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On the Nuclearity of a Dual Space with the Convergence in Probability Topology

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Summary. Let μ be a probability measure on a separable locally convex Fréchet space E and let s_μ denote the topology on E' of the convergence in μ. Then (E', s_μ) is nuclear iff $\mu((E', s_\mu)')=1$.

1. Introduction

Let E be a locally convex separable Fréchet space and μ a full[1] probability measure on E. If we endow the dual space E' with the topology s_μ of the convergence in μ, then $J_\mu=(E', s_\mu)'$ is a linear subspace of E of the form $J_\mu=\bigcup_{n=1}^{\infty} K_n$, K_n convex and compact [9, Prop. 2]. It is called the linear kernel of μ.

The purpose of this paper is to prove that (E', s_μ) is nuclear iff $\mu(J_\mu)=1$. On one hand the above result seems to be very closely related to the celebrated Duality Principle of L. Schwartz [6], on the other it is a completion of a known result (W. Slowikowski [8] and W. Smolenski [9]) which asserts that (E', s_μ) is locally convex iff J_μ is a pre-support of μ (i.e. if μ is scalarly concentrated on convex and compact subsets of J_μ).

As an application we give characterization of nuclear separable subspaces of L_0. We also answer a question of A. Tortrat [10].

2. Preliminaries

In this section we shall fix terminology and notation. We shall also prove two lemmas of which the first is crucial.

If K is an absolutely convex and compact subset of E then E_K denotes a normed space obtained by taking K as a unit ball in span K. Let E_0 be a

[1] μ is full if no proper closed subspace of E is of measure μ one. This assumption is not essential for the results of this paper. However for the sake of simplicity we assume it from now on

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linear subspace of E. E_0 is called conuclear if $E_0 = \bigcup_{n=1}^{\infty} K_n$, K_n absolutely convex and compact, $K \subset K_{n+1}$ and for every n there exits $m > n$ such that the inclusion $E_{K_n} \hookrightarrow E_{K_m}$ is a nuclear operator.

If U is a symmetric neighborhood of 0 in E' then $\hat{E}_U$ denotes a normed space $(E'/\ker p_U, p_U)$, where p_U is the Minkowski functional of U. If $W \subset U$ is another symmetric neighborhood then $\pi_{W,U}$ denotes the canonical projection from $\hat{E}_W$ onto $\hat{E}_U$. If E and F are linear spaces in duality and Z is a subset of E then Z^0 denotes the polar of Z; $Z^0 = \{f \in F\colon |\langle e, f\rangle| \leqq 1$, for every $e \in Z\}$. A cylinder measure μ on E is said to be scalarly concentrated on a family $\mathscr{A}$ of subsets of E if for every $\varepsilon > 0$ there exists $A \in \mathscr{A}$ such that for every $f \in A^0$

$$\mu(\{e \in E\colon |\langle e, f\rangle| \leqq 1) > 1 - \varepsilon.$$

If X and Y are normed spaces and $0 < p < +\infty$, then a linear operator $u\colon X \to Y$ is said to be p-absolutely summing if there exists a constant C such that for each $x_1, \ldots, x_n \in E$

$$\sum_{i=1}^{n} \|u(x_i)\|^p \leqq C \sup_{\substack{x' \in X' \\ \|x'\| \leqq 1}} \sum_{i=1}^{n} |\langle x_i, x'\rangle|^p.$$

u is said to be 0-absolutely summing if there exists $0 < \varepsilon < 1$ and $\delta > 0$ such that if ν is a purely atomic probability measure on X and for every $x' \in X'$

$$\nu(\{x \in X\colon |\langle x', x\rangle| > \delta \|x'\|\}) < \delta$$

then

$$\nu(\{x \in X\colon \|u(x)\| \leqq 1) > 1 - \varepsilon.$$

Lemma 1. *Let K and L be absolutely convex and compact subsets of E and let κ and λ be positive real numbers. Suppose that $\mu(K) > 1 - \kappa$ and that $L \subset (A_\lambda)^0$, where $A_\lambda = \{f \in E'\colon \mu(\{e \in E\colon |\langle e, f\rangle| > 1\}) < \lambda\}$. If $\kappa < \lambda$ then the projection $\pi\colon \hat{E}_{K^0} \to \hat{E}_{L^0}$ is 0-absolutely summing.*

Proof. $\kappa < \lambda$ implies that $L \subset K$ and, consequently, that π is well defined and continuous. Let ε and δ be real numbers such that $0 < \varepsilon < 1$, $0 < \delta < 1$ and $(1-\kappa)(1-\delta) > 1 - \varepsilon\lambda$. Let ν be a purely atomic probability measure on E_{K^0} such that $\forall e \in K$ $\nu(\{f \in \hat{E}_{K^0}\colon |\langle e, f\rangle| > \delta\}) < \delta$. Thus $K \subset \{e \in E\colon \nu(\{f \in E_{K^0}\colon |\langle e, f\rangle| > 1\}) < \delta$. Hence $\mu\{e \in E\colon \nu(\{f \in E_{K^0}\colon |\langle e, f\rangle| > 1\}) < \delta\} > 1 - \kappa$. By the Fubini theorem

$$\mu \otimes \nu \{(e, f)\colon |\langle e, f\rangle| \leqq 1\} > (1-\kappa)(1-\delta) > 1 - \varepsilon\lambda.$$

Using the Fubini theorem once again we get that

$$\nu(A_\lambda) = \nu\{f \in \hat{E}_{K^0}\colon \mu(e \in E\colon |\langle e, f\rangle| > 1\}) < \lambda\} > 1 - \varepsilon.$$

Since $L \subset (A_\lambda)^0$ this implies that $\nu(\{f \in \hat{E}_{K^0}\colon \|\pi(f)\|_{\hat{E}_{L^0}} \leqq 1\}) > 1 - \varepsilon$. So we have proved that there exist $0 < \varepsilon < 1$ and $0 < \delta < 1$ such that for every purely atomic probability measure ν on $\hat{E}_{K^0}$ if for every $e \in K$ $\nu(\{f \in \hat{E}_{K^0}\colon |\langle e, f\rangle| > \delta\}) < \delta$ then $\nu(\{f \in \hat{E}_{K^0}\colon \|\pi(f)\|_{\hat{E}_{L^0}} \leqq 1\}) > 1 - \varepsilon$. Since K is a weakstar dense subset of the unit ball of $(\hat{E}_{K^0})'$ this proves that π is 0-absolutely summing.

Lemma 2. *Let* $E_0=\bigcup_{n=1}^{\infty} K_n$, K_n *absolutely convex and compact, be a linear subspace of* E. *If* E_0 *is conuclear then there exist a probability measure.* $\tilde{\mu}$ *on* E *such that* $\tilde{\mu}(E_0)=1$ *and* $J_{\tilde{\mu}}=E_0$.

Proof. Since E_0 is conuclear we can assume that for every n E_{K_n} is a Hilbert space, $E_{K_n}\subset E_{K_{n+1}}$ and the inclusion i_n: $E_{K_n}\to E_{K_{n+1}}$ is a Hilbert-Schmidt operator (cf. L. Schwartz [7], Part 2, Ch. 4). Let γ_n be the canonical gaussian cylinder measure on E_{K_n}. Then $\tilde{\gamma}_n=i_n(\gamma_n)$ is a probability measure on $E_{K_{n+1}}$ and $J_{\tilde{\gamma}_n}=E_{K_n}$ (because in the case of a gaussian measure the kernel J coincides with RKHS - (Reproducing Kernel Hilbert Space). If we put $\tilde{\mu}=\sum 2^{-n}\tilde{\gamma}_n$ it is easy to see that $\tilde{\mu}(E_0=1)$ and $E_0=\bigcup_{n=1}^{\infty} J_{\tilde{\gamma}_n}\subset J_{\tilde{\mu}}$. This implies that $E_0=J_{\tilde{\mu}}$ (C. Borell, [2]). This completes the proof.

3. Main Result

Theorem 1. *Let* E *be a locally convex separable Fréchet space and* μ *a probability measure on* E. *Then the following conditions are equivalent.*

(a) $\mu(J_\mu)=1$;

(b) (E', s_μ) *is nuclear;*

(c) J_μ *is conuclear and* μ *is scalarly concentrated on convex compact subsets of* J_μ;

(d) (E', s_μ) *is locally convex and there exists a probability measure* $\tilde{\mu}$ *on* E *such that* $J_{\tilde{\mu}}=J_\mu$ *and* $\tilde{\mu}(J_{\tilde{\mu}})=1$.

Proof. We shall prove that (a)$\to$(b)$\to$(c)$\to$(d)$\to$(b) and (c)$\to$(a). (a)$\to$(b): Let $J_\mu=\bigcup K_n$, K_n absolutely convex and compact, $K_n\subset K_{n+1}$. Since $\mu(J_\mu)=1$, the topology s_μ on E' coincides with the topology of the uniform convergence on K_n's ([9], Theorem 2). We have to prove that for every n there exists m such that $\pi_{m,n}$ $\hat{E}_{K_m^0}\to\hat{E}_{K_n^0}$ is nuclear. It is known that the composition of two 2-absolutely summing operators is nuclear (A. Pietsch [5]) and that every 0-absolutely summing operator is p-absolutely summing for every $p>0$ (S. Kwapien [3], cf. L. Schwartz [6]). Thus it is enough to prove that for every n there exists m such that $\pi_{m,n}$ in 0-absolutely summing. But this follows directly from the Lemma 1. Indeed, given n there exist $\alpha>0$ and $0<\lambda<1$ such that $\alpha K_n\subset A_\lambda$. Since $\mu(J_\mu)=1$ there exists m such that $\mu(K_m)>k-\lambda/2$. Applying Lemma 1 we get that π: $\hat{E}_{K_m}\to\hat{E}_{\alpha K_n}$ is 0-absolutely summing, which is equivalent to 0-absolute summability of $\pi_{m,n}$. (b)$\to$(c): Since $J_\mu=(E', s_\mu)'$, nuclearity of (E', s_μ) implies co-nuclearity of J_μ (cf. L. Schwartz [7], Part 2, Chapter 4). On the other hand (E', s_μ)-locally convex implies that μ is scalarly concentrated on convex and compact subsets of J_μ (W. Slowikowski [8]; or [9], Theorem 1). (c)$\to$(d). If μ is scalarly concentrated on convex and compact subsets of J_μ then (E', s_μ) is locally convex (W. Slowikowski [8]; or [9], Theorem 1). The second part of (d) follows directly from the Lemma 2. (d)$\to$(b): From the proofs of previous implications it is clear that $(E', s_\mu)=(E', s_{\tilde{\mu}})$ and that $(E', s_{\tilde{\mu}})$ is nuclear (c)$\to$(a):

This is precisely the celebrated Minlos Theorem (Minlos [4], cf. L. Schwartz [7], p. 233). The proof is finished.

4. Application

Theorem 2. *Let X be a linear space of random variables on a probability space (Ω, P). Let us endow X with the topology s of the convergence in probability and assume that X is separable. Then the following conditions are equivalent:*

(i) (X, s) *is nuclear,*

(ii) *there exists* $\Omega_0 \subset \Omega$, $P(\Omega_0)=1$ *such that the topology s on X coincides with the topology of pointwise convergence on* Ω_0.

Proof. If X is finite dimensional the theorem is trivial. If not let (ξ_n) be a linearly independent sequence of random variables dense in X. Then (ξ_n) defines a measurable mapping T from Ω into the space of sequences R^∞; $T(\omega) =(\xi_n(\omega))$. Let μ denote the image of P under T; $\mu=P\circ T^{-1}$. T induces a map T^* from the dual R_0^∞ of R^∞ into X; $T^*(f)=f\circ T$. It is easy to see that T^* constitutes a linear and topological isomorphism between (R_0^∞, s_μ) and $(\text{span}(\xi_n), s)$. Thus if (X, s) is nuclear, so is (R_0^∞, s_μ), and by Theorem 1 $\mu(J_\mu)=1$. Since $J_\mu=(R_0^\infty, s_\mu)'$ to get (ii) it is enough to put $\Omega_0=T^-(J_\mu)$.

Conversely, if (ii) is fulfilled then $T(\Omega_0)\subset J_\mu$. Hence $\mu(J_\mu)=1$ and, consequently, (R_0^∞, s_μ) is nuclear, which gives (i). This finishes the proof.

5. Remarks

1. There exists a nondegenerate probability measure μ such that $J_\mu=\{0\}$ [9, Example 2]. Thus the local convexity of (E', s_μ) is essential in (d).

2. A linear subspace of E is called μ-Lusin-measurable if it contains convex and compact sets of measure μ arbitrarily close to one. C. Borell [1] proved that the intersection of all μ-lusin-measurable subspaces is equal to the kernel J_μ. Thus, by Theorem 1, the existence of a minimal μ-Lusin-measurable subspace is equivalent to the nuclearity of (E', s_μ).

3. A. Tortrat [10, p. 68], asked whether in every Banach space one can find a probability measure μ such that $\mu(J_\mu)=1$ and μ does not charge finite dimensional subspaces. Lemma 2 gives the existence of a measure with the properties listed above in every topological vector space which contains at least one infinite dimensional convex compact subset. Namely every such space contains a conuclear subspace. Indeed, let K be an infinite dimensional convex compact set. We can assume that K is symmetric. Then the Banach space E_K contains a closed infinite dimensional subspace with a basis (C. Bessega and A. Pelczynski [1]). Let (e_n) be this basis and suppose that $\|e_n\|=1$. It is easy to see that $E_0=\text{span} \bigcup_{k=1}^{\infty} c1 \text{ conv} \left(\left\{\frac{n^{2k}}{n!} e_n\right\}_{n=1}^{\infty}\right)$ is a conuclear subspace of E_K, and, consequently, of a given topological vector space.

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