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A Note on Temporal Games

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Summary. We investigate strategic situations where the zero-sum two-person game in normal form is composed of a sequence of choices where the players are informed about the past and its relations between the orders of realizing them. The minimax theorem is improved for games having cartesian products as strategy sets.

1.

The minimax theorem for a zero-sum two-person game in normal form asserts that the safety levels of both players coincide. The heuristic interpretation of these safety levels depends only on the rules of the game.

In the classic case, that is, when both players realize their respective choices independently upon any information about the opposite's choice, the safety levels have a very well known meaning.

Now, one could associate to the maximin and minimax values somewhat different images as follows: Consider the game played in such a way that the first player is informed on the choice of the second player. Conversely, one has the other situation where the second player knows the choice that has been already made by the first player. In the first case, the minimax value is obviously the value of the game considered with the new rules. In a similar fashion, in the last case the value is equal to the maximin value. Thus, now the saddle point concept can be interpreted as an invariant point with respect to the information on the knowledge of the behavior of both players.

The motive of this paper is to consider a natural extension of that strategic situation involved in the previous consideration, when the game is composed by a sequence of choices over the time of both players knowing the *past*. As a consequence of the presented results the minimax theorem for games where the strategy sets are a cartesian product will be improved.

2.

Let $\Lambda_P = \{\Sigma_1^1, \ldots, \Sigma_{m_1}^1; \Sigma_1^2, \ldots, \Sigma_{m_2}^2; A; P\}$ be a *zero-sum two-person temporal game* defined by a zero-sum two-person game $\{\Sigma_{I_1}, \Sigma_{I_2}; A\}$ where the strategy sets Σ_{I_k} with $k \in \{1, 2\}$ are cartesian products over $I_k = \{1, \ldots, m_k\}$ of the non empty compact and convex subsets Σ_j^k in the $n_{k,j}$-tuple of real numbers $R^{n_{k,j}}$, A is a continuous real function and P a temporal order. A *temporal order* P is a bijective function which assigns to each *time* $t \in T = \{1, \ldots, m_1 + m_2\}$ an element $P(t) \in I_1 \cup I_2$.

From an intuitive point of view, the set T is regarded as the ordered time on which the game is played. Thus, a temporal order P determines the strategy set $\Sigma_{P(t)}$ employed at time $t \in T$ by the $n \cdot P(t) = \{k \colon P(t) \in I_k\}$-th player knowing every previous choice $\sigma_{P(\bar{t})} \in \Sigma_{P(\bar{t})}$ made in the past $\bar{t} \in [1, t) = \{1, \ldots, t-1\}$.

For the sake of simplicity we are only concerned with *monotone* temporal orders, that is, those such that for each $k \in \{1, 2\}$ and each $l \geqq \bar{l}$ in I_k: $P^{-1}(l) \geqq P^{-1}(\bar{l})$. This says that the ordering of choosing is realized in a monotone way. Without any explicit mention, temporal order means monotone too.

Now, we introduce some definitions. Given two temporal orders P and Q, it is said that P *follows* Q and we write $P \succeq Q$, if for each $j \in I_2$: $P^{-1}(j) \leqq Q^{-1}(j)$. Similarly, P *strictly follows* Q, and we write $P \succ Q$ if P follows Q and there is an $j \in I_2$ such that $P^{-1}(j) < Q^{-1}(j)$. $\mathfrak{P} = \{P^1, \ldots, P^r\}$ is called a *chain* if each element is a temporal order and for each $s \in \{1, \ldots, r-1\}$, P^s strictly follows P^{s+1}. The chain $\mathfrak{Q}$ is a *refinement* of $\mathfrak{P}$ if it can be obtained from $\mathfrak{P}$ by inserting some temporal order in the chain $\mathfrak{P}$. A chain is said to be *maximal* if it does not admit any refinement. Given a temporal order P, let us define the t-th associated function P_t: $T \to I_1 \cup I_2$ with $1 \leqq t < m_1 + m_2$ by $P_t = P \cdot \pi_t$ where π_t is a permutation over T inverting the elements t and $t+1$. In other words, $\pi_t(\bar{t}) = \bar{t}$ if $\bar{t} \neq t, t+1$; $\pi_t(t) = t+1$ and $\pi_t(t+1) = t$. If $n \cdot P(t) \neq n \cdot P(t+1)$, then P_t is a temporal order. Indeed, for $k \in \{1, 2\}$ and $j, \bar{j}$ in I_k but different to $P(t)$ and $P(t+1)$, we have

$$P_t^{-1}(j) = \pi_t^{-1} \cdot P^{-1}(j) = P^{-1}(j) \geqq P^{-1}(\bar{j}) = P_t^{-1}(\bar{j})$$

when $j \geqq \bar{j}$. If $j = P(t)$, then $\bar{j} \neq P(t+1)$ since $n \cdot P(t) \neq n \cdot P(t+1)$ and therefore on the one hand

$$P_t^{-1}(j) = t+1 \geqq P^{-1}(\bar{j}) = P_t^{-1}(\bar{j})$$

when $j \geqq \bar{j}$. On the other hand

$$P_t^{-1}(\bar{j}) = P^{-1}(\bar{j}) \geqq t+1 = P_t^{-1}(j)$$

when $\bar{j} \geqq j$, thus, P_t is a temporal order.

Proposition 1. *A chain $\{Q, P\}$ is maximal if and only if there is a $1 \leqq t < m_1 + m_2$ with $n \cdot P(t) = 1$ and $n \cdot P(t+1) = 2$ such that $Q = P_t$.*

Proof. First of all, let $P(t+1) \in I_2$, then

$$t = P_t^{-1}(P(t+1)) < P^{-1}(P(t+1)) = t+1$$

implies $P_t \succ P$, since for all the remaining $j \in I_2$: $P_t^{-1}(j) = P^{-1}(j)$.

Now, suppose that Q strictly follows P, then, there is a $j \in I_2$: $Q^{-1}(j) < P^{-1}(j)$. Let j_0 be the first $j \in I_2$ with this property. From here, for each t satisfying $Q^{-1}(j_0) < t < P^{-1}(j_0)$, we have $Q(\bar{t}) = P(t) \in I_1$. Indeed, suppose there exists a $\bar{t}$ such that $P(\bar{t})$ or $Q(\bar{t})$ belongs to I_2. Then $P(\bar{t}) = Q(\bar{t}) < P(P^{-1}(j_0))$, since j_0 is the first element with the property just mentioned. But $Q(\bar{t}) > Q(Q^{-1}(j_0)) = j_0$, which is impossible. Thus, $n \cdot P(P^{-1}(j_0) - 1) = 1$ and $n \cdot P(P^{-1}(j_0)) = 2$. This implies that the temporal order $P_{P^{-1}(j_0)-1}$ strictly follows P. Therefore $Q^{-1}(j_0) = P^{-1}(j_0) - 1$ must hold and $Q = P_{P^{-1}(j_0)-1}$. Otherwise it would be $Q \succ P_{P^{-1}(j_0)-1}$ which contradicts the fact that the chain $\{Q, P\}$ is maximal. (q.e.d.)

The temporal order P_t can be regarded as *forward* with respect to P at time t. Of course, the corresponding *backward* of P will be the ${}_tP = P \cdot \pi_t^{-1}$ with the property $({}_tP)_t = {}_t(P_t) = P$. Then the previous proposition for backward orders says that $\{Q, P\}$ is maximal if and only if there is a $1 \leqq t < m_1 + m_2$ with $n \cdot Q(t+1) = 1$ $Q(t) = 2$ such that $P = {}_tQ$.

Using this fact, we derive the following useful result:

Proposition 2. *If Q strictly follows P, then there is a maximal chain $\mathfrak{P}$ relative to Q and P (i.e. Q and P are the respective first and last elements of $\mathfrak{P}$).*

Proof. Let

$$\mathfrak{J}=\{j\in I_2\colon Q^{-1}(j)<P^{-1}(j)\}$$

be a non empty set of indices with first element j_0 and from here we know that the chain $\{P_{P^{-1}(j_0)-1}, P\}$ is maximal, since

$$n\cdot P(P^{-1}(j_0)-1)=1 \quad \text{and} \quad n\cdot P(P^{-1}(j_0)-1)=n\cdot(j_0)=2.$$

If $Q^{-1}(j_0)<P^{-1}(j_0)-1$, we repeat the procedure as far as it is obtained a positive integer h_{j_0} such that $Q^{-1}(j_0)=P^{-1}(j_0)-(h_{j_0}+1)$. Thus, the chain

$$\mathfrak{P}^{j_0}=\{P^{j_0}=(((P_{P^{-1}(j_0)-1})_{P^{-1}(j_0)-2})\dots)_{P^{-1}(j_0)-h_{j_0}},\dots,P_{P^{-1}(j_0)},P\}$$

is maximal, since each chain composed by two adjacent terms is maximal. Therefore if $\mathfrak{J}-\{j_0\}=\emptyset$, $P^{j_0}=Q$ and then we are through. Otherwise, define the sets

$$\bar{T}=T-[1,Q^{-1}(j_0)], \quad \bar{I}=I_1-Q(Q^{-1}[1,j_0])\cap I_1$$

and

$$\bar{I}_2=I_2-[1,j_0]=I_2-Q(Q^{-1}[1,j_0])\cap I_2$$

whose numbers of elements are respectively

$$|\bar{T}|=|\bar{I}_1|+|\bar{I}_2|=|T|-|Q^{-1}(j_0)|<m_1+m_2, \quad |\bar{I}_1|\leqq|I_1|-|Q(Q^{-1}[1,j_0])\cap I_1|\leqq|I_1|$$

and

$$|\bar{I}_2|=|I_2|-j_0<|I_2|.$$

Then, a temporal order over T induces a temporal order over $\bar{T}$, namely: its natural restriction and as a consequence, the order $\succ$ for temporal orders on T also induces an order $\bar{\succ}$ on $\bar{T}$. Hence, one is allowed to apply the transfinite induction. Certainly, let $\bar{P}^{j_0}$ and $\bar{Q}$ be the restrictions over $\bar{T}$ of P^{j_0} and Q, respectively. Because $\mathfrak{J}-\{j_0\}\neq\emptyset$, obviously $\bar{Q}\bar{\succ}\bar{P}^{j_0}$ holds true and therefore by well ordering the cartesian product $N\times N$ of the positive integers N by the relation $(u,v)\ll(u',v')$ if $u<u'$ or if $u=u'$, $v<v'$, one has $(|\bar{I}_1|,|\bar{I}_2|)\ll(|I_1|,|I_2|)$. Thus, by the transfinite induction hypothesis, the existence of a maximal chain $\mathfrak{P}(\bar{Q},\bar{P}^{j_0})$ relative to $\bar{Q}$ and $\bar{P}^{j_0}$ over $\bar{T}$ is guaranteed. Now for each $\bar{R}\in\mathfrak{P}(\bar{Q},\bar{P}^{j_0})$ define the extension R over T by $R(t)=Q(t)$ if $t\leqq Q^{-1}(j_0)$ and $R(t)=\bar{R}(t)$ if $t>Q^{-1}(j_0)$, thus the extension chain $\mathfrak{P}(Q,P^{j_0})$ composed by all the corresponding temporal orders is clearly maximal over T. Hence, the chain $\{\mathfrak{P}(Q,P^{j_0}),\mathfrak{P}^{j_0}-\{P^{j_0}\}\}$ is the desired maximal chain over T relative to Q and P. (q.e.d.)

The method of finding the maximal chain just given is constructive, but it is not unique. In fact, one could make a similar analysis getting another maximal chain going backward by *operating on the last element* instead of the first.

3.

Given a temporal order P, let us define recursively an operator

$$\mathfrak{M}_{P,t}\colon\ R^{\Sigma P[1,t]} \to R^{\Sigma P[1,t-1]} \qquad (t\in T),$$

$$\mathfrak{M}_{P,t}(A)(\sigma_{P[1,t]}) = \begin{cases} \max\limits_{s_{P(t)}} A(\sigma_{P[1,t-1]}, s_{P(t)}) & \text{if } n\cdot P(t)=1 \\ \min\limits_{s_{P(t)}} A(\sigma_{P[1,t-1]}, s_{P(t)}) & \text{if } n\cdot P(t)=2 \end{cases}$$

where R^{Σ} indicates the space of *continuous* real functions on Σ. Thus, introducing the operator

$$\mathfrak{M}_P^t = \prod_{r\geqq t} \mathfrak{M}_{P,r} = \mathfrak{M}_{P,t}\,\mathfrak{M}_{P,t+1}\ldots \mathfrak{M}_{P,m_1+m_2-1}\,\mathfrak{M}_{P,m_1+m_2},$$

the *natural value* $v_P(A)$ of the temporal game A played with respect to the temporal order P, is the number

$$v_P = v_P(A) = \mathfrak{M}_P^1(A).$$

Of course, this quantity determines the safety levels for both players when the game A is played as indicated by P.

Proposition 3. *If Q follows P, then $v_Q \geqq v_P$.*

Proof. For a maximal chain $\{P_t, P\}$ consider for fixed $\sigma_{P[1,t-1]}\in\Sigma_{P[1,t-1]}$ the continuous function

$$\mathfrak{M}_P^{t+2}(A)(\sigma_{P[1,t-1]}, \sigma_{P(t)}, \sigma_{P(t+1)})$$

defined on $\Sigma_{P(t)}\times\Sigma_{P(t+1)}$. Then the inequality between the minimax and the maximin values asserts:

$$\mathfrak{M}_{P_t,t}\,\mathfrak{M}_{P_t,t+1}\,\mathfrak{M}_{P_t}^{t+2}(A)(\sigma_{P[1,t-1]}) \geqq \mathfrak{M}_{P,t}\,\mathfrak{M}_{P,t+1}\,\mathfrak{M}_P^{t+2}(A)(\sigma_{P[1,t-1]}),$$

which implies $v_{P_t} \geqq v_P$.

Now, by iteration of this result to every maximal subchain of two elements of the maximal chain $\{Q, \ldots, P\}$, one obtains $v_Q \geqq v_P$. (q.e.d.)

For example, if P^f and P^i are the *final* and the *initial* temporal orders respectively given by $P^f[1, |I_2|] = I_2$ and $P^i[1, |I_1|] = I_1$, then for every temporal order P, we have that

$$v_{P^f} = \min_{s_{I_2}} \max_{s_{I_1}} A(s_{I_1}, s_{I_2}) \geqq v_P \geqq \max_{s_{I_1}} \min_{s_{I_2}} A(s_{I_1}, s_{I_2}) = v_{P^i},$$

holds true and therefore if the minimax theorem is satisfied for the game $\{\Sigma_{I_1}, \Sigma_{I_2}; A\}$:

$$v_{P^f} = v_P = v_{P^i}.$$

This equality expresses the fact that the game is invariant with respect to each temporal form of playing.

From here, a natural question arises: whether the corresponding values for two different temporal orders coincide or not. Indeed, in general this problem could be quite involved. Nevertheless it is certainly an application of the next result.

Given the set

$$U_\lambda = \{\sigma_{P(T)}:\ \lambda \leqq A(\sigma_{P(T)})\}$$

of strategies in $\Sigma_{I_1 \cup I_2}$ with the $\sigma_{P(\bar{T})}$-section

$$U_\lambda(\sigma_{P(\bar{T})}) = \{\sigma_{P(T-\bar{T})}:\ \sigma_{P(T)} \in U_\lambda\} \subset \Sigma_{P(T-\bar{T})}$$

for any subset $\bar{T} \subset T$, let us define recursively the *upper* strategy set

$$U_\lambda^{P(\bar{T})} = \bigcap\!\!\!\!\bigcup_{s_{P(t_1)}} U_\lambda^{P(\bar{T} \cup \{t_1\})}(s_{P(t_1)}) \subset \Sigma_{P(\bar{T})}; \qquad U_\lambda^{P(T)} = U_\lambda;$$

where t_1 is the first element of $T - \bar{T}$, and the symbol $\bigcap\!\!\!\!\bigcup$ represents the union $\cup$ if $n \cdot P(t_1) = 1$ and the intersection $\cap$ when $n \cdot P(t_1) = 2$. In a similar manner, considering the set

$$L_\lambda = \{\sigma_{P(T)}:\ \lambda \geqq A(\sigma_{P(T)})\}$$

and introducing the symbol $\bigcup\!\!\!\!\bigcap$ which means intersection if $n \cdot P(t_1) = 1$ and union if $n \cdot P(t_1) = 2$, we can well define the *lower* sets

$$L_\lambda^{P(\bar{T})} = \bigcup\!\!\!\!\bigcap_{s_{P(t_1)}} L_\lambda^{P(\bar{T} \cup \{t_1\})}(s_{P(t_1)}); \qquad L_\lambda^{P(T)} = L_\lambda.$$

Directly from the definition of the section for upper sets, one obtains that if $t \neq \bar{t}$ are elements of $\bar{T}$, then

$$\sigma_{P(t)} \in U_\lambda^{P(\bar{T})}(\sigma_{P(\bar{T}-\{t\})}, \sigma_{P(\bar{t})}) \quad \text{if and only if} \quad \sigma_{P(\bar{t})} \in U^{P(\bar{T})}(\sigma_{P(\bar{T}-\{\bar{t}\})}, \sigma_{P(t)})$$

and consequently the induction principle guarantees the following equality:

$$U_\lambda^{P[1,t+1]}(\sigma_{P[1,t-1]}, \sigma_{P(t+1)}) = \{\sigma_{P(t)}:\ \mathfrak{M}_P^{t+2}(A)(\sigma_{P[1,t+1]}) \geqq \lambda\} \subset \Sigma_{P(t)}.$$

Analogously, for the lower sets, we have

$$L_\lambda^{P[1,t+1]}(\sigma_{P[1,t-1]}, \sigma_{P(t)}) = \{\sigma_{P(t+1)}:\ \mathfrak{M}_P^{t+2}(A)(\sigma_{P[1,t+1]}) \leqq \lambda\} \subset \Sigma_{P(t+1)}.$$

Now, the next result which is nothing more that the minimax property for the game $\{\Sigma_{P(t)}, \Sigma_{P(t+1)}; \mathfrak{M}_P^{t+2}(A)\}$ remains clear:

Proposition 4. *Given a maximal chain $\{P_t, P\}$, if for fixed $\sigma_{P[1,t-1]}$, $\sigma_{P(t)}$ and $\sigma_{P(t+1)}$ the upper and lower sets*

$$U_\lambda^{P[1,t+1]}(\sigma_{P[1,t-1]}, \sigma_{P(t+1)}) \subset \Sigma_{P(t)}, \qquad L_\lambda^{P[1,t+1]}(\sigma_{P[1,t-1]}, \sigma_{P(t)}) \subset \Sigma_{P(t+1)}$$

are convex for all the λ, then the values v_{P_t} and v_P coincide.

Proof. The properties imply the quasi-concavity of the function $\mathfrak{M}_P^{t+2}(A)$ with respect to $\sigma_{P(t)}$ and the quasi-convexity in $\sigma_{P(t+1)}$. Thus, by virtue of the minimax theorem,

$$\mathfrak{M}_{P_t,t}\,\mathfrak{M}_{P_t,t+1}\,\mathfrak{M}_{P_t}^{t+2}(A)(\sigma_{P_t[1,t-1]}) = \mathfrak{M}_{P,t}\,\mathfrak{M}_{P,t+1}\,\mathfrak{M}_P^{t+2}(A)(\sigma_{P[1,t-1]})$$

and therefore $v_P = v_{P_t}$. (q.e.d.)

4.

Let us consider some interesting applications of the above fundamental proposition for temporal games. First of all, we present this very simple example: $|T|=3$, $|I_1|=1$ and $|I_2|=2$, where the only three possible temporal orders are $P^i \succ P_1^i \succ P^f = P_2^i$, defined respectively as

$$P^i(1)=1\in I_1, \quad P^i(2)=1\in I_2 \quad \text{and} \quad P^i(3)=2\in I_2,$$
$$P_1^i(1)=1\in I_2, \quad P_1^i(2)=1\in I_1 \quad \text{and} \quad P_1^i(3)=2\in I_2,$$
$$P^f(2)=1\in I_2, \quad P^f(2)=2\in I_2 \quad \text{and} \quad P^f(3)=1\in I_1.$$

By iteration of the previous result we will get sufficient requirements for the validity of the equality $v_{P^i}=v_{P_1^i}=v_{P^f}$. In the first place the quasi-concavity in $\sigma_{P^i(1)}$ and the quasi-convexity in $\sigma_{P^i(3)}$ of the payoff function A for fixed $\sigma_{P^i(2)}$ by virtue of the minimax theorem assures the second relation $v_{P_1^i}=v_{P^f}$. On the other hand, since

$$\mathfrak{M}_{P^i}^3(A)(\cdot,\cdot)=\min_{s_{P^i(3)}} A(\cdot,\cdot,s_{P^i(3)}),$$

then by virtue of the quasi-convexity of A, the upper set

$$\begin{aligned} U_\lambda^{P^i[1,2]}(\sigma_{P^i(2)}) &= \{\sigma_{P^i(1)}\colon \min_{s_{P^i(3)}} A(\sigma_{P^i(1)},\sigma_{P^i(2)},s_{P^i(3)})\geqq\lambda\} \\ &= \bigcap_{s_{P^i(3)}} \{\sigma_{P^i(1)}\colon A(\sigma_{P^i(1)},\sigma_{P^i(2)},s_{P^i(3)})\geqq\lambda\} \end{aligned}$$

is convex and the lower set has the following expression:

$$L_\lambda^{P^i[1,2]}(\sigma_{P^i(1)})=\{\sigma_{P^i(2)}\colon \min_{s_{P^i(3)}} A(\sigma_{P^i(1)},\sigma_{P^i(2)},s_{P^i(3)})\leqq\lambda\}.$$

Now, the convexity of the lower set, that is, the quasi-concavity of the minimum function is assured by the following condition: given $\sigma_{P^i(1)}$ for each real $\mu\in[0,1]$, λ and each pair $\sigma_{P^i[2,3]}$ and $\bar\sigma_{P^i[2,3]}$ such that

$$A(\sigma_{P^i(1)},\sigma_{P^i[2,3]})\leqq\lambda, \qquad A(\sigma_{P^i(1)},\bar\sigma_{P^i[2,3]})\leqq\lambda$$

there is an $\tau_{P^i(3)}^\mu$ satisfying

$$A(\sigma_{P^i(1)},\mu\,\sigma_{P^i(2)}+(1-\mu)\,\bar\sigma_{P^i(2)},\tau_{P^i(3)}^\mu)\leqq\lambda.$$

In fact, taking for a fixed $\sigma_{P^i(1)}$ those strategies $\sigma_{P^i(3)}$ and $\bar\sigma_{P^i(3)}$ where the minimum of the function over $\sigma_{P^i(2)}$ and $\bar\sigma_{P^i(2)}$ is respectively reached, that condition immediately implies the convexity of the lower set.

With the above conditions for the payoff function the previous result guarantees the equality $v_{P^i}=v_{P_1^i}$. It follows at once from both equalities an improvement of the minimax theorem for the game $\{\Sigma_{P^i(1)},\Sigma_{P^i(2)}\times\Sigma_{P^i(3)};A\}$. Indeed, the very well know minimax theorem for this kind of game is obtained directly by checking that the quasi-convexity of the payoff function with respect to the variable $(\sigma_{P^i(2)},\sigma_{P^i(3)})$, implies the condition given above, that is the quasi-convexity of the minimum function. An $\tau_{P^i(3)}^\mu$ satisfying that requirement is $\mu\,\sigma_{P^i(3)}+(1-\mu)\,\bar\sigma_{P^i(3)}$.

A further condition less restrictive than the last one which implies the convexity of the minimum function is the following: given $\sigma_{P^i(1)}$ for each $\mu \in [0,1]$, λ and each pair $\sigma_{P^i[2,3]}, \bar{\sigma}_{P^i[2,3]}$ there is an $\tau^{\mu}_{P^i(3)}$ such that

$$A(\sigma_{P^i(1)}, \mu\sigma_{P^i(2)} + (1-\mu)\bar{\sigma}_{P^i(2)}, \tau^{\mu}_{P^i(3)}) \leqq \mu A(\sigma_{P^i(1)}, \sigma_{P^i(2)}, \sigma_{P^i(3)}) + (1-\mu) A(\sigma_{P^i(1)}, \bar{\sigma}_{P^i(2)}, \bar{\sigma}_{P^i(3)}).$$

As a second illustration, this example can be easily extended to a wider kind of temporal games, where $|T| = m_2 + 1$, $|I_1| = 1$ and $|I_2| = m_2$. Here we have only $m_2 + 1$ possible temporal orders, which are of the form P^t defined by $P^{\bar{t}}(t) = t \in I_2$ when $t < \bar{t}$, $P^{\bar{t}}(t) = 1 \in I_1$ and finally $P^{\bar{t}}(t) = t - 1 \in I_2$ for $t > \bar{t}$. We have $P^t \succ P^{\bar{t}}$ if $t > \bar{t}$ and $P^{m_2+1} = P^f$, $P^1 = P^i$.

We are interested to examine the equalities among the values corresponding to the different temporal orders. For this reason, consider a $t < m_2 + 1$. If the payoff function A is quasi-concave with respect to $\sigma_{P^i(1)} = \sigma_{P^t(t)}$ for the fixed remaining strategies, then the upper set

$$U^{P^t[1,t+1]}(\sigma_{P^t[1,t-1]}, \sigma_{P^t(t+1)}) = \bigcap_{s_{P^t(t+1)}} \cdots \bigcap_{s_{P^t(m_2+1)}} \{\sigma_{P^t(t)}: A(\sigma_{P^t[1,t+1]}, s_{P^t[t+2,m_2+1]}) \leqq \lambda\}$$

is convex. On the other hand, under the following modification of the property just used: given $\sigma_{P^t[1,t]}$, for each real $\mu_t \in [0,1]$, λ and each pair $\sigma_{P^t[t+1,m_2+1]}$, $\bar{\sigma}_{P^t[t+1,m_2+1]}$ such that

$$A(\sigma_{P^t[1,t]}, \sigma_{P^t[t+1,m_2+1]}) \leqq \lambda, \qquad A(\sigma_{P^t[1,t]}, \bar{\sigma}_{P^t[t+1,m_2+1]}) \leqq \lambda$$

there is an $\tau^{\mu_t}_{P^t[t+2,m_2+1]}$ satisfying

$$A(\sigma_{P^t[1,t]}, \mu_t\sigma_{P^t(t+1)} + (1-\mu_t)\bar{\sigma}_{P^t(t+1)}, \tau^{\mu_t}_{P^t[t+2,m_2+1]}) \leqq \lambda,$$

the convexity of the lower set

$$L_{\lambda}^{P^t[1,t+1]}(\sigma_{P^t[1,t-1]}, \sigma_{P^t(t)}) = \{\sigma_{P^t(t+1)}: \min_{s_{P^t[t+1,m_2+1]}} A(\sigma_{P^t[1,t+1]}, s_{P^t[t+1,m_2+1]}) \leqq \lambda\},$$

is assured. Thus, under such conditions, by virtue of Proposition 4, the equality $v_{P^t} = v_{P^{\dot{t}}_t}$ holds. We remark that for $t = m_2$ it is exactly the definition of the quasi-convexity in $\sigma_{P^t(m_2)}$.

Now, if the last modified condition is satisfied for every $t < m_2 + 1$, then

$$v_{P^1} = v_{P^2} = \cdots = v_{P^{m_2}} = v_{P^{m_2+1}}$$

is valid, and again the minimax theorem for the game $\{\Sigma_{P^i(1)}, \prod_{1 \leqq t \leqq m_2+1} \Sigma_{P^i(t)}; A\}$ has been improved.

Again, when the payoff function is quasi-concave in the joint variable $\sigma_{P^i[2,m_2+1]}$ then the modified condition is fulfilled for every $t < m_2 + 1$, as one can see immediately.

This example can also be used for temporal games having $|T| = m_1 + 1$, $|I_1| = m_1$ and $|I_2| = 1$. In fact, changing the roles of both players and considering the payoff function $-A$ in the case just considered, one obtains under the condition: given $\sigma_{Q^t[1,t]}$ for each real $\mu_t \in [0,1]$, λ and each pair $\sigma_{Q^t[t+1,m_1+1]}, \bar{\sigma}_{Q^t[t+1,m_1+1]}$ such that

$$A(\sigma_{Q^t[1,t]}, \sigma_{Q^t[t+1,m_1+1]}) \geqq \lambda, \qquad A(\sigma_{Q^t[1,t]}, \bar{\sigma}_{Q^t[t+1,m_1+1]}) \geqq \lambda$$

there exists an $\tau^{\mu_t}_{Q^t[t+1, m_1+1]}$ with the property

$$A\left(\sigma_{Q^t[1,t]}, \mu_t\, \sigma_{Q^t(t)} + (1-\mu_t)\, \bar{\sigma}_{Q^t(t)}, \tau^{\mu_t}_{Q^t[t+1, m_1+1]}\right) \geqq \lambda$$

where the temporal order is given by $Q^{\bar{t}}(t) = t \in I_1$ when $t < \bar{t}$, $Q^{\bar{t}}(\bar{t}) = 1 \in I_2$ and finally $Q^{\bar{t}}(t) = t-1 \in I_2$ if $t > \bar{t}$; and under the quasi-convexity in $\sigma_{Q^t(1)} = \sigma_{Q^t(t)}$ of the payoff function, that $v_{Q^t} = v_{Q^t_t}$. Thus, if the property just explained is satisfied for every $t < m_1 + 1$, then, the respective values of all the temporal orders coincide.

After dealing with these simple cases, finally we are concerned with a very general case based on those examples, because of its own interest.

Let us consider a general temporal game with $|T| = m_1 + m_2$, $|I_1| = m_1$ and $|I_2| = m_2$.

We wish to see that under the following conditions of *minimum-convexity* and *maximum-concavity* all the values coincide.

The temporal game satisfies the maximum-concavity condition if for all the $m_1 \leqq t < m_1 + m_2$, given $\sigma_{P^f[1,t]}$, for each real $\mu_t \in [0, 1]$, λ and each pair $\sigma_{P^f[t+1, m_1+m_2]}$, $\bar{\sigma}_{P^f[t+1, m_1+m_2]}$ such that

$$A(\sigma_{P^f[1,t]}, \sigma_{P^f[t+1, m_1+m_2]}) \geqq \lambda, \qquad A(\sigma_{P^f[1,t]}, \bar{\sigma}_{P^f[t+1, m_1+m_2]}) \geqq \lambda$$

there exists an $\tau^{\mu_t}_{P^f[t+2, m_1+m_2]}$ with the property

$$A\left(\sigma_{P^f[1,t]}, \mu_t\, \sigma_{P^f(t+1)} + (1-\mu_t)\, \bar{\sigma}_{P^t(t+1)}, \tau^{\mu_t}_{P^f[t+2, m_1+m_2]}\right) \geqq \lambda.$$

For all those t this condition implies the convexity of the upper set

$$U_\lambda^{P^f[1,t+1]}(\sigma_{P^f[1,t]}) = \left\{\sigma_{P^f(t+1)} : \max_{s_{P^f[t+2, m_1+m_2]}} A(\sigma_{P^f[1,t+1]}, s_{P^f[t+2, m_1+m_2]}) \geqq \lambda\right\}.$$

The minimum-concavity condition is the following: for all the $m_2 \leqq t < m_1 + m_2$, given $\sigma_{P^i[1,t]}$, for each real $\mu_t \in [0, 1]$, λ and each pair $\sigma_{P^i[t+1, m_1+m_2]}$, $\bar{\sigma}_{P^i[t+1, m_1+m_2]}$ such that

$$A(\sigma_{P^i[1,t]}, \sigma_{P^i[t+1, m_1+m_2]}) \leqq \lambda, \qquad A(\sigma_{P^i[1,t]}, \bar{\sigma}_{P^i[t+1, m_1+m_2]}) \leqq \lambda$$

there is an $\tau^{\mu_t}_{P^i[t+2, m_1+m_2]}$ satisfying

$$A\left(\sigma_{P^i[1,t]}, \mu_t\, \sigma_{P^i(t+1)} + (1-\mu_t)\, \bar{\sigma}_{P^i(t+1)}, \tau^{\mu_t}_{P^i[t+2, m_1+m_2]}\right) \leqq \lambda.$$

Under such a condition the convexity of the lower set for all the $m_1 \leqq t < m_1 + m_2$,

$$L_\lambda^{P^i[1,t+1]}(\sigma_{P^i[1,t]}) = \left\{\sigma_{P^i(t+1)} : \min_{s_{P^i[t+2, m_1+m_2]}} A(\sigma_{P^i[1,t+1]}, s_{P^i[t+2, m_1+m_2]}) \leqq \lambda\right\}$$

is guaranteed.

Consider the chain $\{P^i_{m_1}, P^i\}$. On one hand, by virtue of the minimum convexity property, taking $t = m_2$, we find that the respective lower set of Proposition 4 is convex. On the other hand, the respective upper set is convex since the payoff function has the maximum-convexity property for $t = m_1 + m_2 - 1$, that is, the quasi-concavity with respect to $\sigma_{P^f(m_1+m_2)} = \sigma_{P^i(m_1)}$, and because the minimum of quasi-concave functions is quasi-concave. Thus, $v_{P^i_{m_1}} = v_{P^i}$.

Now, let us examine the chain $\{(P^i_{m_1})^{m_1}_{m_1-1}, P^i_{m_1}\}$. By virtue of the previous example, for each joint strategy $\sigma_{P^i[1,m_1]}$, we have

$$\mathfrak{M}^{m_1+1}_{P^i_{m_1}}(A)(\sigma_{P^i_{m_1}[1,m_1]}) = \max_{s_{P^i_{m_1}(m_1+1)}} \min_{s_{P^i_{m_1}[m_1+2,m_1+m_2]}} A(\sigma_{P^i_{m_1}[1,m_1]}, s_{P^i_{m_1}[m_1+1,m_1+m_2]})$$
$$= \min_{s_{P^i_{m_1}[m_1+2,m_1+m_2]}} \max_{s_{P^i_{m_1}(m_1+1)}} A(\sigma_{P^i_{m_1}[1,m_1]}, s_{P^i_{m_1}[m_1+1,m_1+m_2]})$$

and therefore because of the minimum-convexity property, the minimum is quasi-convex and its maximum maintains this property, the function $\mathfrak{M}^{m_2+1}_{P_{m_1}}(A)$ is quasi-convex in $\sigma_{P^i_{m_1}(m_1)} = \sigma_{P^i(m_1+1)}$. Similarly, by an analogous argument using the second term of the last expression, we obtain the quasi-concavity with respect to $\sigma_{P^i_{m_1}(m_1-1)} = \sigma_{P^i(m_1-1)}$ of the function $\mathfrak{M}^{m_1+1}_{P^i_{m_1}}(A)$, so that $v_{(P^i_{m_1})_{m_1-1}} = v_{P^i_{m_1}}$.

Now, using the induction hypothesis, we admit the minimax theorem:

$$\mathfrak{M}^{m_1}_{P_{m_1,m_1-1}}(A)(\sigma_{P_{m_1,m_1-1}[1,m_1-1]})$$
$$= \max_{s_{P_{m_1,m_1-1}[m_1,m_1+1]}} \min_{s_{P_{m_1,m_1-1}[m_1+2,m_1+m_2]}} A(\sigma_{P_{m_1,m_1-1}[1,m_1-1]}, s_{P_{m_1,m_1-1}[m_1,m_1+m_2]})$$
$$= \min_{s_{P_{m_1,m_1-1}[m_1+2,m_1+m_2]}} \max_{s_{P_{m_1,m_1-1}[m_1,m_1+1]}} A(\sigma_{P_{m_1,m_1-1}[1,m_1-1]}, s_{P_{m_1,m_1-1}[m_1,m_1+m_2]})$$

for fixed $\sigma_{P_{m_1,m_1-1}[1,m_1-1]}$, where $P_{m_1,m}$ indicates $(\ldots((P^i_{m_1})_{m_1-1})\ldots)_m$ with $m<m_1$. Again, the minimum of the first term is quasi-convex in $\sigma_{P_{m_1,m_1-1}(m_1)} = \sigma_{P^i(m_1+1)}$ which implies the same property for the function $\mathfrak{M}^{m_1}_{P_{m_1,m_1-1}}(A)$. On the other hand, the maximun in the second term is quasi-concave in $\sigma_{P_{m_1,m_1-1}(m_1-1)} = \sigma_{P^i(m_1-1)}$ and hence $\mathfrak{M}^{m_1}_{P_{m_1,m_1-1}}(A)$ is too. Therefore Proposition 4 assures the validity of $v_{P_{m_1,m_1-2}} = v_{P_{m_1,m_1-1}}$.

By iteration of this procedure, we obtain the equalities

$$v_{P_{m_1,1}} = \cdots = v_{P_{m_1,m_1-1}} = v_{P_{m_1}} = v_{P^i}.$$

Finally, using again the minimax theorem as induction hypothesis for the function $\mathfrak{M}^2_{P_{m_1,1}}(A)$ it is guaranteed $v_{P_{m_1,1}} = v_{P^f}$, and so we have the desired result.

The well known minimax theorem for games having both strategy sets as cartesian products is derived from the previous improvement recalling that the quasi-convexity property in the joint variable of the second player implies the minimum-convexity property described above and similarly the quasi-concavity in the joint variable of the first player satisfies the maximum-concavity condition.

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