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An Extension of Expectation

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1. Introduction

Let $\mathscr{F}$ denote the convolution semigroup of probability distributions (the operation is the convolution, denoted by $*$, corresponding to the addition of independent random variables). The expectation, denoted by $\mathbf{E}(F)$, satisfies

$$\mathbf{E}(F_1 * F_2) = \mathbf{E}(F_1) + \mathbf{E}(F_2)$$

whenever the expectations on the right-hand side are finite, that is, it is a partial homomorphism. Our aim is to derive the following

Theorem. *There exists a homomorphism $\varphi: \mathscr{F} \to R^+$ (R^+ denotes the additive group of real numbers) such that $\varphi(F) = \mathbf{E}(F)$ for every F having a finite expectation.*

The difficulty is caused by the fact that $\mathscr{F}$ is not cancellative, i.e. $F * F_1 = F * F_2$ does not imply $F_1 = F_2$ (the Khinčin phenomenon, see e.g. Feller [2], p. 479). In Sect. 2 we develop an algebraical method to eliminate this problem, in Sect. 3 we apply it to our case and in Sect. 4 we state several further results and open problems.

2. An Algebraical Lemma

Lemma. *Let S be a commutative semigroup, S' a subsemigroup of S, G a group and ψ: $S' \to G$ a homomorphism. For the existence of a homomorphism φ: $S \to G$ satisfying $\varphi(s) = \psi(s)$ for every $s \in S'$ the following condition is necessary:*

$$\begin{aligned} s s_1 = s s_2 \quad &\textit{implies} \quad \psi(s_1) = \psi(s_2) \\ &\textit{if} \;\; s \in S, s_1, s_2 \in S'. \end{aligned} \tag{1}$$

If G is divisible, then (1) *is also sufficient.*

This may be well-known, but we cannot give any reference.

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Proof. The necessity of (1) is evident. To prove sufficiency write $s_1 \sim s_2$ for s_1, $s_2 \in S$ if there is an $s \in S$ such that $s s_1 = s s_2$. $\sim$ is evidently a congruence relation, and the factor-semigroup $S_0 = S/\sim$ is easily seen to be cancellative. Let α: $S \to S_0$ denote the canonical homomorphism. Our condition (1) just means that ψ is compatible with $\sim$, and hence it induces a homomorphism ψ_0: $S'_0 \to G$, where $S'_0 = \alpha(S')$, such that $\psi = \psi_0 \circ \alpha$.

S_0, being cancellative, can be embedded into a group H and ψ_0 can be extended to a homomorphism ψ_1: $H' \to G$, where H' is the subgroup of H generated by S'_0. As G is divisible, by a theorem of Baer [1] (see also Fuchs [3], Chap. IV. § 21) this homomorphism can be extended to a homomorphism φ_1: $H \to G$, and now we can set $\varphi = \varphi_1 \circ \alpha$.

3. Proof of the Theorem

By the Lemma we have to show only that $F_1 * F = F_2 * F$ implies $\mathbf{E}(F_1) = \mathbf{E}(F_2)$ for F, F_1, $F_2 \in \mathscr{F}$. Denote the characteristic functions of F, F_1, F_2 by f, f_1, f_2 respectively. As the expectations are finite, we have

$$\mathbf{E}(F_j) = -i f'_j(0) \qquad (j = 1, 2). \tag{2}$$

$F * F_1 = F * F_2$ implies $f f_1 = f f_2$. f is continuous and assumes the value 1 at 0, hence we have $f_1(t) = f_2(t)$ for sufficiently small values of t, and therefore

$$f'_1(0) = f'_2(0). \tag{3}$$

(2) and (3) imply $\mathbf{E}(F_1) = \mathbf{E}(F_2)$.

4. Problems and Results

1) It is natural to ask whether such an extension can have any "good" properties. It is easy to see that it cannot be monotonic; however, we proved that it may have the weaker property of being positive for positive variables and negative for negative ones.

2) The additivity of expectation is not restricted to independent variables. Can a real number $\varphi(\xi)$ be assigned to every random variable ξ being additive (i.e. satisfying $\varphi(\xi_1 + \xi_2) = \varphi(\xi_1) + \varphi(\xi_2)$ for all ξ_1 and ξ_2), depending only on the distribution of ξ and coinciding with the expectation if it exists? We conjecture that the answer is negative.

3) Besides its additivity, for independent variables the expectation is also multiplicative. Can an extension have this property as well? That is, together with the additive convolution $*$ we may also consider the multiplicative convolution $\circledast$, defined by $F_1 \circledast F_2 = F$ if there are independent variables having distributions F_1 and F_2, whose product has a distribution F. The operations $*$ and $\circledast$ turn $\mathscr{F}$ into a quasi-ring (not a ring, as $*$ does not have an inverse). Does there exists a mapping φ: $\mathscr{F} \to R$ satisfying (i) $\varphi(F_1 * F_2) = \varphi(F_1) + \varphi(F_2)$, (ii) $\varphi(F_1 \circledast F_2) = \varphi(F_1)\,\varphi(F_2)$, (iii) $\varphi(F) = \mathbf{E}(F)$ whenever the expectation is finite? We

can show that the latter two conditions can be satisfied if (i) is dropped.

4) As the variance (or any other cumulant) is also a partial homomorphism and is completely determined by the behaviour of the characteristic function at the neighbourhood of 0, it can also be extended for every variable. However, if we seek this extension of variance in the form

$$\delta(\xi)=\varphi((\xi-\varphi(\xi))^2) \tag{4}$$

where φ is an extended expectation, we generally fail. Can one construct φ so that the mapping δ defined by (4) will be also a homomorphism? This is connected with problems 2) and 3).

References

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