of the time it takes simple symmetric random walk to travel from 0 to 1, which by the theorem is an infinitely divisible random variable. Similarly, the return time to the origin, whose generating function is $uh(u)$ is also infinitely divisible. By a similar computation, these waiting time distributions for the non-symmetric random walk are also infinitely divisible.

Reference

[1] DWASS, M.: A fluctuation theorem for cyclic random variables. Ann. math. Statistics **33**, 1450—1454 (1962).

Dr. M. DWASS
Dept. of Mathematics
Northwestern University
Evanston, Ill. 60201
USA