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The Central Limit Problem for Mixing Sequences of Random Variables

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Summary. In this paper the central limit problem is solved for sums of random variables having bounded variances and satisfying certain mixing conditions. In case of a stochastic process these mixing conditions essentially say that as time passes events concerning the "future" of the process are almost independent from the events in the "past". It turns out that the class of limit laws for sums of mixing random variables is exactly the same as for the bounded variances case of independent random variables. We also shall give criteria for convergence to any specified law of this class of possible limit laws. Finally we shall derive the central limit theorem involving a kind of Lindeberg-Feller condition and as a corollary thereof a kind of Ljapounov theorem.

1. Introduction

In recent years several authors investigated stochastic processes satisfying certain "mixing conditions". The first paper on this subject was by Rosenblatt [9] who introduced the notion of a stochastic process satisfying a "strong mixing condition": Let $\langle x_n, n=1,2,\ldots\rangle$ be a stochastic process and denote by $\mathfrak{M}_{a,b}$ the σ-algebra generated by the events $1 \leqq a \leqq n \leqq b \leqq \infty$. The process is said to satisfy a "strong mixing condition" if for all $A \in \mathfrak{M}_{1,t}$ $B \in \mathfrak{M}_{t+n,\infty}$

$$|P(AB) - P(A)P(B)| \leqq \alpha(n) \downarrow 0 \qquad (n \to \infty). \tag{1.1}$$

Later several other papers by Rosenblatt and Blum [11], Rozanov [12], Volkonskii and Rozanov [15], Ibragimov [2], Rjauba [9], Statuljavičjus [14], Philipp [4], Serfling [13] and others also dealt with stochastic processes satisfying some kind of mixing conditions. Roughly speaking, all these mixing conditions say that the dependence between the random variables is the weaker the farther they are apart, or else the dependence between the end of the process and its beginning is weak. In the second chapter some of these mixing conditions are discussed in detail.

In most of the papers mentioned above, sufficient conditions are given for the central limit theorem to hold, and it is either assumed that the process is stationary in the weak or the strict sense or else heavy restrictions on the growth of the variances of the partial sums are imposed.

In the present paper the central limit problem is solved for sums of random variables satisfying certain mixing conditions in case that the variances are bounded. It turns out that the class of possible limit laws for sums of mixing random variables is exactly the same as for the bounded variances case of independent random variables. This is, perhaps, surprising at the first glance, since the class of independent random variables is well contained in the class of random variables satisfying the mixing condition under consideration. Moreover, necessary and sufficient conditions are given for convergence to any specified law of the class of possible limit laws.

It seems quite likely that the general case of the central limit problem can also be solved along the present lines but at the cost of a somewhat more involved analysis. I hope to return to this question at some other place.

The fourth and last chapter deals with the central limit theorem itself. A kind of Lindeberg-Feller theorem is derived from the above mentioned general theorem using the standard argument. This theorem gives necessary and sufficient conditions for the central limit theorem to hold without stationarity hypotheses or assumptions on the growth of the variances of the partial sums. Finally some applications are given.

I do not aim at the greatest possible generality but I shall only give a few sample theorems. Moreover, the method used in this paper could be combined with some of the papers cited above to give more general results. I shall not go into the details, however.

The method of proof consists of introducing new random variables which are asymptotically independent and constitute a "nearly weak sense stationary process". This explains why the results obtained resemble of the case of independent random variables.

In subsequent papers [7, 8] I shall prove the law of the iterated logarithm for mixing stochastic processes and give some applications to number theory and analysis, in particular continued fractions, Diophantine approximation and gap series. Another paper [6] deals with the rate of convergence to the normal law.

2. The Mixing Conditions

2.1. Mixing Conditions for Triangular Arrays

Let $\langle x_{Nn},\ n=1,2,\ldots,n_N,\ N=1,2,\ldots\rangle$ be a double sequence of random variables. We shall assume throughout this paper that $n_N\to\infty$ as $N\to\infty$. Denote by $\mathfrak{M}_{ab}^{(N)}$ the σ-algebra generated by the events $\{x_{Nn}<\alpha\}$, $1\leqq a\leqq n\leqq b\leqq n_N$. We shall be concerned with the following four mixing conditions.

(I) For any events $A\in\mathfrak{M}_{1t}^{(N)}$ and $B\in\mathfrak{M}_{t+n,\,n_N}^{(N)}$ we have

$$|P(AB)-P(A)\,P(B)|\leqq\psi(n)\,P(A)\,P(B)$$

with $\psi(n)\downarrow 0$.

(II)

$$\sup_t \sup_{B\in\mathfrak{M}_{t+n,\,n_N}^{(N)}} |P(B\,|\,\mathfrak{M}_{1t}^{(N)})-P(B)|\leqq\varphi(n)\downarrow 0$$

with probability 1.

Condition (II) is equivalent with (for a proof see [2])

(II′) For any events $A\in\mathfrak{M}_{1t}^{(N)}$ and $B\in\mathfrak{M}_{t+n,\,n_N}^{(N)}$ we have

$$|P(AB)-P(A)\,P(B)|\leqq\varphi(n)\,P(A).$$

(III)

$$\sup_t \sup_{A\in\mathfrak{M}_{1t}^{(N)},\ B\in\mathfrak{M}_{t+n,\,n_N}^{(N)}} |P(AB)-P(A)\,P(B)|\leqq\alpha(n)\downarrow 0.$$

(IV) For all choices of integers $r \geqq 1$, $1 \leqq i_1 < \cdots < i_r$, $1 \leqq j \leqq r$ and $p_v \geqq 0$ $(1 \leqq v \leqq r)$, $p > 1$, $q > 1$ with $p^{-1} + q^{-1} = 1$ the mixed moments exist and satisfy

$$|E(x_{Ni_1}^{p_1} \dots x_{Ni_r}^{p_r}) - E(x_{Ni_1}^{p_1} \dots x_{Ni_j}^{p_j}) E(x_{Ni_{j+1}}^{p_{j+1}} \dots x_{Ni_r}^{p_r})| \leqq \beta^{1/p}(i_{j+1} - i_j) \|x_{Ni_1}^{p_1} \dots x_{Ni_j}^{p_j}\|_p \|x_{Ni_{j+1}}^{p_{j+1}} \dots x_{Ni_r}^{p_r}\|_q$$

where $\beta(i) \downarrow 0$.

Of course, condition (III) is a consequence of (II) which in turn is a consequence of (I). The following lemma shows that (IV) is a consequence of (II).

Lemma 1. *Suppose that condition* (II) *is satisfied and that ξ and η are measurable over $\mathfrak{M}_{1t}^{(N)}$ and $\mathfrak{M}_{t+n,n_N}^{(N)}$ respectively. If $E|\xi|^p < \infty$ and $E|\eta|^q < \infty$ with $p, q > 1$ and $p^{-1} + q^{-1} = 1$ then*

$$|E(\xi \eta) - E(\xi) E(\eta)| \leqq 2 \varphi^{1/p}(n) \|\xi\|_p \|\eta\|_q .$$

For a *proof* of Lemma 1 as well as of the following one see [2] and [15].

Lemma 2. *Assume that condition* (III) *is satisfied. If the random variables ξ and η are measurable over $\mathfrak{M}_{1t}^{(N)}$ and $\mathfrak{M}_{t+n,n_N}^{(N)}$ respectively and are essentially bounded then*

$$|E(\xi \eta) - E(\xi) E(\eta)| \leqq 4 \|\xi\|_\infty \|\eta\|_\infty \alpha(n).$$

Lemma 3. *If condition* (I) *is satisfied and if ξ and η are measurable over $\mathfrak{M}_{1t}^{(N)}$ and $\mathfrak{M}_{t+n,n_N}^{(N)}$ respectively then*

$$|E(\xi \eta) - E(\xi) E(\eta)| \leqq \psi(n) E|\xi| E|\eta|$$

provided that they are integrable.

Proof. It is enough to show the lemma for simple functions ξ and η that is for

$$\xi = \sum_i \lambda_i \chi(A_i), \qquad A_{i_1} \cap A_{i_2} = \emptyset \qquad (i_1 \neq i_2),$$

$$\eta = \sum_j \mu_j \chi(B_j), \qquad B_{j_1} \cap B_{j_2} = \emptyset \qquad (j_1 \neq j_2)$$

where all the $A_i \in \mathfrak{M}_{1t}^{(N)}$ and all the $B_j \in \mathfrak{M}_{t+n,n_N}^{(N)}$. But for such ξ and η the lemma follows trivially from (I) – as a matter of fact so does the lemma.

Condition (IV) does not quite follow from (III) because of the form of the error term. On the other hand the following example shows that (IV) does not imply (III).

Example. Let $f(x)$ be any function of bounded variation with period 1. Then the process $\langle f(2^n x), n = 0, 1, \dots \rangle$ is, of course, a strict sense stationary process. We set $x_{Nn} = f(2^n x)$ for $n = 0, 1, 2, \dots, N$, $N = 1, 2, \dots$ (and thus we may suppress the index N in the formulas). As is shown in [5], formula (20) condition (IV) is satisfied with $\beta(\tau) = O(2^{-\tau/2})$ whereas, of course, (III) is not if we choose $f(x)$ to be any function continuous and strictly increasing on $[0, 1)$ and extended with period 1.

There are many examples of processes satisfying (I*), (II*), (III*) and (IV*) (see Section 2.2 below). Of course, if the random variables are independent condition (I) holds whereas (II) holds for the case of m-dependent random variables, and Markov processes satisfying Doeblin's condition (see e.g. Doob [1], p. 221 f.). Some more examples are mentioned in [2].

2.2. *Mixing Conditions for Stochastic Processes*

Let $\langle x_n, n=1,2,\dots\rangle$ be a stochastic process and denote by $\mathfrak{M}_{ab}$ the σ-algebra generated by the events $\{x_n<\alpha\}$, $1\leqq a\leqq n\leqq b\leqq\infty$. Upon setting $x_{Nn}=x_n$ ($n=1,2,\dots,N$; $N=1,2,\dots$) the mixing conditions (I)–(IV) transform to (I*)–(IV*) if we replace n_N by ∞ and suppress the index N in the remaining places.

3. The Central Limit Problem

3.1. *Preliminaries*

Let $\langle x_{Nn}, n=1,2,\dots,n_N; N=1,2,\dots\rangle$ be a double sequence of random variables centered at expectations and with finite variances $\sigma_{Nn}^2=E(x_{Nn}^2)$. Throughout this chapter we shall assume that

$$\sigma_N^2\equiv\max_{1\leqq n\leqq n_N}\sigma_{Nn}^2\to 0 \qquad (N\to\infty), \tag{3.1}$$

$$\Sigma_N^2\equiv E\Big(\sum_{1\leqq n\leqq n_N} x_{Nn}\Big)^2\leqq c<\infty \tag{3.2}$$

where c is a constant not depending on N and that

$$\Sigma_N/\sigma_N\to\infty \qquad (N\to\infty). \tag{3.3}$$

Moreover, we shall suppose that one of the following conditions holds.

(ii) $\langle x_{Nn}\rangle$ satisfies (II) with $\sum_{n=1}^{\infty}\varphi^{\frac{1}{2}}(n)<\infty$.

(iii) $\langle x_{Nn}\rangle$ satisfies (III) with $\sum_{n=1}^{\infty}\alpha^{\frac{1}{2}}(n)<\infty$. The x_{Nn} tend uniformly towards 0 almost surely as $N\to\infty$, more precisely

$$c(N)\equiv\max_{1\leqq n\leqq n_N}\|x_{Nn}\|_\infty\to 0 \qquad (N\to\infty).$$

and

$$\Sigma_N/c(N)\to\infty \qquad (N\to\infty).$$

Obviously if $\langle x_{Nn}\rangle$ satisfies (iii) then (3.1) and (3.3) are automatically satisfied.

To simplify the notation we agree that n may assume the values $1,2,\dots,n_N$ and hence $\sum_n$ stands for $\sum_{n=1}^{n_N}$ and so does $\max_n$ for $\max_{1\leqq n\leqq n_N}$. Further, we omit the index N in the random variables x_{Nn}, y_{Nj}, z_{Nj} defined below. With this convention we write for fixed N

$$X_N\equiv\sum_n x_n=\sum_{j=1}^{l} y_j+\sum_{j=1}^{l+1} z_j\equiv Y_N+Z_N \tag{3.4}$$

where we set

$$\begin{aligned} y_1&=x_1+\cdots+x_{h_1}, & z_1&=x_{h_1}+\cdots+x_{h_1+k},\\ &\cdots\cdots & &\cdots\cdots\\ y_l&=x_{\rho_l+1}+\cdots+x_{\rho_l+h_l}, & z_l&=x_{\rho_l+h_l+1}+\cdots+x_{\rho_l+1},\\ & & z_{l+1}&=x_{\rho_l+1}+\cdots+x_{n_N}. \end{aligned}$$

Here we put

$$\rho_i=\sum_{\nu<i}(h_\nu+k)$$

the integers h_ν and k being at our disposal.

Definition. Suppose that (ii) holds and let $\kappa_N \to 0$ and S_N be any two sequences of real numbers satisfying

$$\xi_N = \kappa_N S_N/\sigma_N^2 \to \infty, \quad \Sigma_N^2/S_N \to \infty, \quad \varphi(\xi_N)\cdot \Sigma_N^2/S_N \to 0. \tag{3.5}$$

Such a pair (κ_N, S_N) we shall call admissible for $\langle x_{Nn}\rangle$ and we shall do so in case that (iii) holds and $\kappa_N \to 0$ and S_N satisfy

$$\zeta_N \equiv \kappa_N S_N/c^2(N) \to \infty, \quad \Sigma_N^2/S_N \to \infty, \quad \alpha(\zeta_N)\cdot \Sigma_N^2/S_N \to 0. \tag{3.6}$$

Admissible pairs always exist. For instance $S_N = \Sigma_N \sigma_N$ and $\kappa_N = (\sigma_N/\Sigma_N)^{\frac{1}{2}}$ do what required since both $\varphi(\tau)$ and $\alpha(\tau)$ are assumed to be monotone and hence satisfy $\tau^2 \varphi(\tau) \to 0$ and $\tau^2 \alpha(\tau) \to 0$ respectively.

Lemma 4. *Suppose that* (iii) *holds and let* (κ_N, S_N) *be any admissible pair for* $\langle x_{Nn}\rangle$. *Then we can represent* X_N *in the form* (3.4) *subject to the following conditions*

$$\begin{aligned} E(y_j^2) = S_N(1+o(1)), \qquad & E(z_j^2) \leqq c_1 \kappa_N S_N, \\ & E(z_{l+1}^2) \leqq S_N(1+o(1)) \end{aligned} \tag{3.7}$$

uniformly in $1 \leqq j \leqq l$. *Here* $c_1 > 0$ *denotes a constant independent of* N *and* j.

$$k = \zeta_N, \tag{3.8}$$

$$l = \eta_N(1+o(1)) \quad \textit{with} \quad \eta_N = \Sigma_N^2/S_N. \tag{3.9}$$

Lemma 5. *Suppose that* (ii) *holds and that* (κ_N, S_N) *is admissible for* $\langle x_{Nn}\rangle$. *Then the conclusions of Lemma* 4 *remain valid except that we have to replace* (3.8) *by*

$$k = \xi_N. \tag{3.10}$$

The *proofs* of the lemmas are similar. Due to Lemma 1 the proof of Lemma 5 is somewhat simpler so that we shall prove Lemma 4 only and indicate the changes to be made for the proof of Lemma 5.

We choose k equal to ζ_N to satisfy (3.8) and the h_j inductively to be the largest integer $h \leqq N$ such that

$$E\left(\sum_{\nu=\rho_j+1}^{\rho_j+h} x_\nu\right)^2 \leqq S_N.$$

At least for large N such a choice is always possible since $\eta_N \to \infty$. (3.7) follows now at once since

$$E(y_j^2) \leqq S_N \leqq E(y_j + x_{\rho_j+1})^2 \leqq E(y_j^2) + 2(E(y_j^2))^{\frac{1}{2}} \sigma_N + \sigma_N^2$$

which in view of (3.6) implies the first part of (3.7). To prove the second part we note that we have for $\mu \neq \nu$ as an application of Lemma 2

$$|E(x_\nu x_\mu)| \leqq 4\alpha(|\nu-\mu|)\, c^2(N). \tag{3.11}$$

Since

$$\sum_n \alpha(n) < \alpha < \infty$$

we obtain upon setting $I_j = [\rho_j + h_j + 1, \rho_{j+1}]$

$$\begin{aligned} E(z_j^2) &\leqq \sum_{i \in I_j} E(x_i^2) + 8c^2(N) \sum_{\nu < \mu \in I_j} \alpha(|\nu-\mu|) \\ &\leqq k(1+8\alpha)\, c^2(N). \end{aligned}$$

This together with (3.6) gives the second part of (3.7). So it remains to show (3.9). We expand

$$\Sigma_N^2 = E\Big(\sum_{j\le l} y_j + \sum_{j\le l+1} z_j\Big)^2$$
$$= \sum_{j\le l} E(y_j^2) + \sum_{j\le l+1} E(z_j^2) + 2\sum_{i<j\le l} E(y_i y_j)$$
$$+ 2\sum_{i<j\le l+1} E(z_i z_j) + \sum_{\substack{i\le l\\ j\le l+1}} E(y_i z_j).$$

Call these sums $\sum^{(1)}, \ldots, \sum^{(5)}$ respectively. From (3.7) we infer that

$$\textstyle\sum^{(1)} = l\, S_N(1+o(1)) \tag{3.12}$$

and

$$\textstyle\sum^{(2)} = O(l\,\kappa_N S_N) + O(S_N) = o(l\, S_N) + O(S_N). \tag{3.13}$$

Consider now a term $E(y_i y_j)$ in $\sum^{(3)}$ with $i<j$. We estimate

$$E(y_i y_j) = \sum_{v\in I_j^*} E(x_v y_i), \qquad I_j^* = [\rho_j+1, \rho_j+h_j]$$

by a simple truncation argument. Let $M>0$ to be chosen later. Using Lemma 1, (3.6) and (3.7) we obtain for fixed $v\in I_j^*$

$$|E(x_v y_i)| \le \Big|\int_{|y_i|>M} x_v y_i\Big| + \Big|\int_{|y_i|\le M} x_v y_i\Big|$$
$$\le c(N)\, S_N\, M^{-1} + 4c(N)\, M\, \alpha(v-\rho_i-h_i)$$

by Cauchy's and Chebyshev's inequalities. We choose $4M^2 = S_N/\alpha(v-\rho_i-h_i)$ so that we get

$$|E(x_v y_i)| \le 4c(N)\, S_N^{\frac12}\, \alpha^{\frac12}(v-\rho_i-h_i). \tag{3.14}$$

Hence

$$\Big|\textstyle\sum^{(3)}\Big| \le 2\sum_{i\le l}\ \sum_{\substack{v=\rho_j+1\\ j>i}}^{n_N} |E(x_v y_i)|$$
$$\le 8c(N)\, S_N^{\frac12}\, l \sum \alpha^{\frac12}(n)$$
$$= O\big(l\, S_N^{\frac12}\, c(N)\big) = o(l\, S_N).$$

In the same way we obtain in view of (3.6)

$$\textstyle\sum^{(4)} = O\big(l\,\kappa_N^{\frac12}\, S_N^{\frac12}\, c(N)\big) = o(l\, S_N)$$

and

$$\sum |E(y_i z_j)| = o(l\, S_N)$$

where the summation is extended over all $i\le l$, $j\le l+1$ with $j\ne i$, $i-1$. For the estimate of these terms we apply Cauchy's inequality and get

$$E(y_i z_i) = o(S_N), \qquad E(y_i z_{i-1}) = o(S_N).$$

Hence

$$\textstyle\sum^{(5)} = o(l\, S_N) + o(S_N).$$

Adding the estimates for $\sum^{(v)}$ $(1\le v\le 5)$ we obtain

$$\Sigma_N^2 = l\, S_N + o(l\, S_N) \tag{3.15}$$

which yields (3.9). This completes the proof of Lemma 4.

The proof of Lemma 5 is essentially the same but requires a few minor changes at these places where the uniform boundedness of the x_{Nn} was used. (3.11) is to be replaced by

$$|E(x_\nu x_\mu)| \leqq 2\varphi^{\frac{1}{2}}(|\nu-\mu|)\,\sigma_N^2$$

which follows from Lemma 1 and similarly we get as a direct application of Lemma 1

$$|E(x_\nu y_i)| \leqq 2\sigma_N S_N^{\frac{1}{2}}\varphi^{\frac{1}{2}}(\nu-\rho_i-h_i)$$

to replace (3.14). We remark in passing that the proof of Lemma 5 could be further simplified, of course, by direct applications of Lemma 1 to $E(y_i y_j)$ and the like. Later we shall use the following

Corollary. *Under the hypotheses of the lemmas we have*

$$E(Z_N^2) = O(\kappa_N \Sigma_N^2)$$

and

$$E(Y_N^2) = \Sigma_N^2 \cdot \big(1+o(1)\big).$$

Proof. With the notation of the proof of Lemma 4 we have

$$\begin{aligned} E(Z_N^2) &= \textstyle\sum^{(2)} + \sum^{(4)} = O(l\,\kappa_N S_N) + O(S_N) + O\big(l\,\kappa_N^{\frac{1}{2}} S_N^{\frac{1}{2}} c(N)\big) \\ &= O(\kappa_N \cdot \Sigma_N^2) \end{aligned}$$

using (3.6). Similarly using (3.9)

$$E(Y_N^2) = \textstyle\sum^{(1)} + \sum^{(3)} = l\,S_N\big(1+o(1)\big) = \Sigma_N^2\big(1+o(1)\big).$$

The proof of the case corresponding to Lemma 5 is the same.

3.2. The Theorems

Let $\langle x_{Nn}\rangle$ be given as in section 3.1 and let (κ_N, S_N) be any admissible pair. According to Lemmas 4 and 5 there is a uniquely determined sequence $y_j = y_{Nj}$ $(1 \leqq j \leqq l)$ associated with (κ_N, S_N). Denote by $F_j = F_{Nj}$ the distribution function of y_j $(1 \leqq j \leqq l)$.

Following the presentation in Loève [3] we say that a sequence of distribution functions F_N converges weakly to a distribution function F, if $F_N \to F$ on the continuity set of F and we say that F_N converges to F completely if $F_N \to F$ weakly and $F_N(\pm\infty) \to F(\pm\infty)$.

Theorem 1. *Let $\langle x_{Nn}\rangle$ be a double sequence of random variables centered at expectations and with finite variances σ_{Nn}^2 satisfying* (3.1), (3.2) *and* (3.3). *Suppose that* (ii) *holds. Then*

1) *the family of limit laws of sequences*

$$\mathfrak{L}\big(\sum_n x_{Nn}\big)$$

coincides with the family of laws of random variables centered at expectations with finite variances and characteristic functions of the form $f = e^{\psi}$, where ψ is of the form

$$\psi(t) = \int (e^{iux} - 1 - i\,u\,x)\,\frac{1}{x^2}\,dK(x)$$

with K continuous from the left and nondecreasing on $(-\infty,\infty)$ and var $K \leqq c < \infty$. *Here c is the constant in* (3.2). *Moreover, ψ determines K and conversely.*

2) *Let* (κ_N, S_N) *be any admissible pair. Then*

$$\mathfrak{L}(\sum_n x_{Nn}) \to \mathfrak{L}(X)$$

with characteristic function necessarily of the form e^{ψ} *if, and only if,* $K_N \to K$ *weakly. Here* K_N *is defined by*

$$K_N(X) = \sum_{j \leq l} \int_{-\infty}^{x} y^2 \, dF_{Nj}.$$

If $\Sigma_N^2 \leqq c < \infty$ *is replaced by* $\Sigma_N^2 \to \sigma^2(X)$ *then* $K_N \to K$ *weakly is to be replaced by* $K_N \to K$ *completely.*

Remark. The statement 2) implies that if $K_N \to K$ is satisfied for some admissible pair (κ_N, S_N) then $K_N \to K$ is automatically satisfied for all admissible pairs. This remark applies for all the theorems of this section.

Theorem 2. *Let* $\langle x_{Nn} \rangle$ *be as in Theorem* 1 *but suppose that, instead of* (ii), *condition* (iii) *holds. Then the family of limit laws of sequences*

$$\mathfrak{L}(\sum_n x_{Nn})$$

is contained in the family of laws of random variables as described in Theorem 1 *under* 1) *and conclusion* 2) *of Theorem* 1 *remains valid.*

As is obvious the possible limit laws of the sums

$$\sum_n x_{Nn}$$

are all infinitely divisible.

Let us take a look at the proof of Theorem A in Loève [3] p. 293 which served as a model for Theorems 1 and 2. First of all Theorem A remains valid if we impose the additional condition

$$\frac{\sum_k \sigma_{nk}^2}{\max_k \sigma_{nk}^2} \to \infty. \tag{3.16}$$

All what we have to check is that the family of possible limit laws of sequences

$$\mathfrak{L}(\sum_n X_{nk})$$

subject to (3.16) does not become smaller. So let us find out which limit laws might occur if (3.16) does not hold. In this case

$$\liminf_n \sum_k \sigma_{nk}^2 = 0$$

which implies

$$\limsup_n P(\sum_k X_{nk} \to 0) = 1.$$

Hence the only limit law towards which the sequence

$$\mathfrak{L}(\sum_k X_{nk})$$

possibly can converge is the degenerate law $\mathfrak{L}(0)$ which, of course, can be obtained as the limit law of a sequence of independent random variables satisfying (3.16).

Simply set

$$x_{Nn} = \frac{1}{N} u_n \qquad n = 1, 2, \ldots, N$$

where u_n is a stationary process of independent random variables centered at expectation and of finite variance.

The second remark is to the effect that the independence of the random variables X_{nk} is expressed as

$$f_n = \prod_k f_{nk}$$

where f_{nk} is the characteristic function of X_{nk}, and f_n is the characteristic function of

$$\sum_k X_{nk}.$$

But inspection of the proof of Theorem A shows that all what is actually needed is

$$f_n = \prod_k f_{nk} + o(1).$$

With these two remarks in mind we see that Theorems 1 and 2 will be proven if we show the following

Lemma 6. *Let* (κ_N, S_N) *be any admissible pair. Then we have with the notation introduced earlier*

$$\limsup_{N \to \infty} \sum_{j=1}^{l} E(y_j^2) \leqq c < \infty, \tag{3.17}$$

$$\max_{1 \leqq j \leqq l} E(y_j^2) \to 0. \tag{3.18}$$

Moreover, for fixed $T > 1$ *we have uniformly in* $|t| \leqq T$

$$E(\exp(i\, t\, Y_N)) = \prod_{j=1}^{l} E(\exp(i\, t\, y_j)) + o(1), \tag{3.19}$$

$$E(\exp(i\, t\, X_N)) = E(\exp(i\, t\, Y_N)) + o(T^2). \tag{3.20}$$

Proof. To prove (3.17) we note that by (3.12), (3.9) and (3.2)

$$\sum_{j=1}^{l} E(y_j^2) = l\, S_N(1 + o(1)) = \Sigma_N^2(1 + o(1)) \leqq c + o(1).$$

(3.18) follows from (3.7) and (3.5) or (3.6). Since (II) implies (III) we get after applying Lemma 2 l times and using (3.5) and (3.6) respectively

$$\begin{aligned} \Big|E(\exp(i\, t\, Y_N)) - \prod_{j \leqq l} E(\exp(i\, t\, y_j))\Big| &\leqq 4\, l\, \alpha(k) \\ &\leqq 4\, \eta_N (1 + o(1))\, \alpha(\zeta_N) = o(1). \end{aligned} \tag{3.21}$$

So it remains to prove (3.20). From the corollary of Lemmas 4 and 5 we know that $E(Z_N^2) = o(\Sigma_N^2)$ and thus

$$E(\exp(i\, t\, X_N)) = E(\exp(i\, t\, Y_N)) \cdot (1 + O(t^2\, E(Z_N^2))) \tag{3.22}$$

which implies (3.20).

The following theorem is a corollary of Theorems 1 and 2. For a proof see Loève [3, p. 295]. Denote by $\mathfrak{N}(0,1)$ the normal law with mean 0 and variance 1.

Theorem 3. *Let* $\langle x_{Nn}\rangle$ *be as in Theorem* 1. *Suppose that either* (ii) *or* (iii) *holds and that* $\Sigma_N \to 1$ $(N\to\infty)$. *Let* (κ_N, S_N) *be any admissible pair. Then*

$$\mathfrak{L}\big(\sum_n x_{Nn}\big) \to \mathfrak{N}(0,1) \quad \textit{and} \quad \sigma_N \to 0 \quad \textit{or} \quad c_N \to 0$$

respectively if, and only if, for any $\varepsilon > 0$

$$\sum_{j\leqq l} \int_{|y|\geqq\varepsilon} y^2\, dF_{Nj} \to 0.$$

Although in this paper we are primarily interested in the case that the random variables are centered at expectations it is, perhaps, worthwhile to remark that Theorems 1 and 2 carry over to case where $E(x_{Nn})$ does not necessarily vanish to yield an analogue of what in [3, p. 294] is called the "extended convergence criterion". It is quite obvious how this extended theorem would read in our case so I shall not give a formal statement but mention only an important particular case, namely the convergence to Poisson law $\mathfrak{P}(\lambda)$.

Theorem 4. *Let* $\langle x_{Nn}\rangle$ *be a triangular array of random variables satisfying* (3.1) *and* $\Sigma_N^2 \to \lambda$ $(N\to\infty)$. *Suppose that either* (ii) *or* (iii) *holds. Let* (κ_N, S_N) *be any admissible pair. Then*

$$\mathfrak{L}\big(\sum_n x_{Nn}\big) \to \mathfrak{P}(\lambda) \quad \textit{if, and only if,} \quad \sum_n E(x_{Nn}) \to \lambda$$

and, for any $\varepsilon > 0$

$$\sum_{j\leqq l} \int_{|y-1|\geqq\varepsilon} y^2\, dF_{Nj}(y+E(y_{Nn})) \to 0.$$

It is perhaps surprising that the theorems do not cover the case of independent variables. As a matter of fact we have $h_j \to \infty$ $(1\leqq j<\infty)$. But it is, of course, possible to formulate the theorems in such a way that they cover the independent case, too.

4. The Central Limit Theorem for Stochastic Processes and Ljapounov's Condition

In this chapter we shall continue to specialize the theorems of the preceding chapter. We start with

4.1. The Lindeberg-Feller Condition

Let $\langle x_n, n=1,2,\ldots\rangle$ be a stochastic process with $E(x_n)=0$ $(n=1,2,\ldots)$. Set

$$\sigma_N^{*2} = \max_{1\leqq n\leqq N} E(x_n^2) \quad \text{and} \quad s_N^2 = E\big(\sum_{n\leqq N} x_n\big)^2.$$

We shall assume throughout this chapter that

$$\frac{\sigma_N^*}{s_N} \to 0 \qquad (N\to\infty).$$

Moreover, suppose that one of the following conditions holds

(ii*) $\langle x_n \rangle$ satisfies (II*) with $\sum_n \varphi^{\frac{1}{2}}(n) < \infty$.

(iii*) $\langle x_n \rangle$ satisfies (III*) with

$$\sum_n \alpha^{\frac{1}{2}}(n) < \infty \quad \text{and} \quad \max_{1 \leq n \leq N} \frac{\|x_n\|_\infty}{s_N} \equiv \frac{c^*(N)}{s_N} \to 0 \qquad (N \to \infty).$$

Upon setting

$$x_{Nn} = \frac{x_n}{s_n} \qquad \text{for } n = 1, 2, \ldots, N$$

we see that (3.1), (3.2) and (3.3) are all satisfied and that in fact $\Sigma_N \equiv 1$ $(N = 1, 2, \ldots)$. Throughout this chapter we call a pair (κ_N, S_N^*) admissible for the process if $(\kappa_N, S_N^* s_N^{-2})$ is admissible for

$$x_{Nn} = \frac{x_n}{s_n} \qquad (n = 1, 2, \ldots, N; \; N = 1, 2, \ldots).$$

According to Lemmas 4 and 5 there are two uniquely determined sequences y_{Nj} and z_{Nj} $(1 \leq j \leq l+1)$ associated with $(\kappa_N, S_N^* s_N^{-2})$. We set $y_j = y_{Nj} s_N$ and $z_j = z_{Nj} s_N$ $(1 \leq j \leq l+1)$ and denote by F_j the distribution function of y_j $(1 \leq j \leq l)$. With this notation we obviously have

Lemma 7. *Suppose that either* (ii*) *or* (iii*) *holds. Then* (κ_N, S_N^*) *is an admissible pair for* $\langle x_n \rangle$ *if, and only if,* $\kappa_N \to 0$, $\xi_N^* = \kappa_N S_N^* / \sigma_N^{*2} \to \infty$, $s_N^2 / S_N^* \to \infty$ *and* $\varphi(\xi_N^*) s_N^2 / S_N^* \to 0$ *or* $\alpha(\xi_N^*) s_N^2 / S_N^* \to 0$ *respectively. Moreover, uniformly in* $1 \leq j \leq l$

$$E(y_j^2) = S_N^*(1 + o(1)), \qquad E(z_j^2) = \kappa_N S_N^*(1 + o(1)), \qquad E(z_{l+1}^2) \leq S_N^*(1 + o(1)),$$

$$k = \kappa_N S_N^*, \qquad l = s_N^2 / S_N^*(1 + o(1)),$$

$$E(Z_N^2) = O(\kappa_N s_N^2) \quad \text{and} \quad E(Y_N^2) = s_N^2(1 + o(1)).$$

Theorem 5. *Let* $\langle x_n \rangle$ *be a stochastic process as described above and suppose that either* (ii*) *or* (iii*) *holds. Then*

$$\mathfrak{L}\left(\frac{1}{s_N} \sum_{n \leq N} x_n\right) \to \mathfrak{N}(0, 1)$$

if and only if, for any admissible pair (κ_N, S_N^*) *and each* $\varepsilon > 0$

$$\frac{1}{s_N^2} \sum_{j \leq l} \int_{|y| \geq \varepsilon s_N} y^2 \, dF_j(y) \to 0 \qquad (N \to \infty).$$

This is an immediate consequence of Theorem 4.

4.2. The Ljapounov Type Conditions

Instead of having necessary and sufficient conditions depending on the choice of certain parameters for the central limit theorem to hold it frequently is more desirable to have only sufficient conditions not depending on such a choice and thus being easier to handle when it comes to applications. One of them is a kind of Ljapounov condition. For convenience we shall assume throughout the rest of the paper that

$$\sup_n \sigma_n \leqq 1 \quad \text{and} \quad s_n \to \infty .$$

Moreover, we shall suppose that one of the following conditions holds:

($\hat{\text{i}}$) $\langle x_n \rangle$ satisfies (I*) with $\sum_n \psi^{\frac{1}{2}}(n) < \infty$;

($\hat{\text{ii}}$) $\langle x_n \rangle$ satisfies (II*) with $\sum_n \varphi^{\frac{1}{5}}(n) < \infty$ and $\sup_n \|x_n\|_4 \leqq 1$.

Theorem 6. *Let $\langle x_n \rangle$ be a stochastic process satisfying the standard conditions. Suppose that ($\hat{\text{i}}$) holds and the fourth moments $E(x_n^4)$ exist and satisfy*

$$\sum_{n=M+1}^{M+H} E|x_n| = o\left(\left(E\left(\sum_{n=M+1}^{M+H} x_n\right)^2\right)^{\frac{3}{2}}\right) \qquad H \to \infty \tag{4.1}$$

and

$$\sum_{n=M+1}^{M+H} E(x_n^4) = o\left(\left(E\left(\sum_{n=M+1}^{M+H} x_n\right)^2\right)^3\right) \qquad H \to \infty \tag{4.2}$$

uniformly in $M = 0, 1, \ldots$. Then

$$\mathfrak{L}\left(\frac{1}{s_N} \sum_{n \leqq N} x_n\right) \to \mathfrak{N}(0, 1).$$

If in particular

$$\sup_n \|x_n\|_\infty \leqq 1$$

then (4.1) *implies* (4.2) *which thus can be omitted.*

Theorem 7. *Let $\langle x_n \rangle$ be a stochastic process with the standard conditions. Suppose that ($\hat{\text{ii}}$) holds and that*

$$\sum_{n=M+1}^{M+H} \|x_n\|_4 = o\left(\left(E\left(\sum_{n=M+1}^{M+H} x_n\right)^2\right)^{\frac{9}{4}}\right) \qquad (H \to \infty) \tag{4.3}$$

uniformly in $M = 0, 1, 2, \ldots$. Then

$$\mathfrak{L}\left(\frac{1}{s_N} \sum_{n \leqq N} x_n\right) \to \mathfrak{N}(0, 1).$$

Remarks. It will be clear from the proof of the theorems that conditions (4.1), (4.2) and (4.3) can be relaxed somewhat at the cost of a restriction of the convergence of $\sum \psi^{1/s}(n)$ and $\sum \varphi^{1/s}(n)$ to values bigger than 2 or 5 respectively.

On the other hand the theorems would become false if (4.1), (4.2) or (4.3) respectively were omitted even if we assumed $\sup \|x_n\|_\infty \leqq 1$. Take for example a stationary process of independent random variables x_n and define a new process $y_n = x_{n+1} - x_n$ which is a stationary process of m-dependent random variables with $m = 1$. (4.1) or (4.3) are no longer valid and in both cases

$$\mathfrak{L}\left(N^{-\frac{1}{2}} \sum_{n \leqq N} y_n\right) \to \mathfrak{L}(0).$$

To prove the theorems it is enough to prove the following

Lemma 8. *Suppose that either the hypotheses of Theorem 6 or Theorem 7 are satisfied and set $s=2$ or 5 accordingly. Then we have for any admissible pair (κ_N, S_N^*) uniformly in $1 \leqq j \leqq l$*

$$E(y_j^4) = o(S_N^{*\,s+2}).$$

We postpone the proof of the lemma and deduce the theorems from it first. Let $\mu_N \geqq E(y_j^4) S_N^{*\,-s-2}$ with $\mu_N \to 0$ so slowly that $\mu_N s_N \to \infty$. Then the pair (κ_N, S_N^*) with $\kappa_N^{s+1} = \mu_N$ and $s_N^2 = S_N^{*\,s+1} \mu_N^{s/(s+1)}$ is admissible for $\langle x_n \rangle$. In fact $\kappa_N \to 0$, $\kappa_N S_N^* \to \infty$, $s_N^2/S_N^* \to \infty$ and

$$\varphi(\kappa_N S_N^*)\, s_N^2/S_N^* = \varphi(\mu_N^{1/(s+1)} S_N^*)\, \mu_N^{s/(s+1)} S_N^{*\,s} \to 0,$$

similarly for ψ. Hence we obtain

$$\sum_{j \leqq l} E(y_j^4) \leqq \mu_N\, l S_N^{*\,s+2} = \mu_N^{1/(s+1)} s_N^4 = o(s_N^4).$$

But this implies for any $\varepsilon > 0$

$$\frac{1}{s_N^2} \sum_{j \leqq l} \int_{|y| \geqq \varepsilon s_N} y^2\, dF_j(y) \leqq \frac{1}{\varepsilon^2 s_N^4} \sum_{j \leqq l} \int_{|y| \geqq \varepsilon s_N} y^4\, dF_j(y) = o(1)$$

and hence Theorem 5 applies.

We prove Lemma 8 at first for the case that the hypotheses of Theorem 6 hold. Lemma 8 is an immediate consequence of

Lemma 9. *Let $\langle x_n \rangle$ be as in Theorem 6 and set for $r=1,2,3,4$*

$$P_r(M,H) = \sum_{n=M+1}^{M+H} E\,|x_n^r|.$$

Suppose that uniformly in $M=0,1,\dots$

$$P_1(M,H) \leqq P_1(H) \quad \text{and} \quad P_4(M,H) \leqq P_4(H) \tag{4.4}$$

for some $P_1(H)$ and $P_4(H)$ depending on H only. Then

$$P_2(M,H) \leqq P_1^{\frac{2}{3}}(H)\, P_4^{\frac{1}{3}}(H) \quad \text{and} \quad P_3(M,H) \leqq P_1^{\frac{1}{3}}(H)\, P_4^{\frac{2}{3}}(H) \tag{4.5}$$

uniformly in $M=0,1,\dots$. Moreover, uniformly in $1 \leqq j \leqq l$

$$E(y_j^4) \ll P_4 + A_1 P_1 + P_1^{\frac{8}{3}} P_4^{\frac{1}{3}} \psi(A_1) + A_2^2 P_1 + P_1^4 \psi(A_2) + P_1^2 + P_1^{\frac{4}{3}} P_4^{\frac{2}{3}}. \tag{4.6}$$

Here the symbol $\ll$ stands for the O symbol; the constants A_0, A_1, A_2 are arbitrary and $P_1 = P_1(h_j)$ and $P_3 = P_4(h_j)$.

Proof. To prove (4.5) we note that for any $a_n \geqq 0$, fixed,

$$\log \sum_{n=M+1}^{M+N} a_n^r = f(r)$$

considered as a function of r is convex from below. In fact we have as a consequence of Hölder's inequality for $0 \leqq s \leqq r$

$$\left(\sum_{n=M+1}^{M+N} a_n^r\right)^2 \leqq \left(\sum_{n=M+1}^{M+N} a_n^{r+s}\right)\left(\sum_{n=M+1}^{M+N} a_n^{r-s}\right)$$

which implies the convexity of $f(r)$. (4.5) follows now immediately. In order to prove (4.6) we expand $E(y_j^4)$ by the multinomial theorem

$$E(y_j^4)=E\Big(\sum_{n\in I_j} x_n\Big)^4=\sum_{\substack{p_1+\cdots+p_4=4\\ p_i\geqq 0}}\frac{4!}{p_1!\ldots p_4!}\sum_{i_1<\cdots<i_4} E(x_{i_1}^{p_1}\ldots x_{i_4}^{p_4}).$$

Here I_j is the set of indices v of x_v defining the y_j and the summation is extended over all i_k $(1\leqq k\leqq 4)\in I_j$ subject to the conditions indicated. We shall estimate only the sum $\sum E(x_{i_1}x_{i_2}x_{i_3}x_{i_4})$, the other sums are treated similarly. We split $\sum$ into two parts $\sum=\sum'+\sum''$ according to whether or not both $i_2-i_1\leqq A_2$ and $i_3-i_2\leqq A_2$ are satisfied. In the first case we have

$$\begin{aligned}\textstyle\sum'&\leqq\textstyle\sum' E|x_{i_1}x_{i_2}x_{i_3}|\,E|x_{i_4}|\,\psi(i_4-i_3)\\&\ll\textstyle\sum' E|x_{i_1}|\,E|x_{i_2}|\,E|x_{i_3}|\,E|x_{i_4}|\,\psi(i_4-i_3)\\&\ll\textstyle\sum' E|x_{i_3}|\,\psi(i_4-i_3)\ll A_2^2P_1.\end{aligned}$$

In $\sum''$ one of the two conditions is violated and we split $\sum''=\sum^*+\sum^{**}$ accordingly. Since $i_3-i_2\geqq A_2$ in $\sum^{**}$ we obtain

$$\begin{aligned}\textstyle|\sum^{**}|&\leqq\textstyle\sum^{**}|E(x_{i_1}x_{i_2})|\,|E(x_{i_3}x_{i_4})|+\sum^{**}E|x_{i_1}x_{i_2}|\,E|x_{i_3}x_{i_4}|\,\psi(A_2)\\&\ll\sum_{i_1<i_2}E|x_{i_1}|\,E|x_{i_2}|\,\psi(i_2-i_1)\cdot\sum_{i_3<i_4}E|x_{i_3}|\,E|x_{i_4}|\,\psi(i_4-i_3)\\&\quad+\textstyle\sum E|x_{i_1}|\,E|x_{i_2}|\,E|x_{i_3}|\,E|x_{i_4}|\,\psi(A_2)\\&\ll P_1^2+P_1^4\,\psi(A_2).\end{aligned}$$

The estimate of $\sum^*$ is similar.

Proof of Lemma 8. From the hypotheses of Theorem 6 it follows that

$$P_1=O(S_N^{*\frac{3}{4}}),\qquad P_4=O(S_N^{*3}).$$

We choose $A_1=A_2=S_N^*$ and obtain the conclusion of Lemma 8 since as before $\psi(S_N^*)=O(S_N^{*-2})$.

In case that, instead of the hypotheses of Theorem 6, those of Theorem 7 are assumed to be valid Lemma 8 is deduced in the exact same way from the following Lemma 10 which then replaces Lemma 9.

Lemma 10. *Let* $\langle x_n\rangle$ *be as in Theorem* 7 *and suppose that uniformly in* $M=0,1,2,\ldots$

$$\sum_{n=M+1}^{M+H}\|x_n\|_4\leqq P(H)$$

for some function $P(H)$ *depending on* H *only. Then we have uniformly in* $1\leqq j\leqq l$

$$E(y_j^4)\ll A_1P+P^3\varphi^{\frac{1}{4}}(A_1)+A_2^2P+P^4\varphi^{\frac{1}{4}}(A_2)+P^2. \tag{4.7}$$

Here we set $P(h_j)=P$ and the constants $A_0, A_1, A_2\geqq 1$ are arbitrary. The *proof* of Lemma 10 is exactly the same as that of Lemma 9 however we have to apply Lemma 1 at those places where we applied Lemma 3. That is where the exponent $\frac{1}{4}$ in $\varphi^{\frac{1}{4}}$ comes from.

4.3. Some Applications

In this section we shall consider two special cases. First we suppose that the random variables x_n are defined as $x_n = \varphi_n - \mu(\varphi_n)$ $(n=1,2,\ldots)$ where φ_n is the indicator function of the event E_n. Then we have the following

Theorem 8. *Let $\langle E_n, n=1,2,\ldots\rangle$ be a sequence of events such that $P(E_n)\to 0$. Let $\mathfrak{M}_{ab}$ be the σ-algebra generated by the E_n $(a\leqq n\leqq b)$. Suppose that* (I*) *is satisfied with $\sum \psi^{\frac{1}{2}}(n)<\infty$ and that*

$$\Phi(N)=\sum_{n\leqq N} P(E_n)\to\infty .$$

Then

$$\mathfrak{L}\Big(\Phi^{-\frac{1}{2}}(N)\Big(\sum_{n\leqq N}\varphi_n-\Phi(N)\Big)\Big)\to\mathfrak{N}(0,1)$$

or else

$$\mathfrak{L}\big(\Phi^{-\frac{1}{2}}(N)\big(A(N,x)-\Phi(N)\big)\big)\to\mathfrak{N}(0,1)$$

where for given x in the sample space $A(N,x)$ denotes the number of integers $n\leqq N$ with $x\in E_n$.

Proof. In order to apply Theorem 6 we have to check condition (4.1). For integer $M,N\geqq 0$ we write

$$\Phi(M,N)=\sum_{n=M+1}^{M+N} P(E_n).$$

Then with $|\theta|\leqq 1$

$$\begin{aligned}
E\Big(\sum_{n=M+1}^{M+N}\varphi_n-\Phi(M,N)\Big)^2 &= \Phi(M,N)+2\sum_{M<m<n\leqq M+N} P(E_m E_n)-\Phi^2(M,N)\\
&=\Phi(M,N)-\Phi^2(M,N)+2\sum_{M<m<n\leqq M+N} P(E_m)\,P(E_n)\big(1+\theta\,\psi(n-m)\big)\\
&=\Phi(M,N)-\sum_{n=M+1}^{M+N}(P(E_n))^2+\theta\sum_{M<m<n\leqq M+N} P(E_m)\,P(E_n)\,\psi(n-m).
\end{aligned}$$

Since $P(E_m)\to 0$ we obtain setting $M=0$ that $s_N^2=\Phi(N)\big(1+o(1)\big)$. Now let m_0 have the property that $P(E_m)\leqq\big(4\sum\psi(n)\big)^{-1}$ for $m\geqq m_0$. Then for $N\to\infty$

$$E\Big(\sum_{n=M+1}^{M+N}\varphi_n-\Phi(M,N)\Big)^2\geqq\tfrac{1}{2}\,\Phi(M,N)\big(1+o(1)\big)$$

uniformly in $M\geqq m_0$. Hence we have uniformly in $M=0,1,\ldots$

$$\Phi(M,N)\ll E\Big(\sum_{n=M+1}^{M+N}\varphi_n-\Phi(M,N)\Big)^2\qquad (N\to\infty)$$

which in view of $E|x_n|\leqq 2P(E_n)$ proves (4.1) and thus the theorem.

Remark. It is not difficult to see that Theorem 8 remains valid if we assume $\sum \psi(n) < \infty$ only. For this purpose we estimate $E(y_i\, y_j)$ in Lemma 5 as a direct application of Lemma 3 without employing the truncation argument. Similarly the estimate of $E(y_j^4)$ in Lemma 9 can be simplified considerably since for example $\sum_4 \ll S_N^{*\,3}$. I shall not go further into the details.

As a second application we suppose that $\langle x_n, n=1,2,\ldots\rangle$ is a weak sense stationary process. Then we have for example

Theorem 9. *Suppose that $\langle x_n, n=1,2,\ldots\rangle$ is a weak sense stationary process with $E(x_n)=0$ and $\sup_n E(x_n^4) \leqq 1$ satisfying condition* (II*) *with $\sum \varphi^{\frac{1}{4}}(n) < \infty$. Then*

$$\sigma^2 = E(x_1^2) + 2\sum_{\nu=1}^{\infty} E(x_1\, x_{\nu+1})$$

exists. Moreover, if $\sigma \neq 0$,

$$\mathfrak{L}\left(\frac{1}{\sigma\sqrt{N}} \sum_{n\leqq N} x_n\right) \to \mathfrak{N}(0,1).$$

Proof. We have

$$\begin{aligned} s_N^2 &= E\big(\sum_{n\leqq N} x_n\big)^2 = N\,E(x_1^2) + 2N\sum_{\nu=1}^{N} E(x_1\, x_{\nu+1}) - 2\sum_{\nu=1}^{N} \nu\, E(x_1\, x_{\nu+1}) \\ &= N\sigma^2 - 2N\sum_{\nu>N} E(x_1\, x_{\nu+1}) - 2\sum_{\nu=1}^{N} \nu\, E(x_1\, x_{\nu+1}) \qquad (4.8) \\ &= N\sigma^2 + O(1) \end{aligned}$$

since

$$\sum_{\nu=1}^{\infty} \nu\, E(x_1\, x_{\nu+1}) \ll \sum_{\nu=1}^{\infty} \nu\, \varphi(\nu) \ll \sum_{\nu=1}^{\infty} \varphi^{\frac{1}{2}}(\nu)$$

using the monotonicity of $\varphi(\nu)$ and $\sum \varphi^{\frac{1}{2}}(\nu) < \infty$. That σ actually exists is contained in the above argument. Hence $\sigma \neq 0$ implies $S_N \to \infty$. Moreover, from (4.8) and the weak sense stationarity

$$\sum_{n=M+1}^{M+N} \|x_n\|_4 \leqq N \ll N\sigma^2 \ll E\left(\sum_{n=M+1}^{M+N} x_n\right)^2.$$

The result follows now from Theorem 7.

Remark. For strict sense stationary processes Ibragimov [2] has Theorem 9 in a slightly sharper form. Instead of Lemma 8 he uses the following one which I was unable to generalize to the nonstationary case. He also has theorems involving condition (III*) only.

Lemma 11 (Ibragimov [2], see also Doob [1], p. 225 and Philipp [4]). *Suppose that the stationary process satisfies* (III*) *and that $E|x_1|^{2+\delta} < \infty$ for some $\delta > 0$. Then*

$$E\big(\big|\sum_{n\leqq N} x_n\big|^{2+\delta}\big) \ll s_N^{2+\delta}.$$

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