Z. Wahrscheinlichkeitstheorie verw. Gebiete
61, 527–539 (1982)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Harmonic Renewal Measures and Bivariate Domains of Attraction in Fluctuation Theory

P. Greenwood[1], E. Omey[2], and J.L. Teugels[2]

[1] University of British Columbia, Vancouver, Canada
[2] Catholic University of Louvain, Dept. Wiskunde, Celestijnenlaan 200 B, B-3030 Louvain, Belgium

Summary. Each probability measure C on a first orthant is associated with a harmonic renewal measure G. Specifically we consider (N, S_N) the ladder (time, place) of a random walk S_n. Using bivariate G we show that when S_1 is in a domain of attraction so is (N, S_N). This unifies and generalizes results of Sinai, Rogosin.

Introduction

Let $\{S_n\}_1^\infty$ be a random walk generated by a random variable with step distribution F. Let $N = \inf\{n\colon S_n > 0\}$ be the first ladder index and S_N the first ladder height. Our object here is to show that if S_1 is in a domain of attraction, then the bivariate (N, S_N) is in an associated bivariate domain of attraction. The marginal results are theorems of Rogosin for N [11] and of Sinai [14] and Rogosin [12] for S_N.

We offer a unified treatment using harmonic renewal measures and double exponential representations. We rely heavily on the univariate results in [6].

§ 1a. Univariate Case: Harmonic Renewal Measure

Since we will heavily rely on the results for the univariate case we now formulate some theorems for that situation. We treat these results in such a way that they are optimally stated for future application in the bivariate case; the main methodology of the proof in the latter situation can be viewed as a generalisation from the univariate to the bivariate case.

So let $X_1, X_2, \ldots$ be i.i.d. with distribution C on $(0, \infty)$. We say that C is in a *domain of attraction* iff we can find constants $a_n \uparrow \infty$ such that $a_n^{-1}(X_1 + \ldots + X_n)$ converges in distribution to a r.v. nondegenerate at 0.

Alternative formulations are contained in the well-known lemma. Lower case letters stand for LST (Laplace-Stieltjes transform) of corresponding capital letters.

0044-3719/82/0061/0527/$02.60

Lemma 1.1. *The following statements are equivalent for* $a_n \uparrow \infty$

(i) $a_n^{-1}(X_1+\ldots+X_n) \xrightarrow{\mathscr{D}} U$, *non-degenerate at* 0;

(ii) $\forall x \geqq 0$: $C^{(n)}(a_n x) \to H(x) = P\{U \leqq x\}$, U *non-degenerate at* 0;

(iii) $\forall s>0$, $n\{1-c(s/a_n)\} \to -\log h(s) \not\equiv 0$.

The conditions of Lemma 1.1 can be transformed into a variety of equivalent statements involving the tail of C or the harmonic renewal measure G associated to C. Recall that $G(x)=\sum_1^\infty n^{-1} C^{(n)}(x)$ and that $1-c(s)=\exp -g(s)$.

Lemma 1.2. *Let* $0<\beta\leqq 1$ *and* L^0 s.v. *Put* $R(x)=x^{-\beta} L^0(x)$. *The following statements are equivalent:*

(i) $\int_0^x [1-C(y)]\, dy \sim x R(x)/\Gamma(2-\beta)$ *as* $x\to\infty$;

(ii) $\forall t>0$: $G(x t)+\log R(x) \to \beta\gamma+\beta\log t$ *as* $x \to \infty$;

(iii) $\forall \theta>0$, $g(\theta s)+\log R(1/s) \to -\beta\log\theta$ *as* $s \downarrow 0$.

Proof. (i)$\Leftrightarrow$(ii): It follows from Theorem 1 and 2 in [6] that (i) is equivalent to

$$G(x)+\log R(x) \to \beta\gamma. \tag{1}$$

Change x into xt and use $R(xt)/R(x)\to t^{-\beta}$.

(ii)$\Leftrightarrow$(iii): Again in [6] it was shown that (1) is equivalent to

$$g(s)+\log R(1/s) \to 0.$$

Put s equals θs to obtain (iii). $\square$

As a guideline for the bivariate case we briefly indicate one possible procedure to link the above two lemmas.

Assume (ii) of Lemma 1.2 holds. Then for any sequence $a_n\uparrow\infty$ and $0<x<u<\infty$

$$\int_x^u G(a_n\, dp) = G(a_n u)-G(a_n x) \to \beta\log\frac{u}{x} \equiv \int_x^u L(dp). \tag{2}$$

But then for $\theta\in(0,\infty)$, $x\in(0,\infty)$

$$g\left(\frac{\theta}{a_n}\right) - G(a_n x) = \int_0^x (e^{-\theta u}-1)\, G(a_n\, du) + \int_x^\infty e^{-\theta u} G(a_n\, du)$$

$$\to \int_0^\infty (e^{-\theta p} - I_{(0,x)}(p))\, L(dp).$$

Or for $\theta\in(0,\infty)$, $x\in(0,\infty)$, (2) implies

$$g\left(\frac{\theta}{a_n}\right) - G(a_n x) \to -\beta(\gamma+\log\theta x) \equiv \gamma(\theta, x) \tag{3}$$

where we used the explicit form of L and one of the possible representations of Euler's constant [5, p. 946].

$$\gamma=\lim_{n\to\infty}\left\{\sum_{m=1}^{n}\frac{1}{m}-\log n\right\}=\int_0^1(1-e^{-w})\frac{dw}{w}-\int_1^\infty e^{-w}\frac{dw}{w}. \tag{4}$$

However (3) implies that for $\theta\in(0,\infty)$

$$g\left(\frac{1}{a_n}\right)-g\left(\frac{\theta}{a_n}\right)\to\gamma(1,1)-\gamma(\theta,1)=\beta\log\theta.$$

Choose a_n in such a way that $g(1/a_n)=\log n$ then using $-g(s)=\log(1-c(s))$ we get

$$\log n+\log(1-c(\theta/a_n))\to\log[-\log h(\theta)]=\beta\log\theta \tag{5}$$

which is (iii) of Lemma 1.1 with $h(\theta)=\exp(-\theta^\beta)$.

The point of the above analysis is that in the bivariate case we do not have such a simple expression for the limit quantities L and h. In the bivariate case a relation of the form (2) is given in Lemma 3.1. The consequence (3) is treated in Lemma 2.2. The analogue of (5) is the content of Theorem 3.2.

§ 1b. Univariate Case: Double Exponential Representations of Infinitely Divisible Distributions

The limit h can be put in a form that generalizes to more dimensions. Suppose that C is infinitely divisible and let $C^{(t)}$ denote the t-th power of C, i.e. the distribution corresponding to the LST $c^t(s)$; equivalently $C^{(t)}$ is the distribution of X_t where X is a process with stationary independent increments such that X_1 has distribution C.

We will use the Frullani integral form of $\log(a/b)$, i.e. for $a>0$ and $b>0$ [5, p. 334]

$$-\log\frac{a}{b}=\int_0^\infty\frac{e^{-aw}-e^{-bw}}{w}\,dw.$$

Write $a=1$ and $b=-\log c(s)$; then successively

$$\begin{aligned}\log[-\log c(s)]&=\int_0^\infty w^{-1}\{e^{-w}-[c(s)]^w\}\,dw\\&=\int_0^\infty w^{-1}\left\{e^{-w}-\int_0^\infty e^{-sz}P\{X_w\in dz\}\right\}dw\\&=\int_0^\infty\int_0^\infty w^{-1}\{e^{-w}-e^{-sz}\}\,P\{X_w\in dz\}\,dw\end{aligned}$$

or

$$\log[-\log c(s)]=\int_0^\infty\int_0^\infty w^{-1}\{e^{-w}-e^{-sz}\}\,C^{(w)}(dz)\,dw. \tag{6}$$

Hence when C is infinitely divisible we have as a companion to

$$\log[1-c(s)] = -\int_0^\infty e^{-sz} G(dx)$$

the "continuous time" representation (6). We call (6) the *double exponential representation* of the distribution C or of its transform c.

It is useful to note at this point that the above derivation of (6) is equally straightforward if C is a distribution on $(-\infty, \infty)$ and c its characteristic function, or, in fact if C is a distribution on a higher dimensional space. The bivariate version of (6) will be used in § 2.

Return to (5). Then as

$$\log\{-\log h(s)\} = \int_0^\infty \int_0^\infty w^{-1}\{e^{-w} - e^{-sz}\} B^{(w)}(dz)\, dw$$

we can write for $s \in (0. \infty)$

$$\log n - \sum_{m=1}^{n} \frac{1}{m} + \int_0^\infty (1-e^{-sx}) \sum_{m=1}^{n} \frac{1}{m} C^{(m)}(a_n dx) - \int_0^\infty e^{-sx} \sum_{m=n+1}^{\infty} \frac{1}{m} C^{(m)}(a_n dx)$$
$$\to -\gamma + \int_0^\infty (1-e^{-sx}) \int_0^1 \frac{dw}{w} B^{(w)}(dx) - \int_0^\infty e^{-sx} \int_1^\infty \frac{dw}{w} B^{(w)}(dx) \tag{7}$$

where we performed some simple algebra on the double exponential representation of $h(s)$.

The limiting operation in (7) constitutes another alternate form of the conditions listed in Lemmas 1.1 and 1.2; it tells us that the convergence of the functional of a sample path of the random walk which can be read from either side of (7) is equivalent to the regular variation of C.

The analogous version of (7) for bivariate C is of additional interest and use because in the bivariate case we do not know how to write the characteristic exponent $-\log h$ for the limiting stable distribution in closed form.

§ 2. Bivariate Case

Let C be a distribution on $(0, \infty) \times (0, \infty)$ such that $C(0+, 0+) = 0$ and let c be its Laplace-Stieltjes transform; let G be the corresponding harmonic renewal measure with $LST\, g$. Hence

$$\begin{aligned}
&\forall \lambda \geqq 0, \mu \geqq 0 \quad && c(\lambda, \mu) = \int_0^\infty \int_0^\infty \exp -(\lambda x + \mu y)\, C(dx, dy) \\
&\forall (x, y) \in \mathbb{R}_+^2 && G(x, y) = \sum_{n=1}^{\infty} \frac{1}{n} C^{(n)}(x, y) \\
&\forall \lambda \geqq 0, \mu \geqq 0 && 1 - c(\lambda, \mu) = \exp - g(\lambda, \mu) \\
&\forall \lambda \geqq 0, \mu \geqq 0 && g(\lambda, \mu) = \int_0^\infty \int_0^\infty \exp -(\lambda x + \mu y)\, G(dx, dy).
\end{aligned} \tag{8}$$

A bivariate distribution C is in the *domain of attraction* of a bivariate stable distribution H if there exist real sequences $a_n \uparrow \infty$, $b_n \uparrow \infty$ such that for $x \geqq 0$, $y \geqq 0$ as $n \to \infty$

$$C^{(n)}(a_n x, b_n y) \to H(x, y)$$

in the sense of convergence of distribution functions. The distribution H is, by definition, *bivariate stable* if it is such a limit.

Conditions equivalent to the bivariate domain of attraction condition in the more general setting where C is a distribution on $\mathbb{R}^2$ and centering constants may also appear, were obtained by Resnick and Greenwood [7]. The following additional result is also useful.

Lemma 2.1. *The following statements are equivalent*

(i) $\forall (x, y) \in \mathbb{R}_+^2$: $C^{(n)}(a_n x, b_n y) \to H(x, y)$,

(ii) $\forall \lambda > 0, \mu > 0$: $n\{1 - c(\lambda/a_n, \mu/b_n)\} \to -\log h(\lambda, \mu)$.

Going from (2) to (3) is generally possible in the bivariate case as shown in the following result. Let G be as defined in (8).

Lemma 2.2. *If for* (x, y, u, v) *satisfying* $0 \leqq x < u$, $0 \leqq y < v$ *but* $x > 0$ *or* $y > 0$

$$\int_x^u \int_y^v G(a_n dp, b_n dq) \to \int_x^u \int_y^v L(dp, dq) \tag{9}$$

as $n \to \infty$, *then for positive* θ, η, x, y *there exists a function* $\gamma(\theta, \eta, x, y)$ *such that*

$$g\left(\frac{\theta}{a_n}, \frac{\eta}{b_n}\right) - G(a_n x, b_n y) \to \gamma(\theta, \eta, x, y). \tag{10}$$

If (9) *holds with* $v = \infty$, *then* (10) *holds with* $y = \infty$.

Proof. The hypothesis implies convergence of the measure G_n associated with $G(a_n x, b_n y)$ on Borelsets in $\mathbb{R}_+^2$.

Let $A(x, y) = (0, x) \times (0, y)$ and $A(x, y)^c = A(\infty, \infty) - A(x, y)$. Then

$$\begin{aligned} g\left(\frac{\theta}{a_n}, \frac{\eta}{b_n}\right) - G(a_n x, b_n y) = & \iint_{A(x,y)} [e^{-\theta u - \eta v} - 1]\, G(a_n du, b_n dv) \\ & + \iint_{A(x,y)^c} e^{-\theta u - \eta v}\, G(a_n du, b_n dv). \end{aligned} \tag{11}$$

Denote the first integral on the right of (11) by $I_n(x, y)$, the second by $II_n(x, y)$.

Pick $\varepsilon > 0$. By assumption for $x > \varepsilon$, $y > \varepsilon$

$$I_n(x, y) - I_n(\varepsilon, \varepsilon) \to \iint_{A(x,y) - A(\varepsilon,\varepsilon)} [e^{-\theta u - \eta v} - 1]\, L(du, dv).$$

But

$$0 \leqq -I_n(\varepsilon, \varepsilon) \leqq \iint_{A(\varepsilon,\varepsilon)} (\theta u + \eta v)\, G(a_n du, b_n dv).$$

For the first term of the integral we write $G_1(x)=\int_0^x u\,G(du,\infty)$; then

$$\int_0^\varepsilon u\int_0^\varepsilon G(a_n\,du, b_n\,dv)\leqq\int_0^\varepsilon u\int_0^\infty G(a_n\,du, b_n\,dv)=a_n^{-1}\,G_1(a_n\,\varepsilon).$$

But G_1 satisfies the conditions of Lemma 3.1.1 in [6]. Hence uniformly in n, $G_1(a_n\,\varepsilon)\leqq a_n\,\varepsilon$ and $|I_n(\varepsilon,\varepsilon)|\leqq\varepsilon(\theta+\eta)$. Letting $n\to\infty$ and then $\varepsilon\downarrow 0$ we find

$$I_n(x,y)\to\iint_{A(x,y)}(e^{-\theta u-\eta v}-1)\,L(du,dv).$$

We pass to II_n, and write $II_n(x,y)=II_n(x_o,y_o)+(II_n(x,y)-II_n(x_o,y_o))$ where (x_o,y_o) will be determined directly. By assumption the second part of $II_n(x,y)$ will converge. Now

$$\begin{aligned}II_n(x_o,y_o)=&\int_{x_o}^\infty\int_0^{y_o} e^{-(\theta u+\eta v)}\,G(a_n\,du, b_n\,dv)\\&+\int_0^\infty\int_{y_o}^\infty e^{-(\theta u+\eta v)}\,G(a_n\,du, b_n\,dv).\end{aligned}\tag{12}$$

The first integral on the right, say J_n, is estimated as follows:

$$\begin{aligned}J_n&\leqq\int_{x_o}^\infty\int_0^\infty e^{-\theta u}\cdot 1\cdot G(a_n\,du, b_n\,dv)=\int_{x_o}^\infty e^{-\theta u}\,G(a_n\,du,\infty)\\&=\theta\int_{x_o}^\infty e^{-\theta u}\{G(a_n\,u,\infty)-G(a_n\,x_o,\infty)\}\,du.\end{aligned}$$

But as proved in Lemma 3.1.1 [6], $|G(u,\infty)-G(v,\infty)|\leqq 1+\log\left|\frac{u}{v}\right|$. Hence uniformly in n

$$J_n\leqq\theta\int_{x_o}^\infty e^{-\theta u}\left\{1+\log\frac{u}{x_o}\right\}du$$

and this can be made as small as we want by taking x_o large enough.

A similar argument estimates the second integral on the right of (12).

This proves the lemma. □

The form of the limit in (10) can be conveniently written as

$$\gamma(\theta,\eta,x,y)=\int_0^\infty\int_0^\infty\{e^{-\theta p-\eta q}-I_{(0,x)\times(0,y)}(p,q)\}\,L(dp,dq).\tag{13}$$

For future reference we write down the double exponential representation for the case when C is infinitely divisible: for $\lambda>0$, $\mu>0$

$$\log\{-\log c(\lambda,\mu)\}=\int_0^\infty\int_0^\infty\int_0^\infty[e^{-w}-e^{-\lambda x-\mu y}]\,C^{(w)}(dx,dy)\frac{dw}{w}.\tag{14}$$

§ 3. The Bivariate Distribution of (N, S_N)

Now we consider a random walk $\{S_n\}_1^\infty$ generated by S_1; we let $N=\inf\{n: S_n>0\}$ be the ladder index and S_N the ladder height. From now on

$$C(x, y)=P\{N\leqq x, S_N\leqq y\}.$$

The Spitzer-Baxter identity provides an identification of the harmonic renewal measure G associated with C in terms of the distributions of $\{S_m, m\in\mathbb{N}\}$. Indeed

$$\log\{1-c(\lambda,\mu)\}=\int_0^\infty\int_0^\infty e^{-\lambda x-\mu y}\, d\left\{\sum_{m=1}^\infty \frac{1}{m} P[S_m\in dy]\,\delta_m(dx)\right\}$$

where δ_m is a unit-step distribution at m. Hence

$$G(dx, dy)=\sum_{m=1}^\infty \frac{1}{m} P[S_m\in dy]\,\delta_m(dx)$$

or in integrated form

$$G(x, y)=\sum_{m=1}^{[x]} \frac{1}{m} P[0<S_m\leqq y]. \tag{15}$$

We first show that if S_1 is in a domain of attraction, G satisfies (9). From the resulting relation (10) we then show that (ii) of Lemma 2.1 is satisfied. On the basis of [6] and Lemma 1.2 one might expect that we should look for a bivariate version of de Haan's result [1] linking the bivariate G to g and hence to c. We leave these questions aside and prove a result which looks like a de Haan-type theorem in two variates, but which is actually univariate along certain curves in 2-space.

We now provide the bivariate analogue of (2).

Lemma 3.1. *Suppose $0<c_n\to\infty$ is chosen in such a way that $S_n/c_n \xrightarrow{\mathscr{D}} X_1$, stable with index α. Let G be defined by (15). Then for $0\leqq x<u<\infty$, $0\leqq y<v\leqq\infty$ but $x>0$ or $y>0$*

$$\int_x^u\int_y^v G(n\,dp, c_n\,dq)\to\int_x^u\int_y^v \frac{1}{t} P\{X_t\in dz\}\,dt \tag{16}$$

where $\{X_t, t\geqq 0\}$ is the associated stable processes.

Proof. First assume $0<x<u<\infty$, $0<y<v\leqq\infty$; then the left hand side of (16) can be written as

$$\int_x^u \frac{n}{[nt]} P\{y<c_n^{-1} S_{[nt]}\leqq v\}\,dt.$$

By assumption $P\{y<c_n^{-1} S_{[nt]}\leqq v\}\to P\{y<X_t\leqq v\}$; since $X_t\overset{d}{=}t^{1/\alpha}X_1$ and X_1 has a continuous distribution, the convergence is uniform [9, p. 139]. Also uniformly $n/[nt]\to 1/t$. From the uniform convergence of the integrands to a continuous limit it follows that the integrals converge.

Let now $y=0$, $v=\varepsilon$, for a fixed $\varepsilon>0$. Then an entirely similar argument yields the required convergence on $[x, u]$ as long as $x>0$.

Finally, let $x=0$, $u=\varepsilon$, $v\leqq\infty$ but $y>0$. The next argument has been inspired by a result of Heyde [8, p. 306] on large deviations. We look for an upper bound on $P[S_m>c_n y]$ where $1\leqq m\leqq[n\varepsilon]$.

Let $Y_k\equiv S_k-S_{k-1}$ $(k\geqq 1)$; take Y_k^s as a symmetrized version of Y_k; finally truncate Y_k^s at $z\equiv c_n y$:

$$Y_k'=\begin{cases} Y_k^s & \text{if } |Y_k^s|\leqq z\\ 0 & \text{if not.}\end{cases}$$

Then

$$P\left\{\left|\sum_{k=1}^m Y_k^s\right|>z\right\}\leqq m\,P\{|Y_k^s|>z\}+P\left\{\left|\sum_{k=1}^m Y_k'\right|>z\right\}.$$

Put for brevity $H(x)=1-F(x)+F(-x)$ and $U(x)=\int_{-x}^{x} y^2\,dF(y)$ where F is the d.f. of $Y_1=S_1$. By a weak symmetrization inequality [10, p. 257] we obtain

$$P\{|Y_k^s|>z\}\leqq 2P\left\{|Y_k|>\frac{z}{2}\right\}\leqq 2H(z/2).$$

By Chebyshev's inequality and part of Heyde's reasoning

$$P\left\{\left|\sum_{k=1}^m Y_k'\right|>z\right\}\leqq z^{-2}E\left|\sum_{k=1}^m Y_k'\right|^2\leqq 2m z^{-2}\int_0^z x\,P\{|Y_1^s|>x\}\,dx$$
$$\geqq 16 m z^{-2}\int_0^{z/2} y\,H(y)\,dy=2m\{H(z/2)+8z^{-2}U(z/2)\}.$$

Combining the above inequalities we get

$$P\left\{\left|\sum_{k=1}^m Y_k^s\right|>z\right\}\leqq 4m\{H(z/2)+2z^{-2}U(z/2)\}.$$

Since S_1 belongs to the domain of attraction of a stable law with index α, $u^2H(u)/U(u)\to(2-\alpha)\cdot\alpha^{-1}$ as $u\to\infty$, [3, p. 313]; moreover $U(u)\sim u^{2-\alpha}L(u)$ for some s.v. L, [3, p. 312] while c_n is determined by $n c_n^{-2}U(c_n)\to 1$. Hence $z^{-2}U(z/2)=y^{-2}c_n^{-2}U(\tfrac{1}{2}y c_n)\sim y^{-\alpha}2^{\alpha-2}n^{-1}$. But then there exists a constant K_1 independent of m, n and ε such that

$$P\left\{\left|\sum_{k=1}^m Y_k^s\right|>z\right\}\leqq K_1 y^{-\alpha}\frac{m}{n}.$$

This is the desired inequality for the symmetrized sequence. Using another weak symmetrization inequality [10, p. 257] we find a constant K_2.

$$P\{S_m-\mu_m>z\}\leqq K_2 y^{-\alpha}\frac{m}{n} \tag{17}$$

where μ_m is a median for S_m.

In order to get rid of μ_m we first note that μ_m/c_m is a bounded sequence; for if not take a subsequence $m'\to\infty$ along which say $\mu_{m'}/c_{m'}\to\infty$. Then by assumption

$$\tfrac{1}{2} \leqq P\{S_{m'} \geqq \mu_{m'}\} = P\{(S_{m'}/c_{m'} \geqq (\mu_{m'}/c_{m'})\} \to P\{X_1 \geqq \infty\}$$

which is a contradiction. Hence for $1 \leqq m \leqq [n\varepsilon]$ and constants K_i $(i=3,4,5)$ as before we have

$$|\mu_m|/c_n = K_3(c_m/c_n) = K_4(c_{[n\varepsilon]}/c_n) \leqq K_5\, \varepsilon^{1/\alpha} \equiv \varepsilon'.$$

Indeed c_n is a r.v. sequence with index α^{-1} and hence is asymptotically monotone. It follows that

$$P\{S_m > c_n y\} \leqq P\{S_m - \mu_m > c_n y'\}$$

where $y' = y - \varepsilon'$. By the first part of the proof we can assume without loss of generality that $y' > 0$ by taking ε small enough. Then by (17) finally for $1 \leqq m \leqq [n\varepsilon]$

$$P\{S_m > c_n y\} \leqq K_2 (y')^{-\alpha} \frac{m}{n}.$$

Returning to (16) we see that

$$\begin{aligned}\int_0^\varepsilon \int_y^v G(n\,dp, c_n\,dq) &= \sum_{m=1}^{[n\varepsilon]} \frac{1}{m} P\{c_n y < S_m \leqq c_n v\} \\ &\leqq \sum_{m=1}^{[n\varepsilon]} \frac{1}{m} P[c_n y < S_m] \leqq K_2 (y')^{-\alpha}\, \varepsilon.\end{aligned}$$

On the right hand side of (16) we remark that by the stability of the limit $\{X_t\}$ there exists a constant K' such that

$$P\{y < X_t\} = P\{y\, t^{-1/\alpha} < X_1\} \leqq K'\, y^{-\alpha}\, t.$$

Hence

$$\int_0^\varepsilon \frac{dt}{t} \int_y^v P\{X_t \in dz\} \leqq K'\, y^{-\alpha}\, \varepsilon.$$

Letting $\varepsilon \downarrow 0$, (16) is fully proved. □

As a consequence of the above lemma we obtain

$$\gamma(\theta, \eta, x, y) = \int_0^\infty \frac{dt}{t} \int_0^\infty \{e^{-\theta t - \eta z} - I_{(0,x)\times(0,y)}(t, z)\}\, P\{X_t \in dz\}.$$

We finally prove our main theorem. We will choose the $\{a_n\}$ sequence on the basis of the domain of attraction condition of N; the sequence $\{b_n\}$ will then automatically be determined.

Theorem 3.2. *Suppose that S_1 is in a domain of attraction of a stable distribution with parameters α, p, q where $0 < \alpha \leqq 2$, but $\alpha \neq 1$ unless $p = q = \frac{1}{2}$. Let $0 < c_n \to \infty$ be chosen in such a way that $S_n/c_n \xrightarrow{\mathscr{D}} X_1$. Then (N, S_N) is in a bivariate domain of attraction and the limiting Laplace-Stieltjes transform is $h(\lambda, \mu)$ where*

$$\log\{-\log h(\lambda, \mu)\} = \int_0^\infty \int_0^\infty \{e^{-w} - e^{-\lambda w - \mu z}\}\, P\{X_w \in dz\} \frac{dw}{w}.$$

Proof. Let $G_N(x)=G(x,\infty)$ be the harmonic renewal measure corresponding to N. Since S_n/c_n converges, N is in a domain of attraction of index $\beta=P(X_1>0)$. See [2, 10] or example 1 in [6]. Note that $\alpha\neq 1$ unless $p=1/2$ is basically used here.

By (ii) of Lemma 1.2 there exists a sequence of constants $0<d_n\to\infty$ such that for $0<y$ and $n\to\infty$

$$G_N(n\,y)-\log d_n\to\beta\,\gamma+\beta\log y.$$

By the definition of G_N then

$$G_N(n)-\log d_n=\sum_{m=1}^{n}\frac{1}{m}P(S_m>0)-\log d_n\to\beta\,\gamma.$$

Define for all $n\geqq 1$, $a_n=n$ and $b_n=c_n$; then by (16)

$$\lim_{n\to\infty}\int_{nx}^{nu}\int_{c_n y}^{c_n v}G(dp,dq)=\int_x^u\int_y^v\frac{1}{t}P\{X_t\in dz\}\,dt$$

and hence for $\lambda>0$, $\mu>0$ by (10)

$$g(\lambda/n,\mu/c_n)-G(n,\infty)\to\gamma(\lambda,\mu,1,\infty).$$

We work towards (ii) of Lemma 2.1. Let us write

$$\begin{aligned}\log d_n-g(\lambda/n,\mu/c_n)=&\left\{\log d_n-\sum_{m=1}^{n}\frac{1}{m}P[S_m>0]\right\}\\&+\sum_{m=1}^{n}\int_0^\infty\int_0^\infty\frac{1}{m}\left\{1-\exp-\left(\frac{\lambda x}{n}+\frac{\mu y}{c_n}\right)\right\}\delta_m(dx)\,P[S_m\in dy]\\&-\sum_{m=n+1}^{\infty}\int_0^\infty\int_0^\infty\frac{1}{m}\exp-\left(\frac{\lambda x}{n}+\frac{\mu y}{c_n}\right)\delta_m(dx)\,P[S_m\in dy].\end{aligned}$$

The first term has limit $-\beta\gamma$ as $n\to\infty$; in the two other terms one uses the effect of the point masses δ_m to write

$$\begin{aligned}&\lim_{n\to\infty}\{\log d_n-g(\lambda/n,\mu/c_n)\}\\&=-\beta\,\gamma+\lim_{n\to\infty}\left\{\sum_{m=1}^{\infty}\int_0^n\int_0^\infty\frac{1}{m}\left\{1-\exp-\left(\frac{\lambda x}{n}+\frac{\mu y}{c_n}\right)\right\}\delta_m(dx)\,P[S_m\in dy]\right.\\&\qquad\left.-\sum_{m=1}^{\infty}\int_n^\infty\int_0^\infty\frac{1}{m}\exp-\left(\frac{\lambda x}{n}+\frac{\mu y}{c_n}\right)\delta_m(dx)\,P[S_m\in dy]\right\}\\&=-\beta\,\gamma+\lim_{n\to\infty}\{G(n,\infty)-g(\lambda/n,\mu/c_n)\}\\&=-\beta\,\gamma-\gamma(\lambda,\mu,1,\infty).\end{aligned}$$

Replace $[d_n]$ by m then for some sequences $0<a'_m\to\infty$ and $0<b'_m\to\infty$

$$\log m+\log\{1-c(\lambda/a'_m,\mu/b'_m)\}\to-\beta\,\gamma-\gamma(\lambda,\mu,1,\infty).\tag{18}$$

It remains to identify the limit. Now by (13)

$$\begin{aligned}\log\{-\log h(\lambda,\mu)\} &= -\beta\gamma - \int_0^\infty\int_0^\infty [e^{-\lambda t-\mu z} - I_{(0,1)}(t)]\, P[X_t\in dz]\,\frac{dt}{t}\\ &= -\beta\gamma + \int_0^\infty\int_0^\infty \{e^{-t} - e^{-\lambda t-\mu z}\}\, P[X_t\in dz]\,\frac{dt}{t}\\ &\quad + \int_0^\infty\int_0^\infty \{I_{(0,1)}(t) - e^{-t}\}\, P\{X_t\in dz\}\,\frac{dt}{t}.\end{aligned}$$

The last integral is easily evaluated using (4); it equals $\gamma P[X_t>0]=\gamma\beta$ since $\beta=P[X_t>0]=P[X_1>0]$. □

According to (14) we expect $\log\{-\log h(\lambda,\mu)\}$ to be of the form given in the double exponential representation for some $H^{(w)}$. Denote by $(Y_1(t), Y_2(t))$ the limiting bivariate stable processes associated with a random walk having steps distributed like (N, S_N). Then we can write

$$\log\{-\log h(\lambda,\mu)\} = \int_0^\infty\int_0^\infty\int_0^\infty [e^{-t} - e^{-\lambda x-\mu y}]\, P\{(Y_1(t), Y_2(t))\in(dx, dy)\}\,\frac{dt}{t}.$$

Theorem 3.2 gives alternatively

$$\log\{-\log h(\lambda,\mu)\} = \int_0^\infty\int_0^\infty\int_0^\infty [e^{-t} - e^{-\lambda x-\mu y}]\, \delta_t(dx)\, P\{X_t\in dy\}\,\frac{dt}{t}.$$

Hence we see that the measures $P\{(Y_1(t), Y_2(t))\in(dx, dy)\}$ and $\delta_t(dx)P\{X_t\in dy\}$ play the same role in the representation of $\log\{-\log h(\lambda,\mu)\}$. This does not mean, of course, that the measures are the same on $\mathbb{R}_+^2$; it actually is the continuous time analogue of the fact that on $(0,\infty)$

$$\sum_{m=1}^{\infty}\frac{1}{m}C^{(m)}(dx) = \sum_{m=1}^{\infty}\frac{1}{m}P[S_m\in dx]$$

where $C(dx)=P[S_N\in dx]$.

The form of $\log\{-\log h(\lambda,\mu)\}$ in the theorem is the form of the characteristic exponent of the time-space maximal process of any Lévy process X_t found by Fristedt [4]. In the present setting, X_t is the limiting stable process associated with the random walk S_n.

Corollary 3.3. *Suppose S_1 is as in Theorem 3.2. Then (N, S_N) is in the domain of attraction of a stable law on $\mathbb{R}_+^2$ with parameters $(\beta, \alpha\beta)$ where $\beta=\frac{1}{2}+\frac{\delta}{2\alpha}$ and with δ defined by*

$$p-q=\left(\tan\frac{\pi\delta}{2}\right)\Big/\left(\tan\frac{\alpha\pi}{2}\right).$$

Proof. The value for $\beta=P[X_1>0]$ is well known. See for example [2, 6]. By (ii) of Lemma 1.2

$$G_N(x)+\log R(x)\to\beta\gamma$$

where $R(x)=\{xL(x)\}^{-\beta}$ for some s.v.L. In the theorem we identified the sequence d_n by

$$G_N(n)-\log d_n \to \beta\gamma.$$

Hence by (ii) of Lemma 1.2 $R(n)\sim 1/d_n$ or $nL(n)\sim d_n^\beta$ or $n\sim d_n^{1/\beta}L^*(d_n^{1/\beta})$ where L^* is the conjugate of L [13]. However we defined m by $m=[d_n]$; hence we obtain similarly that $n\sim m^{1/\beta}L^*(m^{1/\beta})$.

On the other hand S_1 also is in a domain of attraction of a stable law with parameter α; hence for some s.v. L_0

$$c_n=n^{1/\alpha}L_0(n)$$

as $n\to\infty$. Hence in (18)

$$a'_m=n \quad \text{or} \quad a'_m\sim m^{1/\beta}L^*(m^{1/\beta})$$

and

$$b'_m=c_n \quad \text{or} \quad b'_m\sim m^{1/(\alpha\beta)}L_1(m)$$

for some s.v. L_1 [13, p. 18].

Reversing the argument, used above for N, we find that S_N is in the domain of attraction of a stable law with index $\alpha\beta$. □

The above corollary unifies some results of Rogosin [12]. Since $|\delta|\leqq 1-|1-\alpha|$, $0\leqq\beta\leqq 1$ if $0<\alpha<1$ and $1-\frac{1}{\alpha}\leqq\beta\leqq\frac{1}{\alpha}$ if $1<\alpha\leqq 2$. Hence $\alpha\beta\leqq 1$. If $\alpha=2$, S_1 is in the domain of attraction of the normal law and the sample paths are continuous; then $\beta=\frac{1}{2}$ and $\alpha\beta=1$. If $\alpha<2$ and $\alpha\beta=1$ then S_1 is in the domain of attraction of a spectrally negative stable process. If $\alpha(1-\beta)=1$ then the index of S_N is $\alpha\beta=\alpha-1$, a separate result of Rogosin [12].

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Received January 22, 1981; in revised form June 5, 1982