Z. Wahrscheinlichkeitstheorie verw. Gebiete
61, 459–465 (1982)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

On Transience and Recurrence of Generalized Random Walks

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A generalized random walk is a process of partial sums of a stationary and ergodic sequence of random variables. It is remarkable that many results of random walk theory (where the variables are supposed to be independent) carry over to this general setting. In Sect. 1 we give the basic definitions and a survey of some of the main results (which are rather scattered around in the literature). In Sect. 2 we give an example of a symmetric one step transient generalized random walk on $\mathbb{Z}^2$, whose increments have mean zero. In the last section we generalize this construction, obtaining a class of generalized random walks which might be called deterministic, as the corresponding dynamical systems have zero entropy. A related class of random walks is studied in [1]. Thanks are due to Karl Petersen for introducing the author to some of the relevant literature.

§ 1. Generalities

Let A be a locally compact, second countable group with unit e. Although the following can easily be adapted to make sense in general, we shall simplify somewhat by assuming A to be abelian. Then A is metrizable, and we might as well suppose that the topology on A is given by a symmetric invariant metric, determined by a norm $\|.\|$.

Let $(X_n)_{n=1}^{\infty}$ be a stationary and ergodic sequence of A-valued random variables on a probability space $(\Omega, \mathscr{F}, P)$. As usual, we suppose $\Omega = A^{\mathbb{N}}$, with $\mathscr{F}$ the product σ-algebra of the Borel σ-algebra of A, and assume that $X_n = X_1 \circ T^{n-1}$, where $X_1(\omega) = \omega_1$ if $\omega = \omega_1 \omega_2 \dots$. Here $T: \Omega \to \Omega$ is the shift defined by $(T\omega)_n = \omega_{n+1}$. Stationarity of (X_n) is equivalent to shift-invariance of P, and ergodicity to triviality of the σ-algebra of shift invariant measurable sets.

The process $(S_n)_{n=0}^{\infty}$, defined by $S_0 = e$ and $S_n = X_1 + \dots + X_n$ for $n > 0$, is called a *generalized random walk* (GRW) on A (starting at e) ([2, 10, 14]).

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An open neighbourhood base of e is given by $(U_\varepsilon)_{\varepsilon>0}$, where

$$U_\varepsilon=\{x\in A\colon \|x\|<\varepsilon\}.$$

A GRW (S_n) is called *recurrent* ([10]) if

$$P[\exists\, n>0\colon S_n\in U_\varepsilon]=1 \quad \text{for all } \varepsilon>0.$$

If (S_n) is not recurrent, then (S_n) is called *transient*. If follows readily (proving by induction that $P[S_m\notin U_\varepsilon, m\geqq n]=0$ for all $\varepsilon>0$) from the stationarity of (X_n) that (S_n) is recurrent iff

$$P[S_n\in U_\varepsilon\ \infty\text{-often}]=1 \quad \text{for all } \varepsilon>0$$

([10, Cor. 11.2]), and hence that (S_n) is transient iff

$$P[\|S_n\|\to\infty]=1$$

([10, Cor. 11.3]). We remark that in [3] the case is considered where the increments (X_n) are merely stationary.

We shall now define a useful notion for studying transience and recurrence of a GRW.

Let λ be a Haar measure on A. Let $U_\varepsilon(x)=U_\varepsilon+x$ for $x\in A$. Following [7] we define the *range* $R_n=R_n(U_\varepsilon)$ of (S_n) w.r.t. U_ε by

$$R_n=R_n(U_\varepsilon)=\lambda\left(\bigcup_{k=1}^{n} U_\varepsilon(S_k)\right).$$

It is observed in [12, 7] that R_n is a subadditive function:

$$\begin{aligned} R_{n+m}&\leqq\lambda\left(\bigcup_{k=1}^{n} U_\varepsilon(S_k)\right)+\lambda\left(\bigcup_{k=1}^{m} U_\varepsilon(S_{n+k})\right)\\ &=R_n+\lambda\left(\bigcup_{k=1}^{m} U_\varepsilon(S_{n+k}-S_n)\right)=R_n+R_m\circ T^n. \end{aligned}$$

The subadditive ergodic theorem then yields

Theorem 1 [12, 7]. *Let (S_n) be a GRW. Then for all $\varepsilon>0$*

$$\frac{R_n(U_\varepsilon)}{n}\to\int_{U_\varepsilon} P[S_n\notin U_\varepsilon(x), n=1,2,\ldots]\,d\lambda(x)=:C(U_\varepsilon)\quad a.s.$$

If (S_n) is transient, $C(U)$ is called the *capacity* of U. In case $A=\mathbb{Z}$, $C(\{e\})=P[S_n\neq e, n=1,2,\ldots]$, and Theorem 1 has been proved for random walks by Kesten, Spitzer and Whitman ([11, p. 35–40]).

Theorem 2. *A $GRW(S_n)$ is recurrent iff for all $\varepsilon>0$*

$$\frac{R_n(U_\varepsilon)}{n}\to 0\quad a.s.$$

Proof. Suppose (S_n) is recurrent. Let $\varepsilon>0$, $x\in U_\varepsilon$ and let $\delta=\varepsilon-\|x\|$. Then $U_\delta=U_\delta(e)\subset U_\varepsilon(x)$. Hence

$$P[S_n\notin U_\varepsilon(x),\ n=1,2,\dots]\leqq P[S_n\notin U_\delta,\ n=1,2,\dots]=0.$$

So $C(U_\varepsilon)=0$. If on the other hand $C(U_\varepsilon)=0$ for all $\varepsilon>0$, then $0=C(U_\varepsilon)\geqq\lambda(U_\varepsilon)P[S_n\notin U_{2\varepsilon},\ n=1,2,\dots]$. Hence (S_n) is recurrent. □

In [13] there is a neat proof of an interesting result by means of the recurrence criterion furnished by Theorem 2.

Theorem 3 [2, 10, 13]. *Let (S_n) be a generalized random walk on $\mathbb{R}$, with $E|X_1|<\infty$. Then (S_n) is recurrent iff $EX_1=0$.*

Proof. ([13]). If $EX_1\neq 0$, then the transience of (S_n) follows easily from the ergodic theorem.

Suppose $EX_1=0$. Let λ be Lebesque measure. By the ergodic theorem there exists for any $\delta>0$ an $n_0=n_0(\delta)$ such that $|S_k|<k\delta$ for $k>n_0$ a.e.
Hence for all n

$$R_n(U_\varepsilon)\leqq R_{n_0}(U_\varepsilon)+\lambda((-n\delta,n\delta))\leqq 2\varepsilon n_0+2n\delta \quad \text{a.e.}$$

Therefore $\limsup_{n\to\infty}\frac{1}{n}R_n(U_\varepsilon)\leqq\delta$ a.e. As $\delta>0$ can be arbitrarily small, $\frac{1}{n}R_n(U_\varepsilon)\to 0$ a.e. By Theorem 2, (S_n) is recurrent. □

Theorem 3 raises the question whether there exist transient GRW's on $\mathbb{R}^2$ with $EX_1=0$. There is an example in [5], where the increments are even independent, but $\operatorname{var}X_1=\infty$. In [13] there is a rather complicated construction (due to Berbee) of a transient one step GRW in $\mathbb{Z}^2$ with $EX_1=0$. In the next section we present an example of such a GRW, based on a familiar geometrical object: the Peano curve.

§2. The Peano Curve as a Generic Point for a Generalized Random Walk

In the sequel we only consider one step GRW's (i.e., $\|X_1\|\equiv 1$) in $\mathbb{Z}^2$, so we may shrink our probability space to $\Omega=J^{\mathbb{N}}$, where J is a four element set, say $J=\{0,1,2,3\}$, and take X_1 defined by

$$X_1(\omega)=\left(\cos\left(2\pi\frac{\omega_1}{4}\right),\ \sin\left(2\pi\frac{\omega_1}{4}\right)\right)\quad \text{if } \omega=\omega_1\omega_2\dots. \tag{1}$$

If P is the shift-invariant ergodic probability measure determining the GRW, then we denote $(S_n)=(S_n^P)$.

We call (S_n^P) *symmetric* if P is σ-invariant, where $\sigma\colon\Omega\to\Omega$ is defined by

$$(\sigma(\omega))_n=((\sigma(\omega)+1)\bmod 4)_n \qquad \text{for } n=1,2,\dots.$$

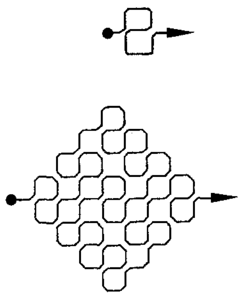

Fig. 1

Consider the usual approximating polygons K_m to the Peano curve [4, p. 400], but unscaled: each curve consists of 9^m line segments of length 1, and K_{m+1} is obtained from K_m by scaling each segment in K_m by a factor 3, and replacing it by a copy of K_1, having the same direction (cf. Fig. 1). Obviously K_{m+1} starts with K_m for all m. We shall define an $\omega^0 \in \Omega$, such that the *finite walk* $(S_n(\omega^0))_{n=0}^{9m-1}$ passes through the consecutive vertices of K_m for all m. Recall that

$$S_0(\omega)=(0,0), \qquad S_n(\omega)=\sum_{k=1}^{n} X_k(\omega)=\sum_{k=1}^{n} X_1(T^{k-1}\omega)$$

for all $\omega \in \Omega$, X_1 as in (1). So our goal will be achieved, if we put

$$\omega_1^0 \omega_2^0 \dots \omega_9^0 = 0\ 1\ 0\ 3\ 2\ 3\ 0\ 1\ 0 =: V, \tag{2}$$

and

$$\omega_{9k-8}^0 \omega_{9k-7}^0 \dots \omega_{9k}^0 = \sigma^j(0)\,\sigma^j(1) \dots \sigma^j(0) = \sigma^j(V), \tag{3}$$

where $j=\omega_k^0$, for $k=1, 2, \dots$.

We say that a word (i.e., a finite string of elements from J) $B=b_1 b_2 \dots b_p$ *occurs* in a word $C=c_1 c_2 \dots c_q$, if $b_1 b_2 \dots b_p = c_{k+1} c_{k+2} \dots c_{k+p}$ for some k. We denote by freq $(B; C)$ the number of occurrences of B in C, divided by the length q of C.

For a word $B=b_1 b_2 \dots b_p$, let

$$[B]=\{\omega \in \Omega\colon \omega_1=b_1, \dots, \omega_p=b_p\}$$

denote a *cylinder*. For each $k \geqq 1$ we define a probability measure P_k on Ω by

$$P_k([B]) = \operatorname{freq}(B; \omega_1^0 \omega_2^0 \dots \omega_k^0) \tag{4}$$

for any cylinder $[B]$. The special structure of ω^0 (in the terminology of Oxtoby [9], ω^0 is a transitive point; see also the next section) brings forth that

$$P([B]) = \lim_{k\to\infty} P_k([B]) \tag{5}$$

exists for all cylinders $[B]$, and that P is a shift-invariant, ergodic probability measure. Let (S_n^P), with increments (X_n), be the generalized random walk determined by P. We first show that $EX_1=0$. This will follow if X_1 takes all values with equal probability, i.e., if $P([j])=\frac{1}{4}$ for all $j\in J$. (A little more work yields that P is symmetric). To see this, note that (3) implies

$$P_{9^{m+1}}([j])=\sum_{i\in J} P_{9^m}([i])\,\pi_{ij} \tag{6}$$

for $j\in J$, where $\pi_{ij}=\operatorname{freq}(j;\sigma^i(V))$. Letting $m\to\infty$ in (6) yields $P([\cdot])=P([\cdot])\,\Pi$ where the matrix $\Pi=(\pi_{ij})$ is double Markov, hence $P([\cdot])=(\frac{1}{4}\,\frac{1}{4}\,\frac{1}{4}\,\frac{1}{4})$. We now show that (S_n^P) is transient. Let $\omega\in\operatorname{Supp}P$. Then $P([\omega_1\omega_2\ldots\omega_k])>0$ for all $k\geqq 1$. By (4) and (5) this implies that $\omega_1\omega_2\ldots\omega_k$ occurs in $\omega_1^0\omega_2^0\ldots\omega_N^0$ for large enough $N=N(k)$. The walk $(S_n(\omega))_{n=0}^k$ is therefore a subwalk of the walk $(S_n(\omega^0))_{n=0}^N$. But from the construction of the Peano curve it is obvious that no lattice point is visited more than twice by $(S_n(\omega^0))$. So this holds for all $\omega\in\operatorname{Supp}P$. Hence (S_n^P) is transient.

§3. A General Construction

It is not the spacefilling property of the Peano curve which makes the associated GRW transient, but rather the fact that it visite no lattice point more than twice. To eludicate this we now consider a more general class of examples, which contains instances both of transience and of recurrence.

Let $J^*=\bigcup_{n=0}^{\infty} J^n$ be the set of finite words over $J=\{0,1,2,3\}$. A map $\theta\colon J^*\to J^*$ such that for all $V,\ W\in J^*$ $\theta(VW)=\theta(V)\,\theta(W)$ is called a *substitution.* The m^{th} iterate θ^m of a substitution θ is defined by $\theta^m(W)=\theta^{m-1}(\theta W)$ for $W\in J^*$, $m=1,2,\ldots$. Suppose 0 is the first symbol of $\theta(0)$, and that r_m, the length of $\theta^m(0)$, tends to infinity as $m\to\infty$. Then θ determines an infinite sequence ω^θ by requiring for $m=1,2,\ldots$

$$\omega_1^\theta\,\omega_2^\theta\ldots\omega_{r_m}^\theta=\theta^m(0). \tag{7}$$

A substitution θ is called *primitive* if for some $m>0$ all $j\in J$ occur at least once in all $\theta^m(i)$, $i\in J$. It is known ([8]) that for primitive θ

$$P_\theta([B])=\lim_{k\to\infty}\operatorname{freq}(B;\omega_1^\theta\,\omega_2^\theta\ldots\omega_k^\theta) \tag{8}$$

exists for any cylinder $[B]$, that P_θ is a shift-invariant and ergodic probability measure, and that for all $j\in J$

$$P_\theta([B])=\lim_{m\to\infty}\operatorname{freq}(B;\theta^m(j)). \tag{9}$$

We now specialize. Let V be any word in J^* starting with 0. Then V determines a substitution θ by requiring for $j\in J$

$$\theta(j)=\sigma^j(V). \tag{10}$$

Unless $V\in\{0\}^*$ or $V\in\{0,2\}^*$, θ is primitive. For the primitive case, let ω^V be the sequence given by (7), P^V the ergodic probability measure given by (8), and

(S_n^V) the associated GRW. From (10) it follows that $\theta^m(j)=\sigma^j\,\theta^m(0)$ for all $m>0$, so (9) implies that P^V is σ-invariant, i.e., (S_n^V) is symmetric. In particular its increments have zero mean. Even if V is such that θ is not primitive, then V still determines a GRW (S_n^V), but (S_n^V) is restricted to the x-axis, and (9) only holds for $j=0$ (in case $V\in\{0\}^*$) or $j=0,2$ (in case $V\in\{0,2\}^*$). We remark that P^V has no atoms, unless ω^V is periodic. (It follows from [6, p. 266] that this is the case – for primitive θ – iff the period is 4, and the repeating word is a permutation of J).

Definition. Let $\omega\in\Omega$ be fixed. The infinite walk $(S_n(\omega))$ is called *resolvable* if it never retraces itself in the same direction, i.e., if $S_n(\omega)=S_{n'}(\omega)$ and $S_{n+1}(\omega)=S_{n'+1}(\omega)$ imply that $n=n'$.

Proposition. *Let V be a word of length at least 2 from J^*. The GRW (S_n^V) is transient iff the walk $(S_n(\omega^V))$ is resolvable.*

Proof. First suppose $(S_n(\omega^V))$ is resolvable. Then $(S_n(\omega^V))$ visits no lattice point more than four times. Hence (cf. the end of Sect. 2), for all $\omega\in\operatorname{Supp}P^V$, the walk $(S_n(\omega))$ has the same property, so (S_n^V) is transient.

Suppose $(S_n(\omega^V))$ is not resolvable. Then for some $k\geqq 1$, the finite walk $(S_n(\omega^V))_{n=0}^{r^k}$ retraces itself, where we put r for the length of V. We may take $k=1$; for $k>1$ consider $V'=\theta^k(0)$, and note that $\omega^{V'}=\omega^V$, $P^{V'}=P^V$. We then have $R_r(\omega^V)\leqq r-1$, and by symmetry, $R_r(\sigma^j\omega^V)\leqq r-1$ for all $j\in J$. Now consider the walk $(S_n(\omega^V))_{n=0}^{r^2}$. Since $\omega_1^V\omega_2^V\dots\omega_{r^2}^V=\theta(\omega_1^V)\dots\theta(\omega_r^V)$, this walk is made up of r walks, each congruent to the walk $(S_n(\omega^V))_{n=0}^{r}$. At least two of these r walks pass through exactly the same points in order, as at least one segment in $(S_n(\omega^V))_{n=0}^{r}$ is traced in the same direction. Therefore $R_{r^2}(\omega^V)\leqq(r-1)\,R_r(\omega^V)\leqq(r-1)^2$. Induction yields that for $m=1,2,\dots$

$$R_{r^m}(\omega^V)\leqq(r-1)^m.$$

This easily implies that $\frac{1}{n}R_n(\omega^V)\to 0$ as $n\to\infty$. Now let $\omega\in\operatorname{Supp}P^V$ be arbitrary. Then $\omega_1\omega_2\dots\omega_{r^m}$ occurs in ω^V (by (8) and (9)). But ω^V is a concatenation of words $\theta^m(j)$, $j\in J$, of length r^m. Consequently $\omega_1\omega_2\dots\omega_{r^m}$ occurs in some word $\theta^m(i)\;\theta^m(j)$, with $i,j\in J$. Therefore $R_{r^m}(\omega)\leqq R_{r^m}(\sigma^i(\omega^V))+R_{r^m}(\sigma^j(\omega^V))=2R_{r^m}(\omega^V)$, and it follows that $\frac{1}{n}R_n(\omega)\to 0$ for all $\omega\in\operatorname{Supp}P^V$. By Theorem 2, (S_n^V) is recurrent. □

We end with some remarks. As the proofs of our assertions are often technical, we omit or merely sketch them.

Remark 1. The following criteria allow one to decide resolvability of $(S_n(\omega^V))$ in most cases.

Circle criterion: If $\|S_r(\omega^V)\|^2<r$, then $(S_n(\omega^V))$ is not resolvable.

Torus criterion: Let $\bar\omega$ be such that $\bar\omega_1\bar\omega_2\dots\bar\omega_{4r}=V\sigma(V)\,\sigma^2(V)\,\sigma^3(V)$. If $(S_n(\bar\omega))_{n=0}^{4r}$ is resolvable on the torus obtained by identifying opposite sides of the square with vertices at $(0,0)$, $S_r(\bar\omega)$, and $S_{2r}(\bar\omega)$, then $(S_n(\omega^V))$ is resolvable.

Remark 2. There exist null recurrent and positive recurrent GRW's (S_n^V). For an example of the latter, consider $V=02$. It is remarkable that although P^V is non-atomic, the GRW is restricted to the points $(0, q)$, where $q \in \{0, \pm 1, \pm 2\}$. (This follows since ω^V is a concatenation of words 02 and 20.) Also, since the word 02020 cannot occur, the GRW returns to the origin within six steps, so (S_n^V) is positively recurrent.

For examples of null recurrent GRW's consider $V=0^p 1230^q$ where $0^p = 0 \dots 0$ (p times), and $p, q \geqq 3$. By the Proposition, (S_n^V) is recurrent. Let $r=p+q+3$ be the length of V, and let Y be the first return time to the origin. Let $\Omega^V = \operatorname{Supp} P^V$. Then $\{\theta^m(\Omega^V), T^{-1}\theta^m(\Omega^V), \dots, T^{-r^m-1}\theta^m(\Omega^V)\}$ is a partition of Ω^V for all m (cf. [6, p. 225]), so these sets have probability r^{-m}. Considering $(S_n(\omega^V))_{n=0}^{r^m}$ for $m=1, 2, \dots$, one sees that

$$[Y=4r^m] = \bigcup_{j=(p-1)r^m}^{pr^m} T^{-j}\theta^m(\Omega^V) \setminus \left(\bigcup_{j=0}^{m-1} [Y=4r^j]\right)$$

for $m \geqq 0$, and that Y cannot have any other value. Let $p_m = P^V[Y=4r^m]$. Then $p_0 = 2/r$, and

$$p_m = \{(r^m+1) - r^m p_{m-1} - \dots - r^m p_0\} r^{-(m+1)},$$

from which we obtain $p_m = \frac{r-3}{r-2} \cdot \frac{1}{r}\left(1-\frac{1}{r}\right)^m + \frac{1}{r-2}\left(1-\frac{1}{r}\right)\frac{1}{r^m}$. So $EY = \sum_{m=0}^{\infty} 4r^m p_m = \infty$, and (S_n^V) is null recurrent.

References

1. Aaronson, J., Keane, M.: The visits to zero of some deterministic random walks. J. London Math. Soc. **44**, 535–553 (1982)
2. Atkinson, G.: Recurrence of co-cycles and random walks. J. London Math. Soc. (2) **13**, 486–488 (1976)
3. Berbee, H.C.P.: Recurrence and transience for random walks with stationary increments. Z. Wahrscheinlichkeitstheorie verw. Gebiete **56**, 531–536 (1981)
4. Blumenthal, L.M., Menger, K.: Studies in geometry. San Francisco: Freeman 1970
5. Chung, K.L., Fuchs, W.H.J.: On the distribution of values of sums of random variables. Memoirs Amer. Math. Soc. No. 6. Amer. Math. Soc. Publ., 1951
6. Dekking, F.M.: The spectrum of dynamical systems arising from substitutions of constant length. Z. Wahrscheinlichkeitstheorie verw. Gebiete **41**, 221–239 (1978)
7. Derriennic, Y.: Quelques applications du théorème ergodique sous-additif. Journées sur les marches aléatoires. Nancy '79. Astérisque **74**, 183–201. Paris: Soc. Math. France 1980
8. Michel, P.: Stricte ergodicité d'ensembles minimaux de substitutions. C.R. Acad. Sci. Paris Sér. A **278**, 811–813 (1975)
9. Oxtoby, J.C.: Ergodic sets. Bull. Amer. Math. Soc. **58**, 116–136 (1952)
10. Schmidt, K.: Lectures on cocycles of ergodic transformation groups. Lect. in Math. 1. Delhi-Bombay-Calcutta-Madras: Mac Millan 1977
11. Spitzer, F.: Principles of random walk. Princeton, New Jersey: Van Nostrand 1964
12. Spitzer, F.: Discussion of a paper by J.F.C. Kingman. Ann. Probab. **1**, 904–905 (1973)
13. Van der Vecht, D.: Recurrence and transience for random walks with stationary and ergodic increments. Report 118. Vrije Universiteit. Amsterdam, 1980
14. Westman, J.J.: Sums of dependent random variables. J. Math. Anal. Appl. **77**, 120–131 (1980)

Received March 18, 1982